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IN

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Chaplain to his Royal Highness the Prince of Wales.

IN TWO VOLUMES.

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CONTENTS.

MECHANICS.

CHAP.
I. *OF matter and its properties* —— Pag. 1.
II. *Of the three laws of motion* —— 11.
III. *Of the compoſition and reſolution of forces* —— 16.
IV. *Of gravity* —— 22.
V. *Of the center of gravity* —— 31.
VI. *Of the mechanical powers* —— 36.
VII. *Of the mutual action of bodies upon each other* —— 66.
VIII. *Of the perpendicular aſcent aud deſcent of heavy bodies* —— 92.
IX. *Of the deſcent of heavy bodies upon inclined planes* —— 106.
X. *Of pendulums* —— 117.
XI. *Of the motion of projectiles* —— 126.
XII. *Of central forces* —— 139.

OPTICS.

CHAP.
I. *Definitions and lemmas* —— 200.
II. *Of light and the cauſe and laws of its refraction* —— 216.
III. *Of the refraction of light when it paſſes through a plane ſurface* 229.
IV. *Of the refraction of parallel rays, when they paſs through lenſes which are either convex or concave* —— 236.
V. *Of the refraction of diverging rays by a convex lens, and of converging rays by a concave lens* —— 263.
VI. *Of the refraction of converging rays by a convex lens and of diverging rays by a concave lens* —— 278.
VII. *Of the principal focus of meniſcuſes and entire ſpheres of either glaſs or water* —— 283.
VIII. *Of the pictures in a dark room* —— 290.
IX. *Of plane viſion* —— 314.
X. *Of refracted viſion through ſingle glaſſes* —— 353.
XI. *Of teleſcopes and microſcopes* —— 380.
XII. *Of the cauſe and laws of the reflection of light* —— 396.
XIII. *Of the reflection of light from plane mirrours* —— 402.
XIV. *Of the reflection of parallel rays from concave and convex mirrours* —— 405.

XV. *Of the reflection of diverging rays from a concave mirrour and of converging rays from a convex one* —— 408.
XVI. *Of the reflection of converging rays from a concave mirrour and of diverging rays from a convex one* —— 417.
XVII. *Of reflected viſion in plane mirrours* —— 419.
XVIII. *Of reflected viſion in concave and convex mirrours* —— 433.
XIX. *Of reflecting teleſcopes and ſome other optical inſtruments* —— 450.
XX. *Of the different refrangibility of light* —— 456.
XXI. *Of the colours of thin tranſparent plates* —— 471.
XXII. *Of the opakeneſs, tranſparency, and colours of natural bodies* - 477.
XXIII. *Of the rainbow* —— 481.

HYDROSTATICS.

CHAP.

I. *Of the properties of fluids* —— 497.
II. *Of the preſſure of fluids upon the bottoms and ſides of the veſſels that contain them* —— 503.
III. *Of the action of fluids upon bodies immerſed in them, and the methods of comparing the ſpecific gravities of bodies* 525.
IV. *Of the general properties of an elaſtic fluid* —— 546.
V. *Of the air* —— 551.
VI. *Of ſound* —— 569.
VII. *Of ſome effects produced by the preſſure and by the elaſticity of the air* —— 613.
VIII. *Of the winds, particularly the trade-winds* —— 627.
IX. *Of the reſiſtence of fluids* —— 637.
X. *Of rain and the barometer* —— 651.
XI. *Of the thermometer* —— 661.
XII. *Of the motion of fluids* —— 665.
XIII. *Of the times in which cylindrical veſſels empty themſelves through holes in their bottoms* —— 674.
XIV. *Of ſpouting fluids* —— 983.
XV. *Of rivers, and the force of ſtreams that run againſt any obſtacle* —— 691.
XVI. *Of capillary tubes* —— 703.

ASTRONOMY.

CHAP.

I. *Of the ſolar ſyſtem* —— 723.
II. *Of the apparent diurnal and annual motions of the ſun, and its real motion upon its own axis* —— 728.

III. *Of the inferior planets* — 757.
IV. *Of the ſuperior planets* — 785.
V. *Of the ſecondary planets* — 815.
VI. *Of eclipſes* — 815.
VII. *Of the fixed ſtars* — 843.
VIII. *Of the Copernican ſyſtem* — 853.
IX. *Of the ſphere* — 857.
X. *Of the equation of time* — 903.
XI. *Of the twilights* — 916.
XII. *Of the parallax* — 922.
XIII. *Of the comets* — 953.
XIV. *Of the diviſion of time* — 968.
XV. *Of the cauſes, which produce the motions of the planets* — 998.
XVI. *Of the lunar inequalities* — 1013.
XVII. *Of the preceſſion of the equinoxes and the figure of the planets* 1066.
XVII. *Of the tides* — 1078.

The reader is deſired to correct propoſition 50 of optics in the following manner.

When parallel rays paſs out of a rarer medium into a denſer through a convex ſurface, the ſemidiameter of the ſphere, of which that ſurface is a ſegment, is to the diſtance of the focus whether affirmative or negative, that is, whether real or virtual, as the ſine of refraction to the ſine of incidence.

A SYSTEM OF
NATURAL PHILOSOPHY.

INTRODUCTION.

IN explaning the appearances of nature, that alone can be called true philoſophy, which is deduced from fact and experience. For as the cauſes of theſe appearances are not arbitrary notions of the minds own making, but are qualities, which really belong to bodies, and which exiſt without us; our knowledge of nature will all be fantaſtical; unleſs we find out by experiment and obſervation what theſe qualities are, and ſo make our knowlege conformable to the reality of things. Indeed ſome bodies are ſo remote, and others are ſo minute, that it is not in our power to make obſervations upon them, and diſcover their qualities: and amongſt thoſe, which are neither removed out of our reach by their diſtance nor hidden from our obſervation by their minuteneſs, there are but few that we can make experiments upon, in compariſon of that almoſt infinite multitude, with which we are ſurrounded. So that if, in reaſoning about natural things, we were to be confined wholly to experience, our knowledge muſt fall ſhort of being general; for it would be confined to thoſe particular bodies only, that we had made experiments upon.

The natural philoſopher therefore, in order to make his knowledge real and univerſal, ſhould obſerve the three following rules.

RULE I.

No more cauſes of natural appearances are to be admitted, than what are real, and what are ſufficient to account for thoſe appearances.

True philoſophy can admit of no qualities in bodies, which have not been found by experience and obſervation really to belong to them: if we go beyound this, if in order to explane the appearances of nature we introduce ſuch qualities, as no experiments have ever diſcovered; however we may amuſe ourſelves with ingenious conjectures and plauſible ſuppoſitions, our knowledge will be fantaſtical, and our beſt digeſted ſyſtems will be only philoſophical romances.

But if we are to admit of no cauſes but ſuch as are proved by experiments to have a real exiſtence; then we are neceſſarily to admit of no more than what are ſufficient to account for the appearances of nature. When we have any appearance of nature to explane, and have by experiment found out a cauſe that is ſufficient to explane it, here we muſt ſtop: for as the appearance does not require us to go farther, if we multiply cauſes beyond this, they muſt be needleſs, the appearance can be explaned without them, and therefore we cannot from that appearance collect the reality of their exiſtence.

RULE II.

Natural effects of the ſame ſort are to be accounted for by the ſame cauſes.

The deſcent of a heavy body towards the earth in Europe is an effect of the ſame ſort with the deſcent of a heavy body towards the earth in America. We ſhould not therefore aſſign different cauſes for ſuch ſimilar effects as theſe. In like manner the aſcent of water into ſmall tubes of glaſs and into the pores of a ſpunge are effects of the ſame ſort: and when we have found out what cauſe produces one of theſe effects, we are to conclude that the other effect is produced by the ſame cauſe. The motion of the moon round the earth, of the ſatellites round jupiter or ſaturn, and of the primary planets round the ſun, are effects of the ſame ſort: therefore whatever we find to be the cauſe of one of theſe motions, the ſame we may conclude to be the true cauſe of the others.

This rule is deduced from the latter part of the foregoing one. Effects of the ſame ſort may be accounted for by the ſame cauſes: for this reaſon we muſt always aſſign the ſame cauſes for effects of the ſame ſort; becauſe otherwiſe we ſhould multiply cauſes without reaſon and ſhould introduce more than are ſufficient to account for the appearances of nature.

RULE III.

Such qualities, as are found invariably to belong to all bodies that we can make experiments upon, are to be looked upon as qualities belonging to all bodies whatever.

Though there are many bodies, upon which we have not, and many, upon which we never can make any experiments; yet if we have frequently made experiments upon other bodies, that fall immediately under our notice, and find them to be invariably endued with certain qualities,

lities, we may be allowed by induction to extend our concluſion to all other bodies, and ſo to make it univerſal.

All bodies that fall under the notice of our ſenſes are extended; and though the conſtituent particles, of which larger bodies are compoſed, are too minute to be perceived either by the ſight or touch; yet it is a fair concluſion, that theſe particles are endued with the ſame property. We find that all bodies, upon which we can make experiments, have ſolidity; and though there are other bodies, upon which we have never made the experiment, though the ſun and planets are ſo remote from us that we cannot make it, yet we may be allowed to conclude that theſe bodies have likewiſe the ſame property. All bodies, that we have obſerved, are found to gravitate towards one another; they endeavour, if they are near the earth, to deſcend by their weight towards the earths center. Now though we have not tryed the experiment upon every ſtone, or upon every piece of lead ,that we ſee, yet from what we have tryed we conclude that all of them have this ſame quality. And if this concluſion is juſt, when it is extended thus far beyond our own obſervation, we may likewiſe extend it ſtill farther. No one, after what he has experienced in all ſorts of bodies, that he has been uſed to, can reaſonably doubt whether other bodies of different ſorts have not gravity as well as theſe: and from what he has obſerved in all bodies where he can make experiments, he may conclude that this property of gravity belongs to all bodies univerſally, and that the moon gravitates towards the earth. This is what we call reaſoning by induction, when upon finding a thing to be true in all inſtances where we have an opportunity of examining it, we conclude it to be true in all inſtances whatever. And it is by the help of this reaſoning that in natural philoſophy we are enabled from a ſufficient number of particular experiments to draw general concluſions.

A SYSTEM OF
NATURAL PHILOSOPHY.

MECHANICS.

CHAP. I.
Of matter and its properties.

1. *By matter is meant an extended, ſolid, and moveable ſubſtance, that has a force of inactivity.*

TO know what extenſion is, we muſt be ſent to our ſenſes for information. It is a ſimple idea, and therefore cannot be defined. All we can do is to expreſs the ſame thing in different words. Thus inſtead of telling our hearers what extenſion is, we only endeavour to ſhew them, where they may find it; when we ſay, that the extenſion of matter is its length, breadth, and thickneſs.

From the reſiſtance, which we find in any one body to the entrance of any other body into the place it poſſeſſes, till it has left that place, we get the idea of ſolidity. This is likewiſe a ſimple idea and incapable of being defined: the beſt and indeed the only way of making it underſtood is to deſcribe, as well as we can, the manner in which we acquire it; and then our ſenſes will lead us to the true meaning of the word. But we muſt be careful to diſtinguiſh between ſolidity and hardneſs. Theſe two words are uſed promiſcuouſly in common life, as if they both ſtood for one and the ſame property. Whereas ſolidity is a property of all bodies; hardneſs only of ſome. Indeed hard and ſoft are names, that we give to things only in relation to the conſtitutions of our own bodies: that being called hard by us, which will put us to pain, ſooner that change figure by the preſſure of any part of our bodies; and that on the contrary, ſoft, which changes the ſituation of its parts upon an eaſy and unpainful touch. Therefore the ſolidity of matter conſiſts in an utter excluſion of all other matter out of the ſpace it poſſeſſes; but hardneſs in a firm coheſion of its parts, making up maſſes of ſenſible bulk, ſo that the whole does not eaſily change figure. We never ſay that air or water are hard, nor do we in our common way of ſpeaking ſay that either of them are ſolid. But yet ſolidity as now explaned is as much a property of air or water as it is of a flint or a diamond. The ſofteſt body in the world will as invincibly prevent the coming together of any two other bodies, if it be not put out of the way, but remain between them, as the hardeſt, that can be found or imagined. He, that thinks otherwiſe, may eaſily ſatisfy himſelf by trying whether he can bring his hands together, when he has nothing between them but a full-blown bladder. The two flat ſides of two pieces of marble will indeed more eaſily approach each other, when there is nothing but a drop of water between them, than they would have done, if there had been a diamond between them. But this is not becauſe the parts of the diamond are more ſolid than thoſe of the water; but becauſe the parts of the water being more eaſily ſeparable from each other, will by a ſide motion be more eaſily removed, and give way to the approach of the two pieces of marble. But if they could be kept from making way by that ſide motion, they would eternally hinder the approach of theſe two pieces of marble, as much as the diamond; and it would be as impoſſible by any force to ſurmount their reſiſtance, as to ſurmount the reſiſtance of the parts of the diamond. Mr. Locke, to prove that water has ſolidity, mentions an experiment that was made at Florence with a hollow globe of gold filled with water and exactly cloſed. For the globe thus filled, being put into a preſs, which was driven by the extreme force of ſcrews, the water made itſelf way through the pores of that very cloſe metal, and finding no room for a nearer approach of its particles within, got to the outſide, where it roſe like a dew, and ſo fell in drops, before the ſides of the globe could be made to yield to the violent compreſſion of the engine that ſqueezed it. This experiment will indeed prove that gold, as cloſe and compact a metal as it ſeems to be, is very full of pores. Or it will prove that water is an incompreſſible fluid; for no force, that could be applyed, was ſufficient to bring the particles nearer together. But how does it demonſtrate that water is ſolid? What, becauſe the ſides of the globe could not be made to yield to the compreſſion, till the water had made itſelf a way through the pores of the gold? I ſuppoſe then, we muſt have granted, that water has not this property, if the force of the ſcrews had made the ſides of the globe yield, whilſt all the water continued within it? And yet, if the globe had been filled with air inſtead of water, notwithſtanding the ſolidity of the air, the ſides would have been preſſed inwards, though none of the air had eſcaped through the pores of the gold. Not becauſe the air has not the property, that we are now deſcribing; but becauſe it is a compreſſible fluid, ſo that the particles may be ſqueezed cloſer together, and the ſame quantity of it may be thus made to fill a leſſer ſpace. The experiment is plainly applyed by this author to an improper purpoſe, and is ſo far from clearing up the idea of ſolidity, that it has only ſerved to perplex and miſlead his readers.

We may venture to ſay that all matter is, by the very eſſence of it, a moveable ſubſtance, without giving ourſelves the trouble to enquire with the metaphyſicians, whether it is not poſſible for matter to be infinite. For though, if it was infinite and ſo filled all place, it could not be moved from one place to another: yet we are to remember, that all our ideas of ſubſtances are fantaſtical, which are made conformable to no pattern exiſting, that we know of, and that conſiſt of ſuch a collection of ideas as no ſubſtance ever ſhewed us united together. This ſort of ideas being made in reference to things exiſting without us, and being intended to be repreſentations of ſubſtances, as they really are, can be no farther real than as they are combinations of ſimple ideas, which are really united and coexiſt in things without us. The metaphyſicians, who determine the eſſences of things by fancy and ſuppoſition without regarding what matter is, perplex themſelves with endeavouring to find out what it might be; and, if they think they can conceive a poſſibility of its being infinite, they immediately determine that a capacity of being moved from place to place is not an eſſential property of it. But our notions in natural philoſophy are not formed by our own imaginations, but are taken from fact; we endeavour to make our ideas conformable to the reality of things, as we find them, and do not amuſe ourſelves with fancying what poſſibly might have been. Therefore as matter is finite, we reckon a capacity of being moved among its eſſential properties, and never give ourſelves the fruitleſs trouble of enquiring, whether the infinity of matter is a poſſible ſuppoſition.

The force of inactivity is that property of matter, whereby all bodies, that are at reſt, reſiſt being moved, and all that are in motion reſiſt being ſtopped in proportion to the quantity of matter contained in them. Matter is not ſimply inactive; it is not merely unable, when at reſt, to put itſelf in motion, and, when in motion, to ſtop itſelf: for if this was all; bodies would only be indifferent as to motion or reſt, and any force however ſmall applyed to any body however large, would be able to change the ſtate of the body; that is, would be able either to move it, if it was at reſt, or to ſtop it, if it was in motion. But bodies, we find, are not only incapable of producing motion or reſt in themſelves, but likewiſe make ſuch a reſiſtance to the production of either, that there is ſome difficulty in cauſing them to change their preſent conditiony ſome force is required both to produce motion in them, and to deſtro; it. And this force muſt be ſo much greater as the quantity of matter is greater in the body, to which the motion is to be communicated; or in which it is to be deſtroyed. If a force that we will call A was ſufficient to move a body as B with a certain degree of velocity; then to move 2B with the ſame velocity 2A or double the force would be requiſite; and 3B or three times the quantity of matter could not be moved with the ſame velocity, unleſs 3A or a triple force was applyed. In the ſame manner; if the force A could ſtop the body B, when it was moving with a certain degree of velocity; to ſtop 2B or twice the quantity of matter moving with an equal velocity the force muſt be doubled; and 3B or three times that quantity, could not be ſtopped unleſs the force, was tripled. The counter force which the bodies exert, the reſiſtance, which they thus make to a change of their preſent ſtate either of moving or reſting, is the property of matter, that we are now deſcribing, and is called the force of inactivity.

To this definition of matter it might be expected that I ſhould add a definition of motion; but I rather chuſe to tranſcribe a paſſage out of Mr. Lockes eſſay concerning human underſtanding as an excuſe for omiting it. The names of ſimple ideas are incapable of being defined. The reaſon whereof is this, that the ſeveral terms of a definition ſignifying ſeveral ideas, they can altogether by no means repreſent an idea, which has no compoſition at all. And therefore a definition, which is properly nothing but ſhewing the meaning of one word by ſeveral others not ſignifying each the ſame thing, can in the names of ſimple ideas have no place. The not obſerving this difference in our ideas and their names has produced that eminent trifling in the ſchools, which is ſo eaſy to be obſerved in the definitions they give us of ſome few of theſe ſimple ideas. For as to the greateſt part of them, even thoſe maſters of definitions were fain to leave them untouched, merely by the impoſſibility they found in it. What more exquiſite jargon could the wit of man invent, than this definition, The act of a being in power as far forth as in power? Which would puzzle any rational man, to whom it was not already known by its famous abſurdity, to gueſs what word it could be the explication of. If Tully aſking a dutchman what beweeginge was, ſhould have received this explication in his own language, that it was Actus entis in potentia quatenus in potentia; I aſk whether any one can imagine he could thereby have underſtood what the word beweeginge ſignified or have gueſſed what idea a dutchman ordinarily had in his mind and would ſignify to another when he uſed that ſound. Nor have the modern philoſophers, who have endeavoured to throw off the jargon of the ſchools, and ſpeak intelligibly, ſucceeded much better in defining ſimple ideas, whether by explaining their cauſes or any otherwiſe. The atomiſts, who defined motion to be a paſſage from one place to another, what do they more than put one ſynonimous word for another? For what is paſſage other than motion? And if they were aſked what paſſage was; how would they better define it than by motion? For is it not at leaſt as proper and ſignificant to ſay paſſage is a motion from one place to another, as to ſay motion is a paſſage from one place to another? This is to tranſlate, and not define, when we change two words of the ſame ſignification one for the other: which, when one is better underſtood than the other, may ſerve to diſcover what idea the unknown ſtands for; but is very far from a definition, unleſs we will ſay, every engliſh word in the dictionary, is the definition of the latin word it anſwers, and that motion is a definition of motus. Nor will the ſucceſſive application of the parts of the ſuperficies of one body to thoſe of another, which the Carteſians give us, prove a much better definition of motion, when well examined.

I have not included diviſibility in the definition of matter, becauſe though all matter is diviſible, yet as this cannot be reckoned a primary property, but is only a conſequence of extenſion, there was no more reaſon for making this a part of the eſſence of matter than innumerable others, which might be named. For ſince a definition of matter would fill a volume, if it was to take in all thoſe properties of it, which follow from the primary ones; many of them muſt be omitted, and moſt of them commonly are omitted. I ſaw no reaſon therefore for including this in particular in the definition of matter; the eſſence of it may be underſtood without this as well as it can without the other properties of the ſame ſort, which inſtead of being made parts of the definition are uſually deduced as conſequences from it. Thus having defined matter to be an extended ſubſtance we may from thence collect the following propoſition relating to its diviſibility.

2. *Matter is infinitely diviſible.*

The meaning of this propoſition ſeems to have been but little underſtood even by thoſe, who have demonſtrated the truth of it. They have not attended to the idea, which the word *infinitely* ſtands for, and by that neglect have laid themſelves open to objections which they knew ſo little how to anſwer, as to make a very judicious and thinking writer ſay that the infinite diviſibility of a finite extenſion, involves us, whether we grant or deny it, in conſequences impoſſible to be explicated, or made in our apprehenſions conſiſtent; conſequences that carry great difficulty and ſome apparent abſurdity.

But if we attend to the idea of infinite as the ſame writer has explained it, we ſhall be leſs perplexed in our enquiries concerning the diviſibility of matter than he ſeems to have been himſelf. Whoever thinks on a cube of an inch diameter, has a clear and poſitive idea of it in his mind, and ſo can frame one of $\frac{1}{2}$ an inch, or $\frac{1}{4}$, or $\frac{1}{8}$ and ſo on, till he has the idea in his thoughts of ſomething very little; but yet reaches not the idea of that minuteneſs, which diviſion can produce. For after he has continued this halving in his mind and has diminiſhed his idea as much as he pleaſes, he has no more reaſon to ſtop, nor is one jot nearer the end of ſuch diviſion, than he was at firſt ſetting out. Nor is it here an inſignificant ſubtilty to ſay, that we are carefully to diſtinguiſh between the infinite diviſibility of matter, and matter infinitely divided. The firſt is nothing but an endleſs progreſſion of the mind in dividing the parts of matter as long as it pleaſes: but to talk of matter as infinitely divided; is either taking it for granted that in our diviſion of it we muſt come to an end at laſt, and this is begging the queſtion; or elſe it is ſuppoſing the mind has already paſſed over and has actually a view of all thoſe parts of matter, which an endleſs diviſion can never totally repreſent to it; which carries in it a plane contradiction.

Plat. I. Fig. 1. will ſhew us that any part of extenſion is infinitely diviſible, or that after we have divided as long as we pleaſe we have no more reaſon to ſtop nor are one jot nearer the end of ſuch diviſion than we were at firſt ſetting out. For let A B be drawn perpendicular to TM, and F G parallel to it: then on C as a center with the opening of the compaſſes C T ſtrike the arc S T L and it will divide the line F G at *p*. If the ſemidiameter had been longer the arc would have approached nearer to the right line A B and would have divided the line F G nearer to F: thus if E is the center R T D paſſing through *o* will be the arc. And in the ſame manner if M was the center the arc would ſtill approach nearer to F. Since therefore as the radius lengthens the line P F is divided into conſtantly leſs and leſs parts by the conſtant approach of the arc to F. And ſince the mind has a power of lengthening the line TM at pleaſure without being ever obliged to ſtop; it follows that the diviſion of P F is endleſs; unleſs we could at laſt come to ſome arc which the ſame right line AB would touch in two points one at T the other at F. But this is impoſſible; Euclid. book III. prop. 2. corol. Therefore any part of extenſion, and conſequently matter, becauſe it is extended, is infinitely diviſible.

Nor can we well conceive it to be otherwiſe. For if any particle of matter is placed upon a plane, the bottom of the particle only can touch the plane and the top will not touch it: becauſe, if it were otherwiſe, the particle would have no thickneſs, and ſo muſt be a mathematical ſurface and not matter. But where there are two parts an upper and a lower, a diviſion is always poſſible. And ſince this would be the caſe, however ſmall the particle was, it follows, that when we have divided as long as we pleaſe, a farther diviſion will be poſſible. Unleſs diviſion could cauſe that matter ſhould ceaſe to be matter by deſtroying the extenſion of it in ſome one of its dimenſions.

3. *The earth attracts all bodies, that are near it: and the force, with which bodies when they are ſo attracted, tend toward the earths center, is called gravity.*

I have not reckoned gravity amongſt the eſſential properties of matter, though we find it in all bodies, upon which we can make experiments. For this I had two reaſons. The firſt is, that no particle of matter can gravitate unleſs there be ſome other particle, which attracts it: ſo that gravity is a relative property, as it carries the mind beyond the particle that is poſſeſſed of it, to ſomething elſe, which acts as if it was the cauſe of gravity, and towards which this gravitation is made. My other reaſon for leaving gravity out of the definition of matter is, that I had reckoned the force of inactivity amongſt the eſſential properties of it: and at the ſame time to have reckoned gravity amongſt them would have been making the parts of the definition inconſiſtent. It would have been ſaying, that matter makes a reſiſtance to motion in all poſſible directions, and yet that in one direction it will begin to move of its own accord. We muſt therefore leave either gravity or the force of inactivity out of our definition; and as the gravity of bodies is a particular exception from the univerſal force of inactivity; there can be no difficulty, when we are defining matter, to determine, which of the two properties we ſhould take into the definition, and which we ſhould leave out of it.

4. *The attraction of coheſion is that force, by which the ſmalleſt, or conſtituent, particles of all bodies tend towards each other, and are united together.*

5. *The repulſive force of matter is that, by which its particles, when they are too far aſunder to attract, are found to repel each other.*

There is a remarkable inſtance of this attractive and repulſive force between the edge of a knife and the rays of light; as they paſs by it, when they are let into a dark chamber through a hole in the window-ſhutter. For thoſe rays, which paſs cloſe to the knife, are bent downwards by it; thoſe, which paſs at greater diſtances being attracted leſs by it, are leſs bent downwards; and thoſe, which paſs at ſtill greater diſtances, are repelled by it, and in their paſſage are bent the contrary way. That there are in nature ſuch forces as theſe is evident enough. The ſeveral particles, which compoſe a ſtone, or a piece of iron, are united by ſome force or other, ſo as to form one compact maſs: and this force is what we call attraction or coheſion. The ſame force is called by the ſame name in all other bodies: and the attraction of coheſion conſidered as a primary law of nature means that force, by which the particles of matter are united to each other. A repulſion likewiſe appears in ſome inſtances. It ſeems to be ſuch a force as this which ſhakes off the particles of light from the ſun, a candle or any other body which emits light. The ſame force appears, when any oily ſubſtance is placed on the ſurface of water; for by a repelling force between the two ſubſtances, the ſurface of the water will be depreſſed about the oil, that is laid on it.

But theſe attractive and repulſive forces, though they may be proved to exiſt in nature, are frequently applyed in philoſophy to caſes, which they do not ſufficiently explane. Aqua fortis will diſſolve iron. If we aſk why? the anſwer immediately is, becauſe the particles of iron are more attracted by thoſe of aqua fortis, than either the particles of iron are by each other or the particles of aqua fortis by each other. For which reaſon the particles of the iron are ſeparated and ſo are thoſe of the aqua fortis by means of this ſtronger attraction; and by the ſame mutual attraction theſe two ſubſtances will be mixed and blended together. But then this aqua fortis will not diſſolve gold? And ſince gold is of a looſer texture; that is, ſince its parts cohere leſs forcibly together than thoſe of iron do, ought not the ſame force, which will ſeparate the particles of iron, to be even more than ſufficient to ſeparate thoſe of gold? Here it is anſwered; there is an attraction between iron and aqua fortis; but none be-

between gold and aqua fortis. If we aſk why aqua fortis, which will not diſſolve gold, will diſſolve ſilver? the anſwer is ready and is the ſame as before; aqua fortis attracts ſilver more than the particles of ſilver attract each other; but either does not attract gold at all, or elſe the mutual attraction of the particles of gold amongſt one another is greater than the force, with which they are attracted by the aqua fortis: and this weaker force cannot ſo overcome the ſtronger as to ſeparate or diſſolve the gold. Aſk again, why aqua regia, which will diſſolve gold, does not diſſolve ſilver? they anſwer again that aqua regia attracts gold, but does not attract ſilver. Theſe anſwers have ſomething in them very like the occult qualities of the Peripatetics. For as they uſed to aſſign as many different occult qualities as there are different appearances to explane; ſo the philoſophers, who give theſe anſwers, introduce as many different ſorts of attractions, as there are bodies to be diſſolved and fluids to diſſolve them.

We may ſhew ſtill farther how unſatisfactory a method this is of accounting for diſſolutions; by aſking thoſe, who maintain it, what they mean by the word *attraction*. Put the word *drawing* inſtead of it, and then we ſhall find that what they ſay may amuſe, but does not inſtruct. I enquire why aqua fortis, for inſtance, will diſſolve iron, or draw the particles of a maſs of iron aſunder. I am anſwered becauſe aqua fortis attracts thoſe particles more than they are attracted by each other, which is juſt the ſame as to ſay that aqua fortis draws them aſunder with a greater force than they draw one another. But this is no better than aſſigning the fact as a reaſon for itſelf: it is ſaying that aqua fortis will draw the parts of iron aſunder, becauſe it has a force ſufficient to draw them aſunder: which is a very unſatisfactory account of the matter, when I wanted, not to be informed that aqua fortis has ſuch a force, for that my ſenſes prove to me, but to know what that force is.

I would here enquire of theſe philoſophers what attraction this is, which diſſolves hard bodies in fluid ones? Is it the attraction of cohesion? If they ſay it is not this ſort of attraction, but quite a different one; I do not ſee how they will get rid of the charge of introducing ſo many different ſorts of attractions into nature, as to make their explication of things by attraction little better than the Ariſtotelian explication of them by occult qualities. But if they ſay that the attraction, to which diſſolutions are owing, is the attraction of cohesion, it will be very eaſy to ſhew that they are miſtaken. I ſuppoſe it will be readily granted that a body is ſo much the harder as the parts of it are more firmly united to each other, or cohere more ſtrongly together. From hence it follows, that thoſe bodies will be the hardeſt, whoſe parts attract each other the moſt forcibly. Now in this account of diſſolution the particles of iron are ſaid to be attracted by aqua fortis more forcibly than they are one amongſt another. Therefore a diſſolution of iron in aqua fortis, that is, a mixture or maſs conſiſting of particles of iron and aqua fortis, ought to be a harder body than iron, or than a maſs, which conſiſts of none but particles of iron. Whereas in fact the diſſolution, inſtead of being harder than iron, is as much a fluid as aqua fortis itſelf.

We may make this argument ſtill more forcible by another inſtance. For as in a diſſolution of aqua fortis and iron there is more of the fluid body than there is of the hard one; it might perhaps be objected, that this is the reaſon why the maſs is fluid. This objection is indeed of very little weight: becauſe the maſs or mixture, if it does not become harder than iron, ought at leaſt to be much harder than aqua fortis alone. But we may quite avoid the objection, if we uſe the inſtance of Epſom ſalts and common water. For any quantity of common water will diſſolve its own weight of Epſom ſalts. So that in the diſſolution or mixture of theſe two bodies, there are juſt as many particles of the ſalts as there are of the water. And as the attraction of coheſion is ſuppoſed more ſtrong between each particle of ſalt and each particle of water, than it is between any two particles of ſalt, it follows that the mixture ought to be harder than ſalt. Whereas it continues as fluid as common water. From whence I think we may conclude that this diſſolution is not owing to the attraction of coheſion.

6. *A body is called an hard one, if its parts make ſo forcible a reſiſtance to every impreſſion, that can be made upon them, as not at all to yield to it.*

7. *A body is called elaſtic, if its parts yield to the impreſſions, that are made upon them; and then reſtore themſelves again to their former ſituation with a force equal to that, which made them change it.*

8. *A fluid is a body, whoſe parts yield to any force impreſſed upon them, and in yielding are eaſily moved one amongſt another.*

CHAP. II.

Of the three laws of motion.

9. *The velocity of a body moving uniformly is directly as the ſpace deſcribed by it, and inverſely as the time of its motion.*

THAT is, the line, which a body runs over in any certain time, will be ſo much the longer, as the body moves faſter, and ſo much the ſhorter as the body moves ſlower. And the time, which it takes up to run over a line of a certain length, will be ſo much the ſhorter as the body moves faſter, and ſo much the longer as the body moves ſlower. Since therefore the velocity is directly as the ſpace deſcribed, when the time of motion is given, and inverſely as the time, when the ſpace is given, it will, where neither of them is given, be in a ratio compounded of both, that is, directly as the ſpace deſcribed and inverſely as the time of motion.

Let the velocity of a body $= v$, the time of motion $= t$, and the ſpace deſcribed $= s$; then from what has been ſaid it follows that v, is always as $s \times \frac{1}{t}$ that is as s multiplyed into $\frac{1}{t}$ or the reciprocal of t, or as $\frac{s}{t}$, or the velocity is as the ſpace deſcribed divided by the time of motion.

10. *The ſpaces deſcribed by bodies moving uniformly are as the velocities and times of motion conjointly.*

That is, the line run over by a body in any certain time will be ſo much the longer as the body moves faſter, and ſo much the ſhorter as it moves ſlower. And this line, where the velocity is given, will be ſo much the longer as the body moves for a greater time, and ſo much the ſhorter, as it moves for a leſs. Therefore the ſpace deſcribed being as the velocity, when the time is given, and as the time when the velocity is given, will be always in a ratio compounded of both or as the time and velocity conjointly. So that if the velocity of a body A is to the velocity of any other body B as 2 to 3, and A moves for 6 minutes and B for 8, then will the ſpaces deſcribed by theſe two bodies be in a ratio compounded of 2 to 3 and 6 to 8, that is, they will be as 2×6 and 3×8, or as 12 to 24.

This propoſition may be deduced from the foregoing one, for ſince v is as $\frac{s}{t}$ it follows that, multiplying both ſides by t, vt is as s.

The ſpaces deſcribed by bodies moving uniformly, that is, the proportions, which the lines they run over bear to each other, may be repreſented geometrically. If the time of A's motion is to the time of B's motion as C D to G H Plat. I. fig. 2. and A's velocity to B's as D L to H R, then will the line deſcribed by A be to the line deſcribed by B as the area of the rectangle under C D, D L, to the area of the rectangle under G H, H R. For the lines deſcribed by the two bodies are to each other in a ratio compounded of the times and velocities, that is, of C D to G H and of D L to H R: and the areas of theſe rectangles being in a ratio compounded of their two ſides are likewiſe as E D to G H, and as D L to H R conjointly. Euclid. book VI. prop. 23.

11. *The firſt law of motion is, that all bodies continue in the ſame ſtate of reſting or of moving uniformly in a ſtrait line: unleſs ſome external force impreſſed upon them makes them alter that ſtate.*

The firſt part of this law, that all bodies continue in a ſtate of reſt, till ſome external force puts them in motion, follows from the very nature of matter. For whatever is capable of moving itſelf, muſt be called by ſome other name, ſince it is a part of the eſſence of matter not only to be inactive or unable to produce motion in itſelf, but even to make a reſiſtance to motion, when communicated to it by any external cauſe. It will be as eaſy to prove the ſecond part of this law to thoſe, who underſtand what this force of inactivity means. Becauſe as all bodies make a reſiſtance to being ſtopped, they cannot be conceived more capable of ſtopping themſelves, when they are in motion, than they are of moving themſelves, when they are at reſt. Some external force is equally neceſſary to produce either of theſe changes in them; and ſuch a force too as is proportionable to the reſiſtance, which they make. This continued motion of bodies would, if they were left to themſelves, be uniform; that is, bodies can neither be accelerated nor retarded, but by ſome external force. For they cannot move faſter without ſome additional motion, nor can they move ſlower without loſing a part of what they had before; therefore ſince they cannot of themſelves either produce motion or deſtroy it, they will perſevere in a ſtate of moving uniformly, till ſome external force makes them change that ſtate. And for the ſame reaſon they will of themſelves move on in a right line, becauſe changing their direction is the ſame thing as moving themſelves another way.

If a veſſel full of water is ſuddenly moved forwards upon a horizontal plane, the water, which hangs looſe in the veſſel or does not adhere to it ſo as to be a part of it, will not have the motion of the veſſel communicated to it immediately, but perſevering in a ſtate of reſt will begin to riſe up againſt that ſide of the veſſel, where the moving force acts, and being by this means made to ſwing in the veſſel will daſh over. But when once the water has the motion of the veſſel communicated to it, and begins to move with a velocity equal to that of the veſſel, then it will perſevere in a ſtate of motion, and if the veſſel is ſuddenly ſtopped, it will run againſt the oppoſite ſide and by riſing up there will be made to ſwing again and to daſh over as before.

When a horſe is running with his full ſpeed, the rider moves along with him the ſame pace: and if the horſe ſtops at once, the rider, unleſs he ſticks cloſe, ſo as to make in a manner one maſs with the horſe, will continue to move forwards and will be thrown out of the ſaddle towards the horſes head. But if a horſe from ſtanding ſtill ſets out on a ſudden to run with his full ſpeed, the rider continuing in a ſtate of reſt will be thrown out of his ſaddle the contrary way.

Whilſt a ſtone is whirled round in a ſling it deſcribes a circle; for at that time it is acted upon by two forces. But if the ſling was to break, by which means one of the forces is removed, the ſtone would then fly off, as by this law it ought to do, in a line, which is a tangent to the circle, that it deſcribed before. The ſtone does not indeed continue to move for ever, for its own weight makes it fall, and by ſtriking againſt the ground it will be ſtopped: but this is agreeable to what the law affirms, for it does not ſtop of itſelf, but is ſtopped by the earth, which is an external force. It is its weight, another external force, which changes the direction of its motion, and makes it, inſtead of going on in a right line deſcribe that curve which brings it to the ground.

12. *The ſecond law of motion is, that the changes made in the motion of bodies are always proportional to the moving forces impreſſed, and are produced in the ſame right line, in which thoſe forces act.*

A change is made in bodies in reſpect of their motion, when they are moved, or ſtopped, or accelerated, or retarded, or have their direction altered: and this law relates alike to all theſe changes. By a moving force, we are not here to underſtand a force that actually produces motion, but a force which might produce motion and which applyed to a body at reſt would produce it. For in this law not only thoſe forces, which move a body or accelerate it, are called moving forces, but thoſe likewiſe which ſtop or retard it. That is in ſhort, all forces are here called moving ones, which produce any change whatever in the motion of bodies.

That all effects are proportional to their adequate cauſes, is an axiom that every one allows to be true. And as the firſt part of this law is nothing but an application of this general propoſition to the particular caſe of motion, we may take the truth of it for granted without giving ourſelves the trouble to find out any formal proof of it.

By the firſt law of motion every moving force produces its effect in a right line, and no moving force can act where it is not, from whence it follows that all changes made in motion, becauſe they are the effects of a moving force, will be produced only in that right line, in which the force is impreſſed.

But before I leave this law, it may not be improper to take particular notice of two conſequences that may be deduced from it, which I ſhall endeavour to explane and eſtabliſh in the two following propoſitions.

13. *The moments of bodies, that is, their quantities of motion, are as the velocities of the bodies, and as the quantities of matter contained in them conjointly.*

The quantity of motion, by the law juſt explaned, is always proportional to the moving force. But whatever degree of our ſtrength is required to throw a ball of lead that weighs 1 pound with a certain velocity, we know that double the force is requiſite to throw a ball of 2 pounds weight with the ſame velocity, and triple the force to throw one of 3 pounds. Therefore, when the velocity is given, the force impreſſed on any body to move it, and conſequently the motion produced in it, is as the quantity of matter, which it contains. We know likewiſe that whatever degree of our ſtrength is required to throw a ball weighing 1 pound with ſuch a degree of velocity, as we will call 1, twice the force would be requiſite to throw the ſame ball with a double velocity, and thrice the force to throw it with a triple velocity. Therefore, when the quantity of matter is given, the force impreſſed on a body to move it, and conſequently the motion produced in it, is as the velocity with which it moves. Since then the moment is as the quantity of matter, if the velocity is given, and as the velocity, if the quantity of matter is given; it follows, that if neither of them are given the moment will be as both of them conjointly, or as the product ariſing from the quantity of matter multiplyed into the velocity.

14. *The moments of bodies are equal, if their velocities are reciprocally as the quantities of matter contained in them.*

For, by prop. 13. if the quantity of matter is called q the velocity v and the moment m then m is always as $q \times v$, that is as the quantity of matter multiplyed into the velocity. But here we ſuppoſe that v is as $\frac{1}{q}$, that is reciprocally as q or the quantity of mater. Therefore $q \times v$ is as $q \times \frac{1}{q}$ or as $\frac{q}{q}$, and ſince $\frac{q}{q}$ expreſſes an invariable quantity the moment muſt be invariable in the caſe ſuppoſed, or when the velocity is reciprocally as the quantity of matter.

Let the weight of a body A be 4 ounces, and that of a body B 10, then if the former moves with 5 degrees of velocity and the latter with 2 their velocities are inverſely as their weights or quantities of matter: for 5 is to 2 as 4 to 10 inverted, or as 10 to 4. The moment of A in this caſe $= 4 \times 5 = 20$ and that of B $= 10 \times 2 = 20$. Where it is obvious, that juſt as much as the moment of A is leſs than that of B, on account of the leſs quantity of matter in A; juſt ſo much the moment of A is greater than that of B on account of the greater velocity, with which A moves: ſo that taking both accounts together the moment of one muſt be equal to that of the other.

15. *The third law of motion is, that reaction is always contrary and equal to action; or, the mutual actions of two bodies upon each other are equal and in contrary directions.*

From the force of inactivity it follows, that whatever change is made in the ſtate of one body A, whether at reſt or in motion, by the action of another B, the ſame change is produced in the ſtate of B by the reſiſtance, which A makes to that change. Thus, if a man in a boat A draws another boat B by means of a rope faſtened to it: the boat A will be equally drawn towards B and they will approach each other with equal quantities of motion; that is, by the laſt propoſition, with equal velocities, if their weights are equal; or, if their weights are unequal, with velocities reciprocally as thoſe weights. From whence we may eaſily ſee the reaſon why by pulling the rope, when it is faſtened to a tree or a poſt upon the bank we ſhall draw the boat up to the ſhore. For the velocities produced on each ſide, that is, in the boat and in the bank will be reciprocally as the quantities of matter. And ſince the quantity of matter in the boat is as nothing compared with that in the earth of which the bank is a part, it follows that the velocity in the bank will be as nothing; and conſequently all the velocity that is produced, will be in the boat.

A perſon, who is in a boat by pulling with ever ſo great a force at a rope, which is tied to the head of it, will not be able to ſtop it, when in motion, nor to move it, when at reſt. For juſt as much as he acts one way with his hands, juſt ſo much he reacts the other way with his feet: and theſe two equal and oppoſite forces deſtroying each other, will leave the boat in the ſame condition of reſting or moving, that it would have been in, if he had not pulled at the rope at all. Whereas, if he had been upon the ſhore, when he pulled the rope, he would then have drawn the boat towards him. For then his reaction would have been exerted upon the ground, where he ſtood; and the boat ſuſtaining only the force of his action would have moved in the direction, in which he pulled it.

If a horſe draws a ſtone forwards, the ſtone draws the horſe back equally: that is, whatever motion the horſe communicates to the ſtone, he will loſe the ſame quantity himſelf. For ſuppoſe him able, if he had no ſtone behind him, to puſh forwards with his breaſt a weight of a 1000*lb.* it is evident, that when he has a ſtone of 400*lb.* to draw, ſo much of his ſtrength will be laid out that way, as to leave him able to puſh forwards with his breaſt no greater weight than $1000 - 400 = 600$*lb.* He has therefore loſt a quantity of motion forwards exactly equal to what he has communicated to the ſtone; or as much as he draws the ſtone one way, ſo much it draws him the contrary, and, theſe equal and contrary forces deſtroying each other, he will go on with the difference.

CHAP. III.

Of the compoſition and reſolution of forces.

16. *A body, that is acted upon by two forces together, will deſcribe the diagonal of a parallelogram in the ſame time that it would have deſcribed the ſides, if the forces had acted upon it ſeparately.*

If there was a force which acting alone upon the body A Plat. I. fig. 3. in the direction AB would carry it to B in a certain time; and if there was another force, which acting alone upon the body A in the direction AC, would carry it to C in an equal time: then, making AB and AC the two ſides of a parallelogram and drawing the diagonal AD, I ſay, that theſe two forces acting together will make the body deſcribe this diagonal in the ſame time that one of them alone would have made it deſcribe A B, or the other of them alone would have made it deſcribe A C.

Let the force, which acts in the line A B, be called X, and that, which acts in the line A C, be called Z; then in a certain time the force X will impell the body through a ſpace equal to A B, or, ſince A C and B D are parallel, through a ſpace equal to the diſtance between thoſe two lines. And becauſe the direction A C, in which the force Z acts, is parallel to B D; therefore by the ſecond law of motion, this force will not hinder or deſtroy the velocity of the body, by which the force X makes it tend towards the line B D. Conſequently at the end of the time the body will be found ſomewhere in that line B D. In like manner the force Z will impell the body in the ſame time through a ſpace equal to A C, or ſince A B and E D are parallel, through a ſpace equal to the diſtance between thoſe two lines. And, becauſe the direction A B, in which the force X acts, is parallel to C D, therefore, as before, this force will not hinder or deſtroy the velocity of the body, by which the force Z makes it tend towards the line C D. Conſequently, at the end of the time the body will be found ſomewhere in that line C D. But it cannot be found at the ſame time both in B D and C D unleſs it is at D the interſection of theſe two lines. And by the firſt law of motion it muſt move in a right line from A to D. Therefore it will have deſcribed the diagonal A D.

If two men ſet upon the oppoſite ſides of a boat, whilſt the boat is going on, and toſs a ball backwards and forwards to each other, they will catch the ball in their turn, juſt as they would have done, if they had both ſtood ſtill upon the ſhore. The ball is here acted upon by two forces: it goes along with the boat, whilſt it flyes through the air, juſt in the ſame manner as it would have done, if it had not been thrown, but had laid ſtill in the boat. This motion common to the ball, the boat, and the men is one of the forces. The other force is that, with which one of the men throws it acroſs to the other. By theſe two forces together the ball will deſcribe the diagonal of a parallelogram, one of whoſe ſides is the line that the boat has deſcribed, whilſt the ball is flying acroſs, and the other ſide is a line drawn from one man to the other. Let A C be the breadth of the boat; let A be the place where one of the men ſets, and C the place where the other ſets: and whilſt one ſide of the boat is moving in the direction A B, and the other ſide of it in the direction C D, let the ball be thrown from A to C with ſuch a velocity as will carry it to C juſt when the boat has got into the ſituation B D. Now

if

if A had not thrown the ball at all, but had continued to hold it in his hand, then it would have moved on along with him and the boat from A to B. This motion, which is common to the boat, the man, and the ball, will continue the ſame when the ball is thrown, that it was whilſt A held it in his hand. For though it might at firſt be produced in the ball by being held in his hand, as he was carryed along with the boat; yet, when once produced it will continue; by the firſt law of motion. But whilſt this force is carrying the ball from A to B, the force with which it is thrown will carry it from A to C or directly acroſs the boat. And at the inſtant it has flown acroſs, the man, who was at C when it was firſt thrown, will be got to D, becauſe at that inſtant the boat will be at BD. Therefore, ſince the ball has gone on in the direction AB, CD, as faſt as the boat has done, and in the mean time has flown acroſs the boat; juſt when the man arrives at D the ball will be there too, and he will catch it in the ſame manner that he would have done, if both he and the perſon who threw it had ſtood ſtill. And any one, who ſtands upon the bank, will in the mean time ſee the ball deſcribe the diagonal AD.

Suppoſe a ruler AC to be carryed with an uniform motion parallel to itſelf, and with ſuch a velocity as would bring it into the ſituation BD in four minutes time. Then a fly ſtanding upon one end of the ruler at A and carryed along with it, would in the firſt minute deſcribe the line AE, in the ſecond EF, in the third FG, and in the fourth GB. But if the fly was in the mean while to walk upon the ruler with an uniform velocity, which would carry it in four minutes the whole length of the ruler or from A to C; then at the end of the firſt minute, or when the ruler has moved one quarter of the way from A to B and is in the line EM, the fly will have walked down one quarter of the length of the ruler and will be at S. In the ſame manner, when the ruler has moved half way and is at FN; the fly will have walked half down it and will be at T. When the ruler is at GO, the fly will be at V. And when the ruler, at the end of the four minutes, is in BD, the fly will have walked the whole length of it and will be at D. But A, S, T, V, and D are all of them in the diagonal AD. And, in the ſame manner, if we had divided the line AB and ruler AC into ever ſo many more parts than four; we might have ſhewn that the fly would at the end of every ſuch part have been in ſome point of the ſame line AD. Therefore the fly, acted upon by two forces, one of which is common to itſelf and the ruler, and the other the force, by which it walks upon the ruler, is made to deſcribe the diagonal of a parallelogram AD in four minutes time; that is, in the ſame time that one of the forces alone would have made it deſcribe AB or one ſide of that parallelogram, and the other alone would have made it deſcribe the other ſide AC.

Here we muſt obſerve that the velocity produced by the compound action of theſe two forces X and Z, that is, of the two forces, of which the former acts in the direction AB and the latter in the direction AC, is proportional to the diagonal or ſpace deſcribed in a given time. Thus the velocity produced by X is to the compound velocity as AB to AD, and the velocity produced by Z is to the compound one as AC to AD. By propoſition 9.

18. *If a body is impelled at the ſame time by two forces, that act in the ſame direction; its velocity will be as the ſum of thoſe forces.*

If two forces, which are to each other as AB to AL, Plat. I. fig. 5. act upon a body at the ſame time; then by the laſt propoſition the velocity produced will be as AR the diagonal of a parallelogram, whoſe ſides are AB and AL. Suppoſe the direction of the force that acts in the line AL to continue the ſame, whilſt the direction of the other force approaches to it; then the velocity produced by the two forces, when they act in the lines AL and AC, will be as the diagonal AF. And when they act in the lines AL and AD, the velocity will be as the diagonal AS. Thus we find that the diagonal lengthens, that is, the velocity of the body increaſes, as the angle at A contained between the directions of the two forces becomes more acute. But when AD lies in the direction AL, or when the two forces act in the ſame direction, then AL and LS will lie ſtrait out, and the diagonal AS by coinciding with them will be equal to AL added to LS, or to AL added to AB, ſince AB is equal to LS. (Euclid. book I. prop.34.) That is, the diagonal, and conſequently the velocity produced, will be as the ſum of the two forces AB and AL, or as theſe two forces added together.

19. *If two forces, that act in contrary directions, impell a body at the ſame time; the velocity produced in it will be as the difference of thoſe forces.*

By the joint action of two forces, as AB and AC, Plat. I. fig. 6. the velocity produced in the body impelled by them will be as the diagonal AD. Let the direction of the force AC continue; then as the direction of the other force grows more oblique to AC, the diagonal ſhortens, that is, the velocity produced by the two forces diminiſhes. Thus, when they act in the directions AC and AR, the diagonal is AO; and when in the directions AC, and AS, it is AG. Till at laſt, when AS and AC lie ſtrait out from each other, that is, when the two forces act in contrary directions,

rections, then, making CE equal to CG, CG will coincide with CE, and the diagonal becomes AE. But becauſe CG = AS, (Euc. book I. prop. 34) therefore CE = AS, and this diagonal AE is equal to CA — AS; that is, to CA with CE or AS taken from it. From whence it appears that the diagonal being the difference of the two forces, the velocity of the body will likewiſe be as that difference.

The body, by the ſecond law of motion, muſt neceſſarily move in the direction of the ſtronger force, if one be ſtronger than the other: but if the two forces are equal, that is, if their difference is equal to nothing, then the velocity likewiſe = 0 and the body acted upon by them will by their joint action be made to reſt.

20. *A motion, though ever ſo ſimple, may be reſolved into the action of two forces: and the directions of the two forces, which might have produced ſuch a motion, may be infinitely varyed.*

Thus a motion in the line AM, Plat. I. fig. 7. may be conſidered as the effect of two forces acting in AL and AN the two ſides of a parallelogram of which AM is the diagonal. For, by prop. 17. theſe two forces would have produced ſuch a motion. It may for the ſame reaſon be conſidered as the effect of two forces acting in AB and AI the two ſides of another parallelogram, of which the ſame line AM is the diagonal. And in the ſame manner the direction of the forces, which might have produced this motion, may be varyed as much as the direction of the two ſides of a parallelogram may, whilſt the ſame line AM continues the diagonal.

The third law of motion alone is ſufficient to explane the manner in which a bird by the ſtroke of its wings is able to ſupport the weight of its body. For if the force, with which it ſtrikes the air below it, is equal to this weight; then the reaction of the air upwards is likewiſe equal to it: and the bird being acted upon by two equal forces in contrary directions, that is, by the reaction of the air and its own gravity, will reſt between them. If the force of the ſtroke is greater than its weight, then the reaction of the air upwards being made greater than the force of gravity downwards, the bird will riſe with the difference of theſe two forces. And if the force of the ſtroke be leſs than its weight, then it will ſink with the difference. By prop. 19.

But to explane the manner, in which a bird by ſtriking its wings downright in the lines DH, ES, OR, IG, Plate I. Fig. 4. perpendicular to the horizon, is able to move itſelf along the line MT parallel to the horizon, we muſt, beſides the third law of motion, have recourſe to the propoſition juſt now explaned. For the wings of a bird are not like the paddles of a boat, they are not ſtruck backwards in order to produce a motion in the contrary direction. But the ſtroke being made in the lines DH, IG, &c, the wings yield to it, and turning round a little, one on the joint I and the other on a joint of the ſame ſort on the other ſide, the ends of them A and F are made to approach each other, and the planes of them become oblique to the line MT. Suppoſe therefore LN to be perpendicular to the ſurface of one wing, when it is bent in the manner we have been deſcribing. Then, ſince by the ſecond law of motion, the action of the wing upon the air is in lines perpendicular to its ſurface, the reaction, by the third law, will be in the line LN or oblique to the line MT, in which the bird is to move. But this oblique force may be reſolved into two others LV and VN, of which LV is perpendicular to the horizon and ſupports the bird, the other VN is parallel to the horizon and will carry it along in the line MT.

21. *If three forces act upon a body at the ſame time, and the body reſts between them; then thoſe forces are to each other as the three ſides of a triangle, that are reſpectively parallel to the directions, in which they act.*

If a body at D, Plat. I. fig. 8. that is acted upon by three forces in the directions DA, DC, and DB, yields to none of them, but is by their joint action made to continue at reſt: then, if we continue AD to E, and draw EF parallel to BD, and EG parallel to CD, the three ſides of the triangle EDF are parallel to the directions of the three forces. For DE is parallel to DA: DF is parallel to DE becauſe it is a part of it: and EF is parallel to BD. I ſay therefore that the forces, which act in the lines DA, DC, DB, are to each other as the ſides of this triangle DE, DF, and EF reſpectively. For ſince the body D reſts, it follows, from propoſition 19, that, if the two forces acting in the lines DB and DC are compounded into one in a direction DE oppoſite to DA, then this compound force muſt be equal to the ſimple one DA. But the three forces DE, DC, and DB, that is, the compound force and the two out of which it is compounded are to each other as DE the diagonal of the parallelogram DGEF, and the two ſides of it DF, and $DG = EF$ reſpectively. By prop. 17. Therefore ſince the compound force DE is equal to the ſimple one that acts in the direction DA, the three forces acting in the directions DA, DC, and DB, are to each other as the three lines DE, DF, and EF, reſpectively, that is as the three ſides of a triangle, which is formed by lines drawn parallel to thoſe directions.

We have an inſtance of a body reſting, whilſt three forces act upon it, in a boys kite as it reſts in the air: for it is acted upon by the wind, by its own weight, and by the ſtring that holds it. There is another inſtance, to which we ſhall have occaſion to apply this propoſition, when we come to ſhew what force is ſufficient to ſupport a heavy body upon an inclined plane.

CHAP. IV.

Of gravity.

22. *Any force, which is propagated through right lines in all poſſible directions from one and the ſame center, is, at different diſtances from that center, inverſely as the ſquares of the diſtances.*

LET A repreſent the center, Plat. I. fig. 9. from which the force is propagated; HLB a great circle in a ſphere, through whoſe ſurface the force is ſpread at the diſtance AB; HGC a great circle in the ſphere, through whoſe ſurface the ſame force would ſpread itſelf at the diſtance AE; and let ALG, ACF, ABE, be ſome of thoſe lines, through which this force is propagated. The greater the ſpace is, through which this force is diffuſed, ſo much the weaker will be its action: that is, the action of it at the diſtance AE will be every where ſo much weaker than it is at the diſtance AB, as the ſpace, through which it ſpreads itſelf at the diſtance AE, is greater than the ſpace, through which it ſpreads itſelf at the diſtance AB. Thus if A was a candle, and the rays of light proceeding from it were to deſcribe the lines ALG, ACF, ABE: it is evident, that the light would be ſtronger at B or L than at E or G; becauſe at E or G the rays are ſpread farther aſunder, and conſequently fewer of them will fall upon any given ſpace than at B or L. Now at the diſtance AE the force is ſpread through the ſurface of a ſphere, whoſe ſemidiameter is AE, and at the diſtance AB it is ſpread through the ſurface of a ſmaller ſphere, whoſe ſemidiameter is AB. Let $AE = D$ and $AB = d$, then ſince the force is greater as the ſpace through which it ſpreads itſelf is leſs, it will be as much greater at the point B than it is at the point E, as the ſurface of a ſphere whoſe ſemidiameter is d is leſs than the ſurface of one whoſe ſemidiameter is D, that is, the force will be as thoſe ſurfaces reciprocally. But the ſurfaces of ſpheres are to each other as the ſquares of their ſemidiameters. The ſurface of the ſphere whoſe ſemidiameter is d, is to the ſurface of the ſphere whoſe ſemidiameter is D, as the ſquare of d to the ſquare of D; that is, as d^q to D^q, or as dd to DD, DD; for theſe two ſquares are expreſſed either of theſe ways. Therefore the force at the diſtances AB and AE, being reciprocally as the ſurfaces through which it is ſpread, is reciprocally as the ſquares of thoſe ſemidiameters, or as $\frac{1}{dd}$ to $\frac{1}{DD}$, or as the reciprocals of dd and DD.

Theſe two fractions $\frac{1}{dd}$ and $\frac{F}{DD}$, are called the recriprocals of dd and DD. Becauſe all fractions, whoſe numerators are given, are inverſely or reciprocally as their denominators. For where the numerator is given, the value of the fraction decreaſes as the denominator increaſes, and vice verſa. It will be eaſy to underſtand this in numbers. For of two fractions $\frac{1}{2}$ and $\frac{1}{4}$, which have the ſame numerator 1, the latter $\frac{1}{4}$ is juſt as much leſs than the former $\frac{1}{2}$, as 4 is greater than 2. That is, by increaſing the denominator from 2 to 4, the fraction decreaſes from $\frac{1}{2}$ to $\frac{1}{4}$; or by doubling the denominator, the value of the fraction becomes but half what it was before.

Or otherwiſe. Every fraction expreſſes a quotient ariſing from the diviſion of the numerator by the denominator. And in all diviſion, where the dividend is given, the quotient is greater as the diviſor is leſs, or leſs as the diviſor is greater: that is, the quotient is reciprocally as the diviſor. Therefore a fraction, where the numerator is given, is reciprocally as its denominator.

23. *The force of gravity from the ſurface of the earth upwards is, at different diſtances from the earths center, as the ſquares of thoſe diſtances inverſely.*

That is, call the gravity of a body 1 when it is at the earths ſurface or at the diſtance of one ſemidiameter from the center: then at the ſeveral diſtances 1. 2. 3. 4. &c. the gravity of it will be as the ſquares of thoſe diſtances inverted. The ſquares are 1. 4. 9. 16. &c. if we invert them, or, which amounts to the ſame thing, take the reciprocals of them, they are 1. $\frac{1}{4}$. $\frac{1}{9}$. $\frac{1}{16}$. &c. and in this proportion the force of gravity decreaſes, whilſt the diſtances from the center increaſe in the proportion 1. 2. 3. 4. &c. For gravity is a force propagated through right lines in all poſſible directions from one and the ſame center: becauſe heavy bodies, which are made to deſcend by this force, deſcribe right lines, and gravitate in all parts of the earth, or on all ſides of the earths center, and by the force of gravity are made to tend towards this center. But by the laſt propoſition every force of this ſort, and conſequently the force of gravity, is, at different diſtances from the center, in a reciprocal duplicate ratio of thoſe diſtances, or as the ſquares of them inverſely.

verſely. The force of gravity below the earths ſurface is varyed by a different law from this: and in order to find out that law, we muſt prove the following propoſition.

24. *If in a concave ſpherical ſurface the particles that compoſe it attract with forces, which, at different diſtances from them, are as the ſquares of the diſtances inverſely; that is, if they attract according to the above-mentioned law of gravity; then a body within the ſphere will remain at reſt wherever it is placed.*

It is ſelf-evident that a body, Plat. I. fig. 10. placed at C the center of the hollow ſphere IHQLK would reſt, becauſe it would be equally attracted all round, and thoſe equal and contrary attractions would deſtroy each other. If it was placed any where elſe, as at P, then indeed it would be nearer to one ſide HI of the ſphere, than to the other ſide KL, and would, by the laſt propoſition, be attracted more towards H than towards K, upon account of the diſtance PH being leſs than the diſtance PK. But the quantity of matter or number of attracting particles, and conſequently the attraction, will be greater towards K than towards H. If therefore the attraction towards K on account of the quantity of matter is juſt as much greater than the attraction towards H, as it is leſs on account of the greater diſtance, then it will follow that the body P will reſt, becauſe it is equally attracted both ways.

Let PK = D and PH = d, then, by the laſt propoſition, the force, with which the body is attracted towards K, will be to the force, with which it is attracted towards H, as the ſquares of PK and PH inverſely, or as $\frac{1}{DD}$ to $\frac{1}{dd}$ on account of the different diſtances from K and H. If we imagine in the ſpherical ſurface a circle of matter whoſe diameter is KL, and from K through I draw KPH, and from L in the ſame manner draw LPI; then between theſe two lines KH and LI, a circle of matter, whoſe diameter is KL attracts the body one way, and another circle of matter whoſe diameter is IH oppoſite to the former attracts it the other. Now if L is taken infinitely near to K, and I infinitely near to H, the two triangles KPL and HPI will be ſimilar, and their ſides (Euc. book 6. prop. 4.) being proportional, it will be as PK = D to PH = d, ſo KL to IH. But theſe oppoſite circles, whoſe diameters are KL and IH, or whoſe diameters are to each other as D to d, contain quantities of matter proportional to their areas, becauſe we either conſider the ſpherical ſurface as having no thickneſs, or elſe, which amounts to the ſame thing, ſuppoſe the thickneſs of it to be the ſame in every part. And ſince the areas of circles are to each other as the ſquares of their diameters, (Euc. book 12. prop. 2.) it follows that the quantity of matter and conſequently the force, by which the body is attracted towards K, is to the force, by which it is attracted towards H, as DD to dd. Thus the attraction towards K, on account of the greater quantity of matter on that ſide, has been ſhewn to be as DD, but on account of the bodys greater diſtance as $\frac{1}{DD}$, therefore on both accounts together as $DD \times \frac{1}{DD} = \frac{DD}{DD} = 1$: whilſt the attraction towards H on account of the leſs quantity of matter on that ſide is as dd, on account of the body's leſs diſtance as $\frac{1}{dd}$, therefore on both accounts together as $dd \times \frac{1}{dd} = \frac{dd}{dd} = 1$. But if the attraction towards K $= 1$, and the attraction towards H likewiſe $= 1$, then they are equal to each other, and the body when placed at P will continue at reſt.

25. *The force of gravity below the earths ſurface is, at different diſtances from the center, directly as thoſe diſtances.*

That is, the gravity of a body at P, plate I. fig. 11, is as much greater than the gravity of the ſame body at Q would be, as PC is greater than QC. Or, if a body was to fall through a hole bored from the earths ſurface at B to its center at C, the gravity of it would conſtantly diminiſh, till at laſt, when it came to the center, or when its diſtance from the center $= 0$, the gravity likewiſe would $= 0$.

This perhaps will appear probable, if the reader conſiders what would be the caſe, when the ſtone is at the center. There is then juſt as much of the earth on any one ſide of it, as there is on the oppoſite ſide. And conſequently it is equally attracted all ways, and theſe equal forces in contrary directions will deſtroy each other, and cauſe the ſtone to reſt; that is, the ſtone will not be moved any way by the force of gravity, or will not gravitate at all. But when the ſame ſtone is at the ſurface, it is attracted only one way, only towards the earths center; and when it is any where between the ſurface and the center, it is attracted both towards the center, and by as much of the earth as it has left behind it in falling. From whence it is evident that the gravity of the ſtone muſt be greateſt of all at the ſurface; becauſe when it is there, the whole earth attracts it downwards: whereas when it is any where elſe, a part of the earth attracts it downwards, and a part attracts it upwards. But ſince the weight of the ſtone is nothing at the center, and

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is greateſt of all at the ſurface; it follows that the force of gravity decreaſes, as the ſtone falls downwards from the ſurface towards the center.

Let $PC = D$ and $QC = d$; then, by propoſition 23, the gravity of the body, when at P, is to the gravity of it, when at Q, as $\frac{1}{DD}$ to $\frac{1}{dd}$. But when the body is at P all that part of the earth, which is included between ABEM and IHKL is a hollow ſphere, and ſuch a ſphere, by the laſt propoſition, attracts it equally all ways; therefore the body is made to gravitate towards the center only by the attraction of the ſolid ſphere IKHLC. And when it is at Q, all that part, which is included between ABEM and QR attracting equally all ways, it will be made to gravitate towards the center only by the attraction of the ſolid ſphere QRC. Now the gravity of the body cauſed by the ſphere IKHLC is greater than the gravity cauſed by the ſphere QRC, for their attractions, and conſequently the weights produced by thoſe attractions, are to each other as their ſolid contents, that is, (Eucl. book 12. prop. 18.) as the cubes of their ſemidiameters, or as PC^c to QC^c or as DDD to ddd. Therefore the gravity of the body, when at P, is on account of its diſtance from C as $\frac{1}{DD}$; on account of the quantity of matter contained in the ſolid ſphere, which attracts it, as DDD, and on both accounts together as $DDD \times \frac{1}{DD} = \frac{DDD}{DD}$, or as the cube of the ſemidiameter divided by the ſquare of it, which gives D or the ſemidiameter itſelf in the quotient. And for the ſame reaſon the gravity of it, when at Q is as $ddd \times \frac{1}{dd} = \frac{ddd}{dd} = d$. That is, the gravity of the body at P is to its gravity at Q as D to d or as PC to QC; or as its diſtance from the center.

But ſuppoſe there was a hole bored from the earths ſurface to its center, might we not aſk why a ſtone or any other heavy body, would fall down ſuch a hole till it came to the center? For ſince we have juſt now ſeen, that the gravity or weight of the ſtone is greateſt of all when the ſtone is at the ſurface, and decreaſes, whilſt it is falling, ſo as to be nothing at the center: ought not the ſtone to ſtop at that place where its gravity is greateſt, that is, at the ſurface, and not fall towards the center, where its gravity is nothing? But the anſwer to this is obvious. The ſtone gravitates only towards the center when it begins to fall. And afterwards when it is below the ſurface it gravitates both towards the center, which is below it, and towards that part of the earth, which is above it, and beyond which it has fallen. This makes its weight leſs as it deſcends; for when it begins to gravitate both ways, it will move one way only with the difference of theſe two forces. And when the ſtone gets to the center, this difference is nothing, for it is equally attracted all ways, and conſequently will not move any way. But notwithſtanding in the whole courſe of its fall it has gravitated towards the ſurface as well as towards the center; yet it could not either ſtop at the ſurface, or ever move upwards from the center; becauſe of theſe two forces that which acted towards the center was always the greateſt. So that though the ſtone has leſs weight as it gets nearer to the center, yet this weight always tends downwards; and the ſtone will always fall downwards: only it falls with the greateſt force, when it is at the ſurface, and with a leſs force as it approaches the center.

26. *All bodies near the earths ſurface, when they fall perpendicularly by their own weight or by the force of gravity, and meet with no reſiſtance, move with equal velocities.*

In the air indeed a piece of lead will fall faſter than a feather, but this is owing to the reſiſtance which they meet with from the air. For in an exhauſted receiver, if they are both let fall from the top of it at the ſame time, they will both come to the bottom together; and therefore having fallen from the ſame height in the ſame time they muſt have moved with equal velocities.

27. *The weights of bodies, or their gravities near the ſurface of theearth, are as the quantities of matter contained in them.*

Any one body is juſt as much heavier than any other, as it contains a greater quantity of matter. For, by propoſition 26, all bodies put in motion by the force of gravity move with equal velocities. Therefore, by prop. 13, becauſe the velocities are given, the moments are as the quantities of matter. And ſince, by the ſecond law of motion, the moments are as the moving forces, it follows that the moving forces are likewiſe as the quantities of matter. But when bodies fall by the force of gravity, the moving forces are their own weights. Therefore their weights are as their quantities of matter.

28. *The preſſure of a fluid upon the conſtituent particles of bodies is not ſufficient to account for their gravity.*

A piece of lead in the form of a bullet has the ſame weight, that it has, when beaten out into a thin plate. Whereas, if gravity had been owing to the preſſure of a fluid, by increaſing the ſurface, we ſhould find the preſſure, and conſequently the gravity ariſing from that preſſure, increaſed at the ſame time. It is to no purpoſe to ſay here that the fluid is ſo ſine as to penetrate the pores of the lead and to preſs only upon the ſolid parts of it; and therefore ſince there is the ſame number of ſolid parts in the bullet as there is in the plate, the preſſure ought to be the ſame. For they, who know any thing about the preſſure of fluids, underſtand, that more parts are expoſed to this preſſure in the plate than in the bullet. Suppoſe for inſtance that in any one line drawn through the bullet perpendicular to the horizon there are 2000 ſolid parts one under another, then the whole column of fluid, which preſſes upon theſe 2000 particles is no more than what reſts upon the uppermoſt particle in the line. But imagine, when the lead is beaten out into a thin plate that this line is reduced to ten other ſhorter lines, in each of which there will be no more than 200 particles one under another: then it is plain, that each of theſe lines bearing the column that preſſes upon the higheſt particle will be as much preſſed as the 2000 were preſſed before, and conſequently that the weight of theſe 2000 particles would be ten times greater in the plate than it was in the bullet, even though the fluid preſſed upon the ſolid parts only. Or ſuppoſe we turn the edge of the plate of lead upwards; then a fluid which preſſes downright, will certainly act with leſs force to make the lead deſcend, than it would have done if the flat ſide of the plate had been upwards. And yet we find that the weight of the plate is the ſame in both poſitions of it.

29. *The whole world is not full of matter, but there are in nature ſpaces, that are quite empty, or mere vacuities.*

For if all parts of the world were full of matter, then all bodies of the ſame ſize muſt contain equal quantities. A cubic inch of cork for inſtance muſt contain juſt as much matter as a cubic inch of lead does. And ſince the weight of all bodies, by propoſition 27, is as the quantity of matter contained in them, the cork muſt be as heavy as the lead. But their weights, we find, are different: from whence we muſt conclude that there are in ſome bodies interſtices void of all matter, and that the lighter body muſt have more of theſe interſtices than the heavier has. It cannot be replyed here, that the cork would be lighter than the lead without ſuppoſing any mere vacuities in either of them; if the pores of the cork inſtead of being quite empty were filled with a finer and more ſubtil matter than thoſe of the lead. Indeed this anſwer would deſtroy our former reaſoning, if it could be made appear that the weight of matter, the quantity of it being given, could be diminiſhed by the fineſs or ſubtilty of its parts. But ſince all matter, that we can make experiments upon, whether the parts of it be fine or coarſe is found to gravitate in proportion to its quantity; though the pores of the cork were filled with a finer fluid than thoſe of the lead, yet both being full of matter they muſt, as their dimenſions are equal, contain equal quantities of it, and therefore one muſt neceſſarily weigh juſt as much as the other.

In this chapter the word gravity is uſed in three different reſpects, or gravity is ſpoken of as capable of being greater or leſs in reference to three different meaſures. In proving the 25 propoſition, the gravity of a body cauſed by the ſphere IKHLC, Plat. I. fig. 11. is affirmed to be greater than the gravity cauſed by the ſphere QRC. Here gravity is conſidered as greater or leſs in proportion to the cauſe, by which it is produced. It is referred to the ſame meaſure when we ſay that gravity would be greater than it is, upon ſuppoſition that the quantity of matter in the earth was double or triple what it is at preſent.

By the 23 propoſition it appears, that the force of gravity decreaſes as the ſquares of the diſtances from the center increaſe; ſo that at diſtances, which from the ſurface upwards are in the proportion of 1, 2, 3, 4, the forces of gravity will be as 1, $\frac{1}{4}$, $\frac{1}{9}$, $\frac{1}{16}$. Here the cauſe producing gravity, that is, the earth and the quantity of matter contained in it, continues the ſame, and yet the gravity propagated from it is ſo varyed as to be greater at ſome diſtances from the center and leſs at others. Therefore it is here referred to ſome meaſure different from the former, and that is the velocity, which the force of it would produce in a given time. For when we ſay that gravity at the diſtance of one ſemidiameter from the center is four times greater than at double that diſtance, or as 1 to $\frac{1}{4}$ we mean that in any certain time, as a ſecond for inſtance, the velocity, which the force of gravity would produce in a body at the diſtance of one ſemidiameter, is four times what it would produce in the ſame body at double that diſtance. Gravity is referred to the ſame ſtandard, when the weight of a body falling perpendicularly is ſaid to be greater than its weight if it falls down an inclined plane. For though it may begin to fall from the ſame height in both caſes, yet a greater velocity is produced in it, in a given time, when it falls perpendicularly, than when it falls upon an inclined plane: and in reference to this velocity the gravity in one caſe is ſaid to be greater than the gravity in the other.

According to propoſition 27, the gravity of bodies is proportional to the quantities of matter contained in them: ſo that, according to propoſition 29, a cubic inch of lead has more gravity than a cubic inch of cork, becauſe the former contains a greater quantity of matter than the latter. The word gravity is here referred to a third meaſure different from both the others. For the quantity of matter in the earth or force propagating gravity is the ſame: and, by 26 propoſition, the lead and the cork would both fall with equal velocities. But becauſe they fall with equal velocities, their quantities of motion produced in the ſame time will be different, ſince their quantities of matter are unequal. And, as the force of gravity produces a greater quantity of motion in the lead than in the cork, in this reſpect it is ſaid to be greater.

The following definitions will inform us by what name gravity is to be called in each of theſe references.

30. *When gravity is conſidered as greater or leſs in reference to the efficacy of the cauſe, which produces it; then it is called the abſolute force of gravity.*

31. *When the velocity, produced in a given time, is the meaſure by which gravity is ſaid to be greater or leſs, then it is called the accelerating force of gravity.*

32. *When gravity is conſidered as greater or leſs in proportion to the quantity of motion, which it produces in a given time, then it is called the moving force of gravity.*

33. *Gravity is not ſuch a quality, as the occult ones made uſe of in philoſophy by the Ariſtotelians;*

It differs from thoſe qualities in two reſpects. The firſt is that the Ariſtotelians make uſe of almoſt as many different occult qualities as there are appearances in nature to be explaned; and ſo every particular effect has its particular cauſe aſſigned for it. But gravity is an univerſal quality, which belongs to all matter whatever, and is applyed by the Newtonian philoſophy to the explication of innumerable appearances. The ſecond difference between gravity and occult qualities is, that theſe qualities are, as their name implys, occult ones; they cannot be made the object of our ſenſes, nor is their exiſtence proved by any experiments. But gravity is a well-known quality, which we may perceive by our ſenſes, and whoſe exiſtance in nature is confirmed to us by conſtant experience. The Newtonians therefore in the uſe which they make of gravity

PLATE I.

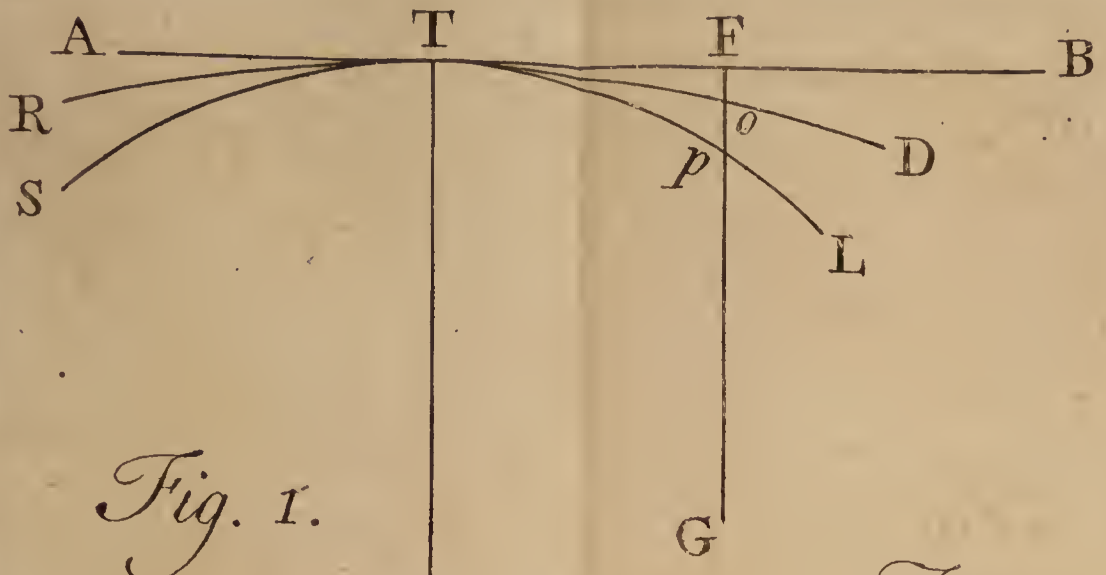


Fig. 1.

Fig. 3.

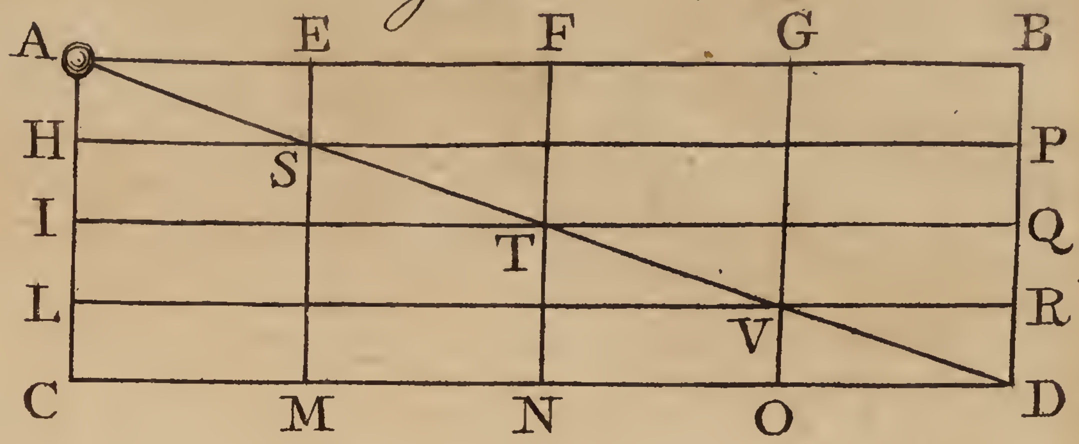
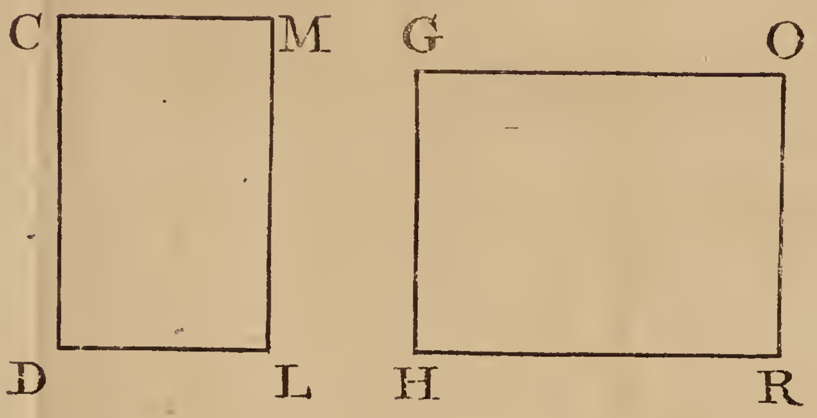


Fig. 2.



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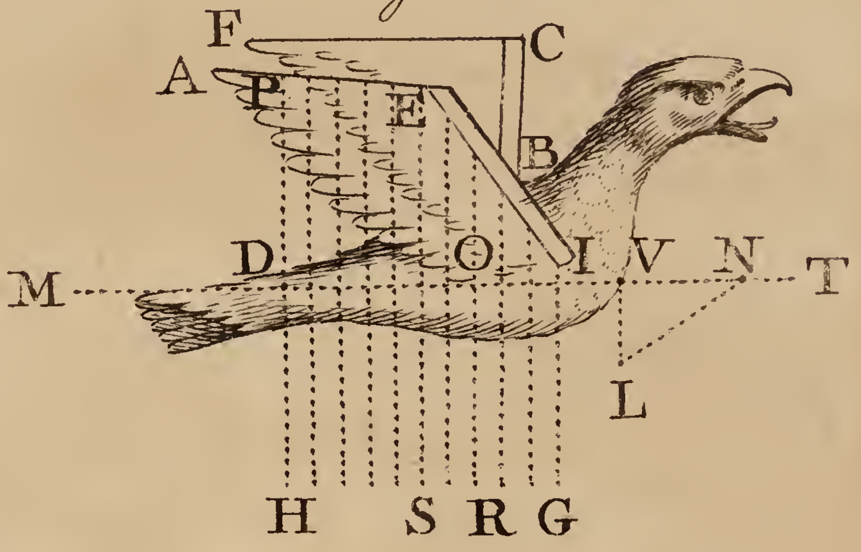


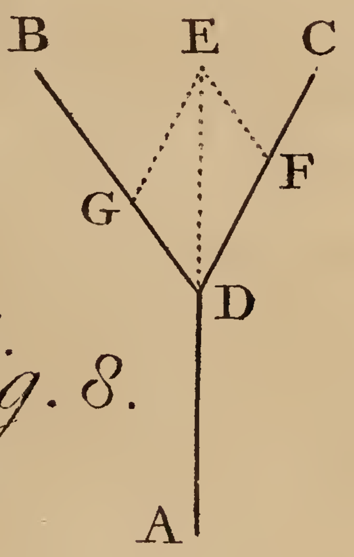
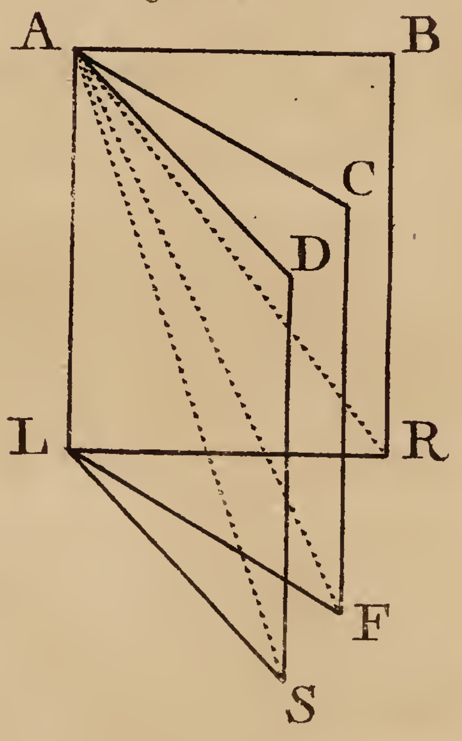


Fig. 8.

Fig. 5.



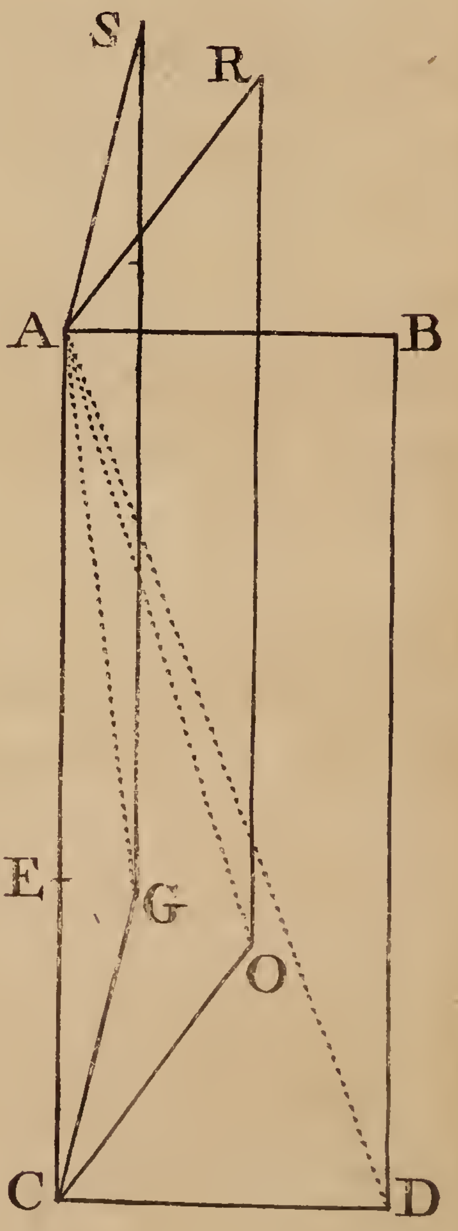


Fig. 6.

Fig. 9.

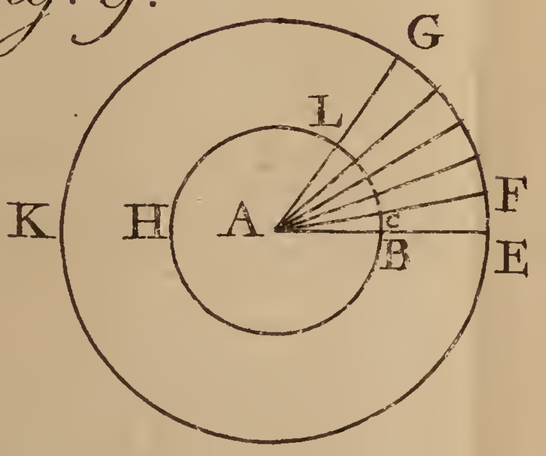


Fig. 7.

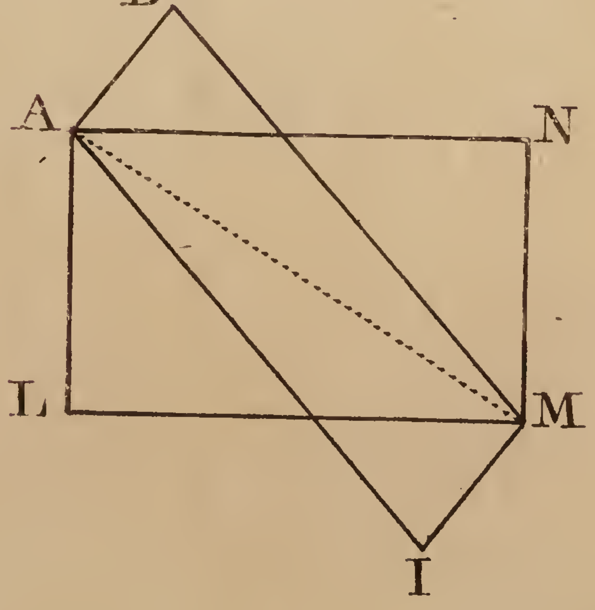
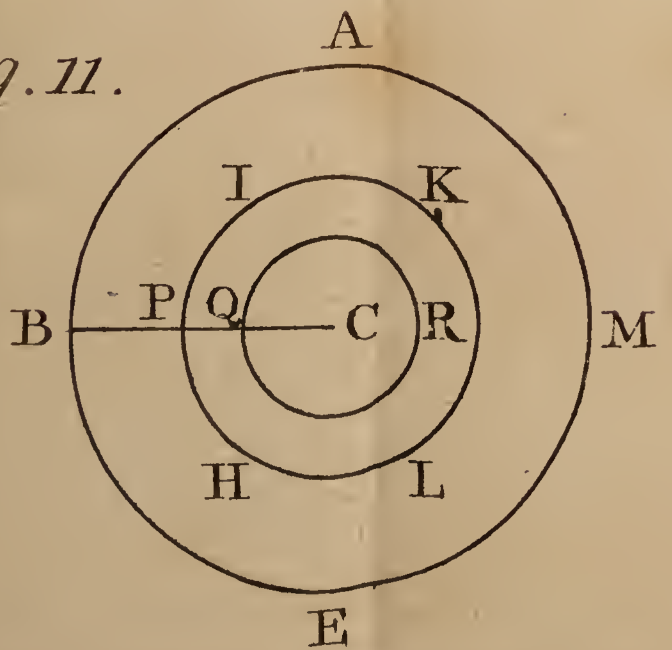


Fig. 11.



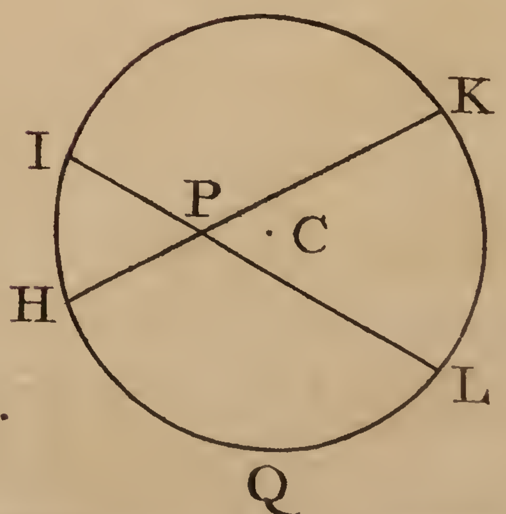


Fig. 10.

gravity do not admit of more cauſes of natural things, than are both true and ſufficient for explaning their appearances; and thus they obſerve a principal rule of true philoſophy, which the Ariſtotelians neglect.

CHAP. V.

The center of gravity.

34. *The center of gravity in any body is a point on each ſide of which all the parts of the body are in equilibrio.*

35. *The center of motion is that point round which a body does or may move.*

36. *The line of direction is that, in which a weight or power acts.*

37. *A body is moved by its own weight in the ſame manner that it would be, if the whole weight of it acted in the center of gravity.*

FOR if the whole weight acted in that center only and was not diffuſed through the whole maſs, then the parts on one ſide of the center could never by their weight put the parts on the other ſide of it into motion, ſince by the ſuppoſition, there would be no weight on either ſide. But by the definition of the center of gravity the parts of a body on each ſide its center of gravity balance each other, ſo that thoſe on one ſide can never by their gravity put thoſe on the other ſide into motion. Therefore, as to producing motion in a body, the gravity of it acts in the ſame manner, when diffuſed through every part of the body, as if it were all accumulated in one point ſo as to act in its center of gravity only. From hence it follows that the line of direction of a weight is drawn from its center of gravity to the earths center.

38. *The weight of a body will make it fall, unleſs its center of gravity is ſupported.*

This would certainly be true if all its weight acted in the center of gravity. For then ſupporting this center and ſupporting the weight of the body, or, keeping it from falling, would be the ſame thing. And it is no leſs true though the weight is diffuſed through every part of the body. Since by the laſt propoſition the motion produced in a body by its weight will be the ſame upon either ſuppoſition.

39. *The weight of a body can never move it in ſuch a manner as to make its center of gravity aſcend.*

Becauſe a body is acted upon by its weight juſt as it would be if the whole weight was united in the center of gravity. And if it was all united there, we then may be ſure that gravity could never make it aſcend, unleſs an impulſe downwards, as gravity is, could make that point, upon which ſuch an impulſe acts, move upwards.

40. *Bodies will fall by their weight, when they are ſet upon an horizontal plane, unleſs the line of direction, in which that weight acts, paſſes through their baſe.*

For when the line of direction paſſes through the baſe, then the weight of the body, becauſe it acts in that line, preſſes upon the baſe, and will keep the body firm upon it. Whereas, if the line of direction is without the baſe, then the weight, will draw the body off from the baſe, or will make it fall. Thus the body ABDE Plat. II. fig. 2. whoſe center of gravity is C will ſtand firm; becauſe the weight of it acts in the line CO, as if it was all united in the ſingle point C. But a force, which preſſes the baſe DE downwards cannot poſſibly make it riſe, as it muſt do before the body can fall either towards K or any other way. But the body *abde*, whoſe center of gravity is *c* will be thrown down by its weight. For the weight acting in the line *co* on one ſide of the baſe will make the body turn round upon the point *e*, that is, will make the baſe *de* riſe and the body fall towards *k*.

Set one leg of the compaſſes at E and with the opening CE ſtrike the arc CFK. As the body cannot fall towards K without turning round on the point E, the center of gravity C muſt in the fall deſcribe the arc CFK, but that it cannot do unleſs it riſes, for in that arc the point F is higher than C. Therefore ſince the weight of a body cannot make its center of gravity move upwards, neither can it make this center deſcribe the arc CFK, that is, it cannot throw the body down. If in the ſame manner with the opening of the compaſſes *ce* we ſtrike the arc *cfk*; where the line of direction falls without the baſe: then the point *f* in this arc and every other point will be lower than *c*; ſo that the center of gravity can deſcend, whilſt it deſcribes the arc *cfk*. And there is nothing in the line of direction to ſupport it; therefore it will deſcend, ſo as to turn the body round upon the point *e*, that is, ſo as to make it fall towards *k*.

A wall or any other building may lean a great way before it falls. For its gravity will not make it fall; till it leans ſo much as to carry the line of direction beyond the foundation. For this reaſon it will ſtand the longer and may lean ſo much the more for having a wide foundation.

For a man to ſtand upright the line of direction of his weight muſt paſs through his baſe, which is a parallelogram, or ſome other quadrilateral figure, contained between his two feet and two lines, of which the firſt is drawn acroſs from one foot to the other at his toes and the ſecond at his heels. It is therefore eaſier to ſtand upon both feet than it is upon one: becauſe the center of gravity muſt be brought directly over the one foot that we ſtand upon, and the line of direction muſt be kept within the compaſs of this ſmaller baſe. When a man holds any conſiderable weight in his left hand, his center of gravity, that is, the center of gravity of himſelf and the weight conſidered as one maſs, will be carryed too far towards the left, unleſs by leaning the contrary way he reduces it again and brings it to ſuch a poſition as will make the line of direction paſs through his baſe.

When we are ſetting on a chair, our center of gravity is upon the ſeat, and the line of direction falls behind our baſe. Therefore in order to riſe, we lean forwards, that we may bring the line of direction towards our feet; and we draw our feet backwards at the ſame time, that we may carry our baſe towards the line of direction: and when by both theſe motions the center of gravity is reduced ſo as to be directly over our feet, we are able to raiſe ourſelves upright.

In walking, the foot, from which we ſet off, is our baſe at firſt; till by turning the ball of it round we have thrown the center of gravity forwards beyond it, by which means we ſhould throw ourſelves down, if we did not take up the other leg from the ground, and ſet it before us ſo as to catch ourſelves upon it, as upon a prop or ſupport to prevent us, from falling. This is taking one ſtep. And in order to take a ſecond, the center of gravity muſt be brought directly over this prop or ſupport; that is, the foot, which we put out before us, muſt be made our baſe; that ſo in taking a ſecond ſtep we may ſet off from it. We do this by turning the ball of the other foot ſtill farther round ſo as to puſh the ground that we ſtand on back with our toe, and the ground by its reaction puſhes our center of gravity forwards; till the line of direction is got to the place whither we wanted to reduce it. Therefore in walking the line of direction paſſes through each foot alternately, and if we ſet one foot directly before the other in every ſtep that we take, then this line will move ſtrait forwards: but if we ſtraddle as we walk; then the line of direction does not go on evenly, but is carryed out firſt towards the right ſide, and then towards the left, at the ſame time that it goes forwards: which is the reaſon, why thoſe who ſtraddle in their walk, as fat people commonly do, are obſerved to waddle. Indeed every body waddles either more or leſs, only they who walk pretty cloſe do it ſo little as not to be perceived.

In ſkaiting the line of direction paſſes through the foot on which the ſkaiter moves; and as he raiſes his other foot from the ice, he ſcrapes it in a direction contrary to that, in which he deſigns to move; and the reaction of the ice upon it puſhes him forwards.

In jumping the center of gravity is both made to riſe from the ground and is likewiſe carryed forwards at the ſame time. The manner in which the center of gravity is made to riſe will be beſt underſtood if we firſt explane ſomething of the ſame ſort in a more ſimple inſtance. If you ſet one end of a bow-ſtick on the ground and with your hand bend down the other end of it; upon letting this end go, the bow flyes out at once, and jumps from the ground. For the ſticks elaſticity unbends it equally both ways, and as the action of one end is downwards and preſſes upon the ground, the reaction will be upwards and the center of gravity will have a motion communicated to it in that direction, and therefore by the firſt law of motion, will continue to riſe: and the ſtick will by this means be carryed off from the ground, till its weight puts a ſtop to the farther aſcent of it, and makes it fall back again. The force of the muſcles is what in jumping ſupplys the place of elaſticity. For by bending the joynts of the two heels, the two knees, and the two thighs there are ſix bows bent; and if all of theſe are ſuddenly ſtraitened by the force of the muſcles the center of gravity of the body will have a motion upwards communicated to it, and the body itſelf will by this means be made to riſe from the ground. If we are to jump directly upwards, then we puſh the ground, on which we ſtand, directly downwards. But if we would move obliquely ſo as to riſe and to jump forwards at the ſame time, then we puſh the ground obliquely backwards with both feet together, in the ſame manner as we puſh it with one foot in walking.

The reader may here conſider, what it is that makes the difference between walking and running. If he imagines that theſe two modes of motion differ only in the degree of ſwiftneſs, and that as walking is a ſlow motion, ſo running is a quicker motion of the ſame ſort, he is much miſtaken. For theſe two motions differ in ſort rather than in degree. Indeed they do not always differ in degree, as any ones experience, will inform him: for there is no man who is not able to go one pace which he would call walking, that will carry him on faſter than another pace, which he would call running. We can walk quick and run ſlow; and, if we pleaſe, we can rid ground faſter by walking quick, than we do by running ſlow. This makes it evident that walking is not a ſlow ſort of running, nor running a quick ſort of walking. They are two entirely different modes of motion, and though we were to run or walk, with juſt the ſame degree of ſwiftneſs, yet every man would be able to tell us, when we walk and when we run. Walking is moving forwards by ſteps, ſo as never to have both legs off the ground at the ſame time. Running is moving forwards by jumps. Indeed in running we do not take our jumps with both legs cloſe together, but with one of them before the other: but I call it jumping becauſe in throwing ourſelves forwards both legs are off the ground at once.

41. *Some heavy bodies ſlide down an inclined plane; others roll down it.*

A body is ſaid to ſlide down, if it deſcends upon the ſame baſe conſtantly, or with the ſame ſide towards the plane. It is ſaid to roll down, if in its deſcent the baſe is conſtantly changing. Therefore, when a heavy body is upon an inclined plane, if the line of direction falls within the baſe of the body, it will ſlide down. For, by propoſition 40, the weight of the body will not throw it off the baſe. But if the line of direction falls beyond the baſe, then the body in its deſcent cannot continue long on the ſame baſe; but the ſide which is towards the plane will be conſtantly changing; that is, the body will roll down. Thus both the bodies A and B Plat. II. fig. 3. will move down the plane H I. Becauſe their centers of gravity can deſcend. But they will not move down in the ſame manner. For *c r* the line of direction paſſes through the baſe *d e* of the body A; therefore, in moving down, the body will always keep upon the ſame baſe or will ſlide down. But the line of direction *c s* paſſes beyond the baſe *f g* of the body B; therefore it will be thrown off this baſe and will change it for *g t*; and as this ſort of motion will continue through the whole deſcent, the body will roll down.

42. *A heavy body, which is ſuſpended on a center of motion, will reſt, if the center of gravity is directly under the center of motion, otherwiſe the weight of the body will make it move.*

If a heavy body P Plat. II. fig. 1. hangs by a ſtring on the center of motion C, the ſtring, by which it hangs, acts in the direction PC, and the force of it muſt be equal to the weight of the body, otherwiſe it would break. And when the body is at P, theſe two equal forces act in contrary directions, the ſtring in the line PC, and the weight in the line PL: therefore they will deſtroy each other, and the body will reſt beween them. But if the body is at *p*, then one of the forces acts in the line *p* C, and the other in the line *p* L. And though theſe forces are

equal,

equal, yet, becauſe they act obliquely, the body by, propoſition 16, will begin to move in the diagonal of a parallelogram, one of whoſe ſides is in the direction p C and the other in the direction p L. This proof may eaſily be made univerſal. For in all caſes, where a body is ſuſpended, the force, which ſuſtains it, muſt be equal to its weight. And, when the center of gravity is directly under the center of motion, theſe two equal forces act in contrary directions, for which reaſon the body will reſt. But in any other ſituation, theſe forces are oblique to each other: and therefore, though they are equal, the body will be made to move in the diagonal of a parallelogram, of which the two forces are the ſides.

CHAP. VI.

Of the mechanical powers.

THE ſubject of this chapter has given the name of *mechanics* to one branch of natural philoſophy. Mechanics ſtrictly ſpeaking ſhould treat of nothing but ſuch machines or inſtruments as aſſiſt us in lifting weights or overcoming reſiſtances. But becauſe the effects of theſe inſtruments cannot be underſtood unleſs the laws of motion are firſt explaned; the laws of motion and, in conſequence, the application of theſe laws to the motion of bodies in general is made the ſubject of mechanics.

There are five ſimple machines. A lever, a wheel and axle, a pully, a ſcrew, and a wedge. Some writers reckon up more, and others reduce them to fewer. But this diviſion will anſwer our purpoſe as well or better than any other.

43. *A lever is a bar, which turns round upon a prop or center of motion, and is applyed either to raiſe weights or to overcome reſiſtances.*

A lever is commonly called a hand ſpike, when it is made of wood; and a crow, when of iron. In theory we ſuppoſe, that the bar itſelf has no weight at all, and that no force whatever can bend it.

44. *By the intenſity of a power is meant the abſolute force of it, or the force it will have, when applyed immediately to a weight, ſo that the power and weight move with the ſame velocity.*

45. *The moment of a power, like that of a weight, is found by multiplying its intenſity into its velocity.*

46. *A lever is ſaid to be of the firſt kind, when the prop or center of motion* C *Plat.* II. *fig.* 4. *is between the weight* A *and the power* B. *Of the ſecond kind, when the weight* W *Plat.* II. *fig.* 5. *is between the center of motion* C *and the power* P. *Of the third kind, when the power* P *Plat.* II. *fig.* 6. *is between the center of motion* C *and the weight* W.

47. *In turning a lever upon the center of motion, the velocity of each point in the lever, or of any weight or power applyed to that point, will be as the diſtance of it from the center of motion.*

In turning the lever ACB Plat. II. fig. 4. upon the center till it comes into the poſition *a*C*b*; the point A deſcribes the arc A*a*, and the point B deſcribes the arc B*b* in the ſame time. Now it is evident that B*b* is longer than A*a*. Therefore the point B muſt move faſter or muſt have a greater velocity than the point A, becauſe B runs over the longer line B*b* in the ſame time that A runs over the ſhorter line A*a*. And the velocity of B muſt be juſt as much greater than that of A, as B*b* is longer than A*a*: for otherwiſe B could not deſcribe B*b* in the ſame time that A deſcribes A*a*. Now B*b* is juſt as much longer than A*a*, as BC is longer than *a*C: for as B*b* contains A*a* ſix times, ſo BC contains AC ſix times. And B moves juſt as much faſter than A, as B*b* is longer than A*a*, that is, as BC the diſtance of B from the center is greater than AC the diſtance of A from the center; or the velocity of B bears the ſame proportion to that of A, that BC bears to AC.

Or otherwiſe: the velocities of theſe two points A and B, being, by propoſition 10, as the ſpaces, which they deſcribe in the ſame time, that is, as A*a* to B*b*, are likewiſe as AC to BC or as their reſpective diſtances from the center. For A*a* bears the ſame proportion to B*b* that AC bears to BC. Becauſe the arcs A*a* and B*b* ſubtend the angles AC*a* and BC*b*, which are vertical ones and therefore equal, Eucl. b. I. p. 15. from whence it follows that the arcs being ſimilar, Eucl. b. VI. p. 33. corol. 2. are to each other as the ſemidiameters of their reſpective circles; that is, A*a* the ſpace deſcribed in a given time by the point A or the velocity of A is to B*b* the ſpace deſcribed in the ſame time by B or to the velocity of B, as AC the diſtance of A from the center is to BC the diſtance of B from the center.

Let a power, ſuch as the force of a man be applyed at the point B and move along with it, whilſt a weight, that is faſtened at the other end of the lever, moves along with the point A. Then, ſince the power moves as faſt as B does, and the weight no faſter than A, the velocity of the power is to that of the weight as BC to AC, or as their reſpective diſtances from the center of motion.

We have here ſuppoſed the power and weight to be on the oppoſite ſides of the center of motion, which is the caſe only in a lever of the firſt kind. For in levers both of the ſecond and third kind, the power and weight are always on the ſame ſide the center. Thus let

CB Plat. II. fig. 4. with the prop at one end of it be the lever. Then, if the power acts at B, and the weight at F, it is a lever of the ſecond kind: but if the power acts at F and the weight at B, it is a lever of the third kind. And in either caſe the velocity of B is to that of F, as BC to FC, the reſpective diſtances of thoſe two points from the center. For, as before, the velocities of B and F are as the arcs B*b* and F*f*, which are deſcribed by them in the ſame time: and, as theſe arcs ſubtend the ſame angle at C, they are ſimilar. Therefore B*b*, or the velocity of B, is to F*f*, or the velocity of F, as BC to FC.

48. *The comparative velocity of a power or weight is to be eſtimated by the perpendicular diſtance of their line of direction from the center of motion.*

If BCD Plat. II. fig. 7. is a bent lever; and the power by means of a rope faſtened at D acts in the direction DF, then the velocity of the power is as CG a line drawn from the center of motion C perpendicular to AF the line of direction, in which the power acts. For ſince, by the ſecond law of motion, the power acts only in the line AF, the perpendicular diſtance of this line from the center, or CG muſt likewiſe be the diſtance of the powers action from it.

In the foregoing propoſition, I eſtimated the comparative velocity of a power by the diſtance between the center and that point, where the power is applyed; and the comparative velocity of a weight by the diſtance between the center and that point, where the weight is ſuſpended. Thus Plat. II. fig. 4. the velocity of a power applyed at B was ſaid to be to that of a weight ſuſpended at A, as BC to AC. But if the power pulls downright, then BE is its line of direction: and though the weight deſcribes A*a* yet the diſtance, to which it riſes from the earths center in riſing from A to *a*, is only AD and in ſuch a line as AD, or in the line of its direction, the weights velocity is to be eſtimated. Therefore the perpendicular diſtances of theſe lines of direction from the center of motion being CE and CD, the velocity of the power, by this propoſition, is to that of the weight as CE to CD. But whenever the points of application and ſuſpenſion A and B are not altered by altering the poſition of the lever from ACB an inclined one to *a*C*b* an horizontal one, the comparative velocities are the ſame in reſpect of each other, whether they are eſtimated by this proportion, or by that laid down in the laſt propoſition, becauſe CB is to CA as CE to CD. For, ſince CE and CD are the perpendicular diſtances of BE and AD from the center, in the triangle BEC the angle at E is a right one, and ſo is the angle at D in the triangle ADC. The angle at C likewiſe in one of theſe triangles is equal to the angle at C in the other. And, becauſe there are two angles in one equal to two angles in the other, therefore the remaining angle at B is equal to the remaining angle at A. Eucl. b. I. prop. 32. corol. 2. From hence it follows, Eucl. b. VI. p. 4. that theſe two triangles being equiangular, their ſides about the equal angles will be proportional, CB to CA as CE to CD.

49. *A power and a weight, which act upon a lever, are in equilibrio, when the power is to the weight, as the diſtance of the weight from the center to the diſtance of the power from the center.*

For ſince their velocities by prop. 47. are to each other as their reſpective diſtances from the center; the power by what is here ſuppoſed is to the weight as the weights velocity to the powers velocity; therefore, Eucl. b. VI. prop. 16. the power multiplyed into its own velocity, or the moment of the power, is equal to the weight multiplyed into its own velocity, or to the moment of the weight: that is, the power and weight are in equilibrio. Or otherwiſe, when the power is to the weight, as the weights diſtance from the center to the powers diſtance from the ſame, then the power is to the weight as the weights velocity to the powers velocity; or the power and weight are to each other as their reſpective velocities inverted, that is, as their velocities reciprocally, therefore, by prop. 14. 45. their moments are equal.

This may perhaps be made more intelligible, if we apply it in ſome particular inſtances to a lever of each kind. A power which is equivalent to 100 pounds in weight, can lift no more than ſuch a weight, when applyed immediately to it. Becauſe in raiſing the weight to any height the power and weight both move through equal ſpaces in the ſame time, that is, they both move with the ſame velocity. But if the lever AB is divided by the prop C Plat. II. fig. 4. in ſuch a manner that the arm CB is to the arm CA as 6 to 1: then the ſame power applyed at B will be in equilibrio with a weight of 600 pounds ſuſpended at A. And, if the power is ever ſo little increaſed, it will raiſe the weight. For the moment at one end of the lever is equal to the powers intenſity multiplyed into its own velocity or diſtance from the center $= 100 \times 6 = 600$, and the moment at the other end is the ſame being equal to the weight multiplyed into its own velocity or diſtance from the center $= 600 \times 1 = 600$.

Upon a lever of the ſecond kind Plat. II. fig. 5. if CP the diſtance of the power is to CW the diſtance of the weight from the center as 5 to 1, then a power equivalent to 100 pounds, if applyed at P, will be in equilibrio with a weight of 500 pounds ſuſpended at W. For the power multiplyed into its own velocity or diſtance $= 100 \times 5 = 500$, and the weight multiplyed into its own velocity or diſtance $= 500 \times 1 = 500$. Therefore their moments are equal. And if the power is ever ſo little increaſed it will raiſe the weight by turning the lever round upon the center of motion.

Upon a lever of the third kind, Plat. II. fig. 6. if CP the diſtance of the power is to CW the diſtance of the weight from the center as 3 to 5 then a power equivalent to 500 pounds, if applyed at P, will be in equilibrio with a weight of 300 pounds ſuſpended at W. For the power multiplyed into its own velocity or diſtance $= 500 \times 3 = 1500$, and the weight multiplyed into its own velocity or diſtance $= 300 \times 5 = 1500$. Therefore their moments are equal.

I muſt deſire the reader to take notice, that though a power and weight will reſt, when they are in equilibrio, yet it is not abſurd to reaſon concerning their velocity, and to ſay that they are in equilibrio, becauſe they are to each other as their reſpective velocities reciprocally. For here we do not mean the velocities with which the power and weight move; that indeed might be abſurd as neither of them move at all: but we mean the velocities, with which they muſt begin to move, if they were to move. Thus, if the center of motion divides a lever of the firſt kind in the proportion of 1 to 6; a power, which is equivalent to 100 pounds, applyed at the end of the levers longer arm, will be in equilibrio with a weight of 600 pounds ſuſpended at the end of the ſhorter: and therefore the power and weight will reſt. And when, in giving a reaſon for the equilibrium, we ſay that the power and weight are to each other as their reſpective velocities reciprocally: we do not mean that the power moves with ſix times the velocity of the weight; ſince neither of them are ſuppoſed to move at all: but that the weight cannot begin to move without making the power move at the ſame time with ſix times its own velocity. From whence it follows, that the weight cannot move in one direction, unleſs it can make the power move in the contrary direction with a moment equal to its own. And ſince equal forces in contrary directions deſtroy each other, neither the weight can move the power nor the power move the weight: therefore both of them will reſt, that is, they will be in equilibrio.

50. *When two weights ſuſpended upon a lever are in equilibrio; their common center of gravity is ſupported by the prop.*

When inſtead of applying a power at one end of a lever and hanging a weight upon the other, we hang a weight upon both ends; there will be an equilibrium, if the weights, or quantities of matter, are to each other inverſely as their diſtances from the prop, that is, inverſely as their velocities. For it will be eaſy to underſtand, from what has been ſaid already, that in this poſition their moments will be equal. Now as the center of gravity in a ſingle body is the point on each ſide of which all the parts of the body are in equilibrio; ſo the common center of gravity of two bodies is that point on each ſide of which the two bodies are in equilibrio; or on each ſide of which the quantities of motion are equal. But, by the laſt propoſition, the quantities of motion in the lever are equal on each ſide of the prop, when the two bodies are at diſtances inverſely as their weights. Therefore if a line be drawn from the center of gravity of one body to the center of gravity of the other, the point of this line, which is over the prop, will be their common center of gravity. Or; ſince the diſtance of each weight from the prop is inverſely as its quantity of matter, the common center of gravity of the two bodies is a point which divides the diſtance between them in the inverſe proportion of their quantities of matter.

Some of thoſe, who object to our conſidering the velocity of weights or powers, when theſe weights and powers do not move, if they are called upon to prove that two unequal weights will reſt, when they are ſuſpended upon a lever at diſtances from the prop, which are as the weights inverſely, ſay, that in theſe circumſtances the common center of gravity is ſupported, and therefore the weights ought to reſt. But by examining this reaſon we ſhall find it proves (as the logicians ſay) in a circle. Suppoſe I aſk why two unequal weights, one of which is ſuſpended at *a*, Plat. II. fig. 4. and the other at *b*, will reſt, if the diſtances C*a* and C*b* are as the weights inverſely: it is juſt the ſame as if I had aſked why the two unequal weights in theſe circumſtances balance each other or are in equilibrio. And can it be a ſatisfactory anſwer to tell me, that C, which is the common center of gravity of the two weights is ſupported? for might not I aſk my inſtructers, how they prove that C is the common center of gravity? If they anſwer, that on each ſide of C the two weights are in equilibrio; this is what I charged them with, it is proving in a circle. For firſt they tell me, that the two weights are in equilibrio, becauſe C is the center of gravity, and is ſupported; and then they tell me that C is the center of gravity, becauſe the two weights are in equilibrio. But if they give me the true reaſon why C is the common center of gravity of the two bodies at *a* and *b*, and ſay that the quantities of motion on each ſide of C are equal; then they do the ſame thing themſelves which they blame in others: for whilſt they call it abſurd in us to talk of the velocity of bodies, which are at reſt, they themſelves talk of the motion of bodies, which are at reſt. And, unleſs they will conſider the velocities of bodies in equilibrio, I know not by what means they would be able to prove their aſſertion and ſhew that the quantities of motion on each ſide of C are equal, when the quantities of matter at *a* and *b* are unequal.

51. *A common balance will not be in equilibrio, unleſs the weight in one ſcale is equal to that in the other.*

The beam of a common balance is a lever of the firſt kind, which is divided into two equal parts or arms by the center of motion. And, ſince the diſtance of each point of ſuſpenſion, or of each ſcale from the center of motion is equal, therefore the velocities on each ſide are equal. From whence it follows, by propoſition 13, that the moment of each ſcale will be as the quantity of matter contained in it; and conſequently that the moments will not be equal, or there will be no equilibrium, unleſs the quantity of matter or weight in one ſcale is equal to the quantity of matter or weight in the other.

For this reaſon it is, that when you want to weigh different quantities of any thing, you muſt uſe a different ſtandard. Thus to weigh a pound, you uſe a pound ſtone, or ſtandard weight of a pound; to weigh 14 pounds, you uſe a ſtandard of that weight, and to weigh 100 pounds you uſe a third ſtandard, the weight of which is 100 pounds. It is otherwiſe in ſteelyards, where one and the ſame ſtandard may be applyed to eſtimate, and, by the conſtruction of the inſtrument, may be made to balance very different weights. The reaſon of this will appear under the following propoſition.

52. *In the ſteelyard different weights may be in equilibrio with the ſame ſtandard: and the weight to be eſtimated is as the diſtances of the ſtandard from the center of motion, when there is an equilibrium.*

The ſtandard 1 Plat. II. fig. 8. may be conſidered as a power acting on the arm CP of the lever WP. And as the ſtrength of a man applyed at P and a weight at W, ſo likewiſe two weights one at P inſtead of a power, and the other at W are in equilibrio, when they are to each other inverſely as their diſtances from the center of motion

The weight of 1 pound at the diſtance of 4 has 1 for the quantity of matter and 4 for the velocity, and conſequently has a moment $=1\times4=4$, and the weight of 4 pounds at the diſtance 1 has 4 for the quantity of matter and 1 for the velocity, and conſequently has a moment $=4\times 1=4$, therefore their moments are equal. But the ſame ſtandard of one pound at the diſtance 3 will have a moment $=1\times 3=3$; and therefore will be in equilibrio with a weight of only 3 pounds ſuſpended at W or at the diſtance 1 on the other arm of the ſteelyard. For the ſame reaſon if the ſtandards diſtance from the center of motion $=2$ it would be in equilibrio with no greater weight than 2 pounds at W or at the diſtance 1 on the other arm. And if the ſtandards diſtance was 1, or equal to the weights diſtance, then to make an equilibrium their quantities of matter muſt be equal.

The moment of the ſtandard, which is ſuſpended on one arm of the ſteelyard, is as its quantity of matter multiplyed into its diſtance from the center of motion; and ſo likewiſe is the moment of the weight, which is ſuſpended on the other arm. Now on the ſide of the ſtandard the quantity of matter is given; therefore the moment is as the diſtance from the center, or as the velocity, by propoſitions 47. 13. And on the ſide of the ſteelyard where the thing to be weighed hangs, the diſtance of the weight and conſequently its velocity is given. Therefore the moment is as the quantity of matter by propoſition 13. From whence it follows, that when theſe two moments are equal, that is, when there is an equilibrium, the quantity of matter in the weight on one ſide is as the diſtance of the ſtandard from the center of motion on the other ſide.

53. *The beam of a balance will not reſt in any poſition but a horizontal one, when the weights in the oppoſite ſcales are in equilibrio with each other.*

The center of motion *m* of the balance Plat. II. fig. 9. does not coincide with the common center of gravity *g*. But, when the beam is in the horizontal poſition AB, the center of gravity *g* is directly under the center of motion *m*. For this reaſon, according to propoſition 42, the whole will reſt. But, if the beam is inclined in the poſition HI, then the center of gravity *g* is moved from under the center of motion towards the ſide I, that is towards the end of the beam which is raiſed. Therefore, by the ſame propoſition, the end I will deſcend till the center of gravity comes under the center of motion: and, by the velocity acquired in deſcending, it will ſink ſtill farther; ſo that the end H will be raiſed above the level; and the center of gravity will be moved from under the center of motion towards the ſide H: for which reaſon H will deſcend as I did before. And the beam will keep vibrating in this manner, each end of it alternately aſcending and deſcending; till by the reſiſtance of the air and the friction on the center, all its motion is loſt.

54.

54. *A man ſtanding in one ſcale, and in equilibrio with a weight in the oppoſite ſcale, cannot increaſe his own weight by pulling the ropes, to which his ſcale hangs; nor can he diminiſh it by thruſting directly upwards againſt the point of ſuſpenſion.*

By pulling downwards he hangs either a part of his own weight, or elſe the whole of it to the ropes: for by, the third law of motion, the reaction of the ropes upwards, is juſt equal to the force with which he pulls downwards: for inſtance, if he weighs 160 pounds and pulls with a force equal to this weight, then the reaction of the ropes will ſuſtain his whole weight, and no part of it will preſs upon the ſcale. If he pulls with a force equal to 80 pounds, then the ropes ſuſtain half his weight, and the ſcale is preſſed by the other half. And ſince in all caſes a part of his weight equal to the force, with which he pulls, is taken out of the ſcale and is hung to the rope; the point of ſuſpenſion will be always preſſed by the ſame quantity; by his whole weight in the ſcale, when he does not pull at all; but by the remainder of his weight in the ſcale and by the part which he hangs to the rope, when he does pull; and theſe two together are equal to his whole weight.

It can ſcarce be asked why he may not pull with a force greater than his own weight. Becauſe, I think, it is obvious, that if he weighs only 160 pounds and ſhould pull downwards with a force greater than this, as ſuppoſe with a force equal to 170 pounds; then, the reaction of the ropes upwards being likewiſe equal to 170 pounds, he would raiſe his feet from the ſcale. Becauſe he is drawn upwards by this reaction, or by a force greater than his own weight. And when his feet are raiſed from the ſcale and he hangs at the rope by his hands, it is plain that the rope, and conſequently the point of ſuſpenſion bears juſt his weight and no more.

If he puſhes directly againſt the point of ſuſpenſion, then his reaction downwards upon the ſcale will, by the third law of motion, be equal to his action upwards. And, theſe two equal and contrary forces deſtroying each other, the ſcale will be preſſed more downwards than it is upwards only by his own weight, juſt as it is, when he does not puſh at all.

55. *A man ſtanding in one ſcale and in equilibrio with a weight in the oppoſite ſcale, by puſhing any part of the beam, except his own point of ſuſpenſion, upwards, will make himſelf heavier, and by pulling any part of it downwards will make himſelf lighter.*

Let C Plat. II. fig. 10. be the center of motion, AB the beam and D the ſcale in which the man ſtands. Firſt by puſhing upwards with a cane or by any other means againſt the point H, which is on the ſame ſide of the center with himſelf, he will make his own ſcale preponderate, and the end A of the beam will deſcend. For by puſhing againſt H his action upwards and his reaction downwards are equal, according to the third law of motion. But his action upwards is applyed at the diſtance HC from the center, and his reaction downwards being againſt the ſcale, where he ſtands, is applyed as the point of ſuſpenſion, or at the diſtance AC. And, ſince the moments of theſe two counter-actions are as their reſpective diſtances from the center, by puſhing againſt the point H he produces a greater moment downwards, than upwards; and upon that account the ſcale, where he ſtands, will perponderate, in the ſame manner, as if he had been made heavier. In this caſe the two unequal moments of action and reaction act on the ſame ſide of the center of motion. And therefore the ſcale D preponderates with a force equal to the difference of theſe moments.

But ſecondly, ſuppoſe the man to puſh upwards againſt the point K on the contrary ſide of the center. Then his action upwards is applyed at the diſtance KC from the center, and his reaction downwards at the diſtance AC as before. But both his action upwards will raiſe the end CB, that is, will make the ſcale D, where he ſtands, preponderate, and ſo likewiſe will his reaction downwards. In this caſe the ſcale D will deſcend with a force equal to the ſum of the two moments of his action and reaction.

If the man has a rope faſtened at H and pulls it downwards; then the reaction of the rope upwards takes a part of his weight out of the ſcale D, and applyes it to the beam at the point H, or at the diſtance HC from the center of motion; that is, at a diſtance leſs than AC, by which means, the moment of this part of his weight will be diminiſhed; and the oppoſite ſcale will preponderate in the ſame manner, as if he was made lighter.

Or ſecondly, if his rope was faſtened at K on the contrary ſide of the center, then by pulling at it the reaction of it takes a part of his weight out of the ſcale at D and applyes it at K on the other ſide of the center. Therefore, the moment on his own ſide being diminiſhed and that on the contrary ſide increaſed, the ſcale E will preponderate, and the ſcale D where he ſtands will riſe.

56. *A balance may be a deceitful one, though the beam will hang even; that is, will be in equilibrio either without the ſcales, or when the ſcales are empty.*

Let the center of motion divide the beam unequally, ſuppoſe in the proportion of 10 to 9. Or, let one arm of the balance be 10 inches long and the other 9. Then if the quantity of matter in each arm is as the length of it inverſely, by prop. 49. they will be in equilibrio. And, if the weights of the empty ſcales or diſhes are likewiſe made inverſely as their diſtances from the center, there will be an equilibrium, when they are ſuſpended upon the beam. Thus ſuppoſe one of the ſcales or diſhes to weigh 9 ounces, and let that hang at the arm which is 10 inches long: the moment of this ſcale, when it is empty, will be as $9 \times 10 = 90$. Suppoſe the other ſcale to weigh 10 ounces, and let that hang at the end of the arm which is 9 inches long: the moment of it, when empty, will be as $10 \times 9 = 90$. Therefore, as their moments are equal, they will be in equilibrio.

But a balance made in this manner will be a deceitful one. And if the ſeller puts the ſtandard weight into the ſcale, which hangs nearer to the center, the buyer will be cheated one part in ten of the weight, which he ought to have. Thus, if he ought to have a pound, he would inſtead of it have only $\frac{9}{10}$ of a pound. For, as what he buys is put into the ſcale, which hangs farther from the center; the greater velocity of it will make up for the defect in quantity, and the moment of it will be equal to that of the ſtandard, though the quantity of matter is leſs. A pound ſtandard or $\frac{10}{10}$ of a pound, if put into the ſcale, which is at the diſtance of 9 inches from the center, produces a moment as $10 \times 9 = 90$. And $\frac{9}{10}$ of a pound, if put into the other ſcale at the diſtance of 10 inches from the center, produces a moment equal to the former, ſince it is as $9 \times 10 = 90$.

For the ſake of making this the more intelligible I have ſuppoſed the difference of the two arms to be ſo great in proportion to their whole length that the eye would diſcover the cheat. But the difference may be ſo ſmall in proportion, that it could not be diſcovered any other way than by changing ſcales. And if the balance is a falſe one, this method will certainly deſtroy the equilibrium. For, when the pound ſtandard or $\frac{10}{10}$ of a pound is put into the ſcale, whoſe diſtance from the center is 10 inches, the moment will be as $10 \times 10 = 100$. And when $\frac{9}{10}$ of a pound, that is, the thing to be weighed, are put into the ſcale whoſe diſtance is 9 inches, the moment will be only as $9 \times 9 = 81$.

57. *When two men carry a weight, which hangs between them upon a pole; this pole is a lever of the ſecond kind. And the moment which each man ſuſtains will be inverſely as the diſtance of the weight from him.*

If we ſupport any weight upon a pole, which we hold out at arms length, and no perſon has hold of the other end of the pole; then the nearer the weight is to us, ſo much leſs is the moment, which we ſuſtain. For this is a lever of the third kind. Our own ſhoulder is the center of motion, and to raiſe the weight we muſt turn the lever, which is the pole and our own arm together, round upon this center, and muſt raiſe the farther end of the pole. For this reaſon by bringing the weight nearer to us, we bring it at the ſame time nearer to the center of motion, and conſequently diminiſh its moment. Therefore in this caſe the moment is directly as the diſtance of the weight from us.

But, when another perſon has hold of the other end of the pole, it is changed into a lever of the ſecond kind. For the center of motion is his hand, if he holds the pole in his hand, or his ſhoulder, if he reſts it there. And to raiſe the weight we muſt raiſe the end, which is next to us, by turning it round upon that center. The weight therefore is between the power and the prop. And, ſince the center of motion is in this caſe beyond the weight, the farther the weight is from us, the nearer it is to the center, and therefore the moment of it is the leſs. That is, the moment ſuſtained diminiſhes as the diſtance from us increaſes, and conſequently is as the diſtance inverſely.

From hence it appears, that when two men of unequal ſtrength ſupport a weight upon a pole; each of them will bear a moment in proportion to his ſtrength; if the diſtance of the weight from each of them is inverſely as his ſtrength.

58. *Several mechanical inſtruments are levers of the firſt or of the ſecond kind: but ſcarce any inſtance can be found, where a lever of the third kind is uſed.*

Levers are uſed to overcome reſiſtances as well as to raiſe weights. For a pair of ſciſſors is a double lever of the firſt ſort; the power is applyed at the bow or handle of each lever, the rivet is the common center of motion, and the thing to be cut, or reſiſtance to be overcome, is placed on the other ſide of this center. In a pair of ſciſſors both the levers are moveable, but in a braſiers ſheers one of them is faſtened to a couple of ſtrong ſtandards in a block, and a man by preſſing upon the long handle of the other ſheer or lever, which is moveable upon the center, is able to overcome the reſiſtance of a plate of copper a quarter of an inch thick ſo as to cut it.

A cutting knife uſed by patten-makers to cut their wood is moveable on a joynt or center at one end, whereby it is faſtened to a plank. The wood to be cut is placed on this plank between the joynt and the handle where the power acts. Therefore it is a lever of the ſecond kind applyed to overcome the reſiſtance of the wood. The oar of a boat is a lever of the ſecond kind: for the water by its reſiſtance makes a ſort of prop for one end of the oar, whilſt the power acts at the other end, and the boat or weight to be moved, is between them.

A lever of the third kind is ſeldom uſed but from neceſſity. For the power in this lever is always nearer the center of motion than the weight is: and conſequently in moving the inſtrument the power will always have a leſs velocity than the weight has. This is a mechanical diſadvantage, which makes it neceſſary, when ſuch a lever is uſed, to apply a power that would be more than ſufficient to lift the weight or to overcome the reſiſtance without any lever at all. In raiſing a ladder, the weight of it is lifted at firſt by a lever of the ſecond kind, and afterwards by one of the third kind. The end of the ladder, which reſts upon the ground, is the prop or center of motion; the man, who is to raiſe it, applys the power at the other end; and its center of gravity, or the weight to be raiſed, is between the power and the prop. But as the ladder riſes the man walks under it towards the prop. And when he has got nearer to the prop than the center of gravity is; he acts upon a lever of the third kind, and accordingly finds the difficulty of raiſing the ladder increaſe as he gets nearer to the prop. A pair of ſheep-ſheers are two levers of the third kind. The wool or reſiſtance is applyed at one end; the bow-ſpring at the other end is the common center of motion; and the hand or power is applyed between them.

59. *When a weight inſtead of being ſuſpended by a rope, is faſtened to the lever; the point, upon which the center of gravity acts, may be conſidered, as the point of ſuſpenſion. And, if the center of gravity is above the lever, this point by inclining the lever is made to approach the lower end, but if below it, then it is made to approach the higher end.*

When the lever, Plat. II. fig. 11. with the weight *c* faſtened above it is in an horizontal poſition; the line of direction is *cp*: and *p* is the point, on which the center of gravity acts. When the lever is inclined in the poſition S M, *e* the point, through which the line of direction paſſes, is nearer to the lower end of the lever S than *p* is. And when it is inclined the other way, in the poſition TR, *a* the point, through which the line of direction paſſes, is likewiſe nearer to the lower end R than *p* is. From hence by a bare inſpection of the figure it appears, that when a weight, which has its center of gravity faſtened above a lever of the firſt kind, is below the horizontal level, as the point of action by approaching the lower end of the lever gets farther from the center of motion, the velocity and conſequently the moment of the weight will be increaſed. But when the weight is above the level, as the point of action by approaching the lower end of the lever gets nearer to the center of motion, the velocity and conſequently the moment of the weight will be diminiſhed.

But if the center of gravity *c* is faſtened below the lever, and *p* is the point of action, whilſt the lever Plat. II. fig. 12. is in the horizontal poſition AL; then by inclining it into the poſition KR, *i c* will be the line of direction, and *i* the point of action will be nearer to R, the higher end of the lever, than *p* is. Alſo by inclining it into the contrary ſituation DF, *ce* will become the line of direction, and *e* the point of action will be nearer to the higher end of the lever than *p* is. From hence again it will appear by a bare inſpection of the figure, that when a weight, which has its center of gravity faſtened below a lever of the firſt kind, is below the horizontal level, the point of action, by approaching the higher end of the lever, gets nearer to the center of motion, by which means the moment of the weight is diminiſhed. But when the weight is above the level, the point of action by approaching the higher end of the lever, gets farther from the center of motion and the moment of the weight is increaſed.

Nothing of this ſort will happen, when the weight is ſuſpended by a rope: becauſe the point of ſuſpenſion or point of action is not altered by inclining the lever; ſince, by propoſition 42, as the weight ſwings, the center of gravity will always keep directly under the ſame point. Neither would the point of action be altered by inclining the lever, though the weight was faſtened to it, if the center of gravity was neither above the level nor below it, but in the ſame horizontal plane with it.

The reader will eaſily perceive that what I have proved in a lever of the firſt kind is equally true in both the other kinds of lever. I ſhall apply it in the inſtance of two men carrying a weight between them. When two dray-men carry a barrell on a coul-ſtaff, to which it is ſuſpended by a chain, the point on which the weight acts is not altered by inclining the ſtaff, and therefore in going either up or down hill, there will be no inconvenience ſuffered, nor advantage gained, either by the firſt man or by the laſt. But if they carry the barrell upon two dogs, one of which is faſtened at each end, then the weight does not ſwing, and the center of gravity is below the lever. Therefore the point, on which the weight acts, will by inclining the lever be made to approach the higher end. And in going down hill, the firſt man by having this point removed from him, will, according to propoſition 57, be eaſed of a part of his burden, and the laſt man will have an equal part added to his. When two chair-men are going down hill, if the center of gravity of the weight, which they carry, is above the plane of their poles, the point of action will approach the lower ends of the poles, and the man that walks firſt will have the diſadvantage. If the center of gravity is below that plane, the point of action will approach the higher ends, and the man that walks laſt will have the diſadvantage. But, if the center of gravity is in the plane of the poles, neither of them will ſuffer any diſadvantage by inclining the poles or by going down hill. Becauſe the point of action, notwithſtanding the poles or levers are inclined, will be the ſame, as if they lay horizontally, or as if the men walked upon level ground.

60. *The quantity of motion produced by a power, which acts upon a lever, is always proportional to the moving force.*

We ſay indeed that the weight, which a man is able to lift by the help of a lever is greater than his ſtrength. But by this we only mean that the weight is greater than he could raiſe with his ſtrength directly applyed to it, in ſuch a manner as to make his power and the weight that is to be lifted, move with the ſame velocity. And what enables him to raiſe it with the help of a lever is, that by ſuſpending the weight at the ſhorter arm of the lever and applying his ſtrength at the longer arm, the weight is made to move ſlower than the power. And though the moment of the weight, when it moves with the ſame velocity as the power does, might be greater than his ſtrength; yet the proportion, which the velocity of the weight bears to that of the power, may, by means of the lever, be diminiſhed, till the moving force, which he exerts, ſhall be equal to the moment of the weight.

No motion therefore is produced by the lever itſelf: the only uſe of it is to make the weight move ſlower than the power. So that by means of this mechanical inſtrument we can indeed lift a greater weight than we could without it: but at the ſame time we produce only the ſame quantity of motion. For juſt as much as the moment we can raiſe with a lever exceeds in weight, ſo much it falls ſhort in velocity of the moment we could raiſe without one. What has been here ſaid concerning the lever, I muſt deſire the reader to apply, as he goes along, to all the other mechanical powers.

61. *In a wheel and axle, the velocity of the power is to the velocity of the weight, as the circumference of the wheel is to the circumference of the axle.*

This machine is much uſed and is applyed in very different ſhapes. Let the axle A Plat. II. fig. 13. and the wheel CD be faſtened together, ſo that one cannot turn round but the other muſt turn round with it. Suppoſe a power or a ſmaller weight to be applyed at a rope P, which winds about the circumference of the wheel, and a weight to be ſuſpended at W by another rope, which winds about the circumference of the axle the contrary way. To turn the wheel once round, by pulling at the rope P, as much rope muſt be drawn off as winds once about the wheel, that is, as much as is equal to the circumference of the wheel. Therefore, when it is turned once round, the power, which acts at P, muſt have run through a ſpace equal to the length of rope, that it has drawn off the wheel; or through a ſpace equal to the wheels circumference. But whilſt the wheel is turning once round, the axle A will turn once round along with it, and the rope, by which the weight is ſuſpended, will wind once about the axle. Upon this account the weight W will be raiſed through a ſpace equal to as much rope as will go once round the axle; or through a ſpace equal to the circumference of the axle. Since therefore in the time that the machine is turned once round, the ſpace, which the power deſcribes, is equal to the circumference of the wheel; and the ſpace, which the weight deſcribes, is equal to the circumference of the axle; and ſince, by propoſition 10, the velocities are as the ſpaces deſcribed in the ſame time; it follows, that the velocity of the power is to the velocity of the weight, as the circumference of the wheel to the circumference of the axle.

In all the variety of ſhapes, in which this machine is uſed, there is the ſame proportion between the velocities of the power and weight. Suppoſe that the power does not act by a rope which winds round the wheel, but that it is a mans ſtrength applyed immediately to the handles K, E, F, G, H, I. If the man firſt lays hold of the handle H, and puſhes it down to the place G, his hand will have run through the ſpace HG or the ſixth part of a circle, and the handle I will be brought down to the place of H. He then lays hold of I and puſhes it down to G as be-

before, by which means his hand will have gone through another ſixth part of a circle. And thus by laying hold of every one of theſe ſix handles in its turn, and puſhing it down to G he will turn the wheel once round, and his hand or the power, will have deſcribed $\frac{6}{6}$ or the whole circumference of a circle, whoſe diameter is IF. This circle is to be looked upon as the circumference of the wheel; though the wheel is then an imperfect one, for the handles are only the ſpokes of it. And ſince, whilſt the power is deſcribing this ſpace, the weight has riſen, as before, through a ſpace equal to as much rope as winds once round the axle, it follows that, in this inſtance likewiſe, the velocity of the power is to the velocity of the weight as the circumference of the wheel to the circumference of the axle.

62. *In a wheel and axle, the power and weight are in equilibrio; if the power is to the weight, as the circumference of the axle to the circumference of the wheel.*

For then, by the laſt propoſition, the power is to the weight, as the weights velocity, to the powers velocity. And if there be four quantities, of which the firſt is to the ſecond as the third to the fourth, then the firſt multiplyed into the fourth is equal to the ſecond multiplyed into the third. Eucl. book VI. prop. 16. Therefore the power multiplyed into its own velocity, or the moment of the power, is equal to the weight multiplyed into its own velocity, or to the moment of the weight.

Or otherwiſe, ſince the power is to the weight as the velocity of the weight to the velocity of the power; therefore the power and weight are to each other as their velocities inverted, and conſequently, by propoſition 13, their moments are equal.

We may apply this in the following inſtance. Let the circumference of the wheel be to that of the axle as 12 to 1, and ſuppoſe a weight of 1 pound ſuſpended at P, Plat. II. fig. 13. and another weight of 12 pounds ſuſpended at W, then P or 1 pound, is to W, or 12 pounds, as the circumference of the axle 1, or W's velocity, to the circumference of the wheel 12, or P's velocity. Therefore $1 \times 12 = 12$, which is the quantity of matter in P multiplyed into its velocity, or P's moment, is equal to $12 \times 1 = 12$ which is the quantity of matter in W multiplyed into its own velocity, or to W's moment.

In uſing this machine the weight is not always ſuſpended to the axle by a rope, but is ſometimes faſtened to the axle itſelf. This is the caſe of a bell, the weight of which is moved, in ringing it, by a wheel and axle.

And, in once turning the wheel round, the velocity of the bell is as the circumference deſcribed by its center of gravity.

In a crane wheel on the other hand the power is not applyed to the wheel by means of a rope; nor does it act upon any handles or ſpokes. But as the man or the horſe within the wheel ſteps forwards, the part, upon which he treads, by this means becomes the heavieſt part of the wheel, and deſcends till it is the loweſt. Thus he keeps going on, and by treading upon every part of the wheels circumference in its turn he walks over that whole circumference, and brings the wheel once round.

A capſtan is a cylindrical axle perpendicular to the horizon, which is turned by four levers or bars that croſs it. Theſe are the ſpokes of an imperfect wheel. And the velocity of the power applyed to them is as the circumference, which it deſcribes in one revolution of the capſtan.

63. *The motion of a watch is made uniform by means of the fuſee; though the force of the ſpring is not always the ſame, but is greateſt of all when the watch is juſt wound up, and keeps decreaſing as the watch goes down.*

From propoſition 61 it follows, that the velocity of a weight or of any other force applyed to the axle of a wheel, will increaſe as the circumference of the axle increaſes. Thus Plat. II. fig. 13. a weight, or any other force, acting upon the axle B will have a greater velocity than one which acts upon the axle A. If the force was applyed to the circumference T V or R S as to an axle, its velocity would be farther encreaſed. Now the fuſee of a watch is an axle in the ſhape of a cone. And the ſpring by means of the chain acts upon the ſmalleſt part of this axle, when the watch is juſt wound up. As the watch goes down the chain is applyed to different parts of this cone, and to the thickeſt part of all when the watch is juſt down. Therefore as the elaſticity or abſolute force of the ſpring decreaſes, the velocity, with which it acts to move the watch increaſes by being applyed to a greater part of the axle in proportion: and conſequently, if the fuſee is regulated as it ought to be, the moment produced in the wheels and in the hands of the watch will be always the ſame.

65. *An upper pulley, that only turns round upon its center and does not move with the weight, gives no mechanical advantage to a power.*

If a weight F Plat. II. fig. 14. is ſuſpended at one end of a rope running acroſs the pulley E D, which turns on the center C, but is fixed to the beam A in ſuch a manner as neither to move upwards nor downwards;

and if another weight B is ſuſpended at the other end of the rope : then it is evident that B cannot deſcend through any ſpace B G but F muſt riſe through an equal ſpace, becauſe all the rope, which is drawn to the ſide of B in deſcending, is taken from the other ſide where F hangs; that is, as much as the ſide D B is lengthened, ſo much the ſide E F is ſhortened. Therefore the two weights move with equal velocities, and conſequently, by propoſition 13, will not be in equilibrio, or have equal moments, unleſs their quantities of matter are equal. In the ſame manner, if inſtead of a weight at B a power had been applyed there, we might have proved that the velocity of the power would be equal to that of the weight, and that a power, which could lift the weight F with ſuch a pulley as this, might lift it without any pulley at all.

If a rope is thrown acroſs a pulley faſtened at the top of a room, and a maſs of lead is hung at one end of the rope ; a man, who pulls at the other end, cannot raiſe the maſs, if it weighs more than he does himſelf. For ſuppoſing him to weigh 160 pounds; and the lead to weigh 200 pounds. Then if he pulls with a force leſs than his own weight, it is evident that he cannot raiſe the lead. Becauſe a force leſs than 160 pounds cannot raiſe a weight of 200. It is likewiſe evident that he cannot raiſe it, if he pulls with a force equal to his own weight. And if he pulls with a force greater than his own weight, inſtead of raiſing the lead, he will draw himſelf up from the ground. For, by the third law of motion, the reaction of the rope upwards is equal to his action downwards. Therefore if he pulls downwards with a force equal to 170 pounds weight, which is 10 pounds more than he weighs himſelf, the reaction of the rope upwards is equal to 170 pounds or 10 pounds more than his weight. And, ſince the rope reacts upon him with a force greater than his own weight, he will riſe with the difference.

If the maſs of lead was to weigh juſt as much as the man himſelf weighs, or 160 pounds, he could not raiſe it. Becauſe, if he pulls with a force leſs than his own weight he cannot raiſe a weight equal to his own, or 160 pounds. And if he pulls with a force exactly equal to his own weight, he will hang by his hands at one end of the rope, and the maſs of lead, which hangs at the other end, will be in equilibrio with him, and conſequently will not riſe. But if he pulls with a force greater than his own weight, then, inſtead of raiſing the lead, he will draw himſelf up from the ground as before.

But ſuppoſe inſtead of the maſs of lead that there was a nooſe at that end of the rope, and that the man was to put his foot into this nooſe, whilſt

PLATE II.

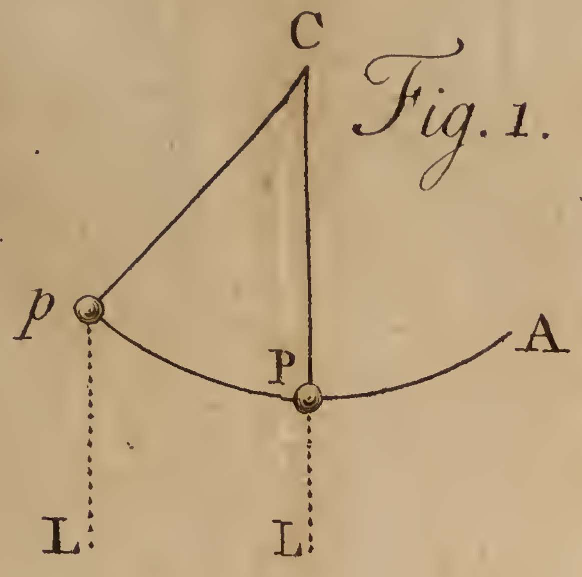


Fig. 1.

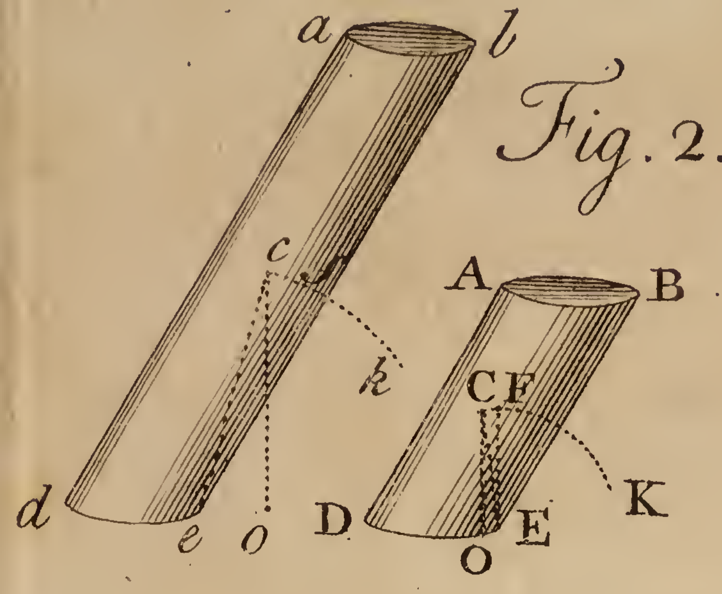
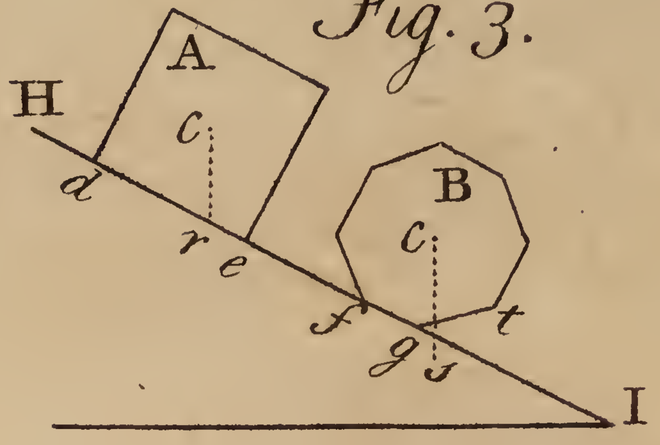


Fig. 2.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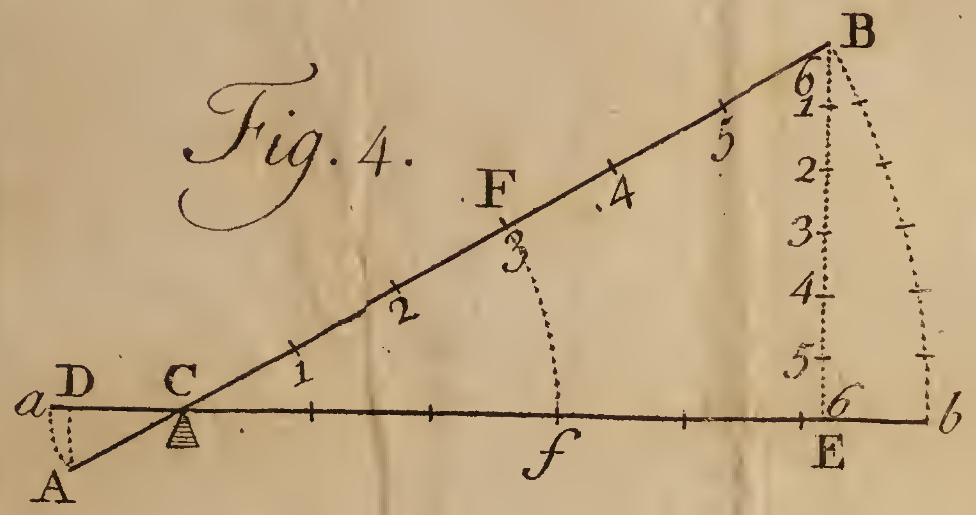
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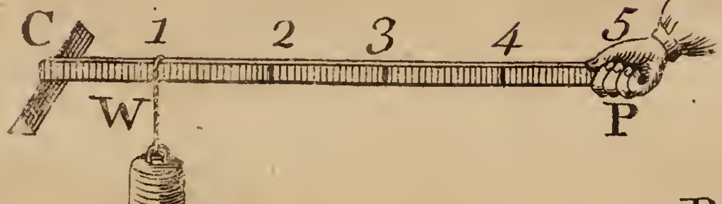
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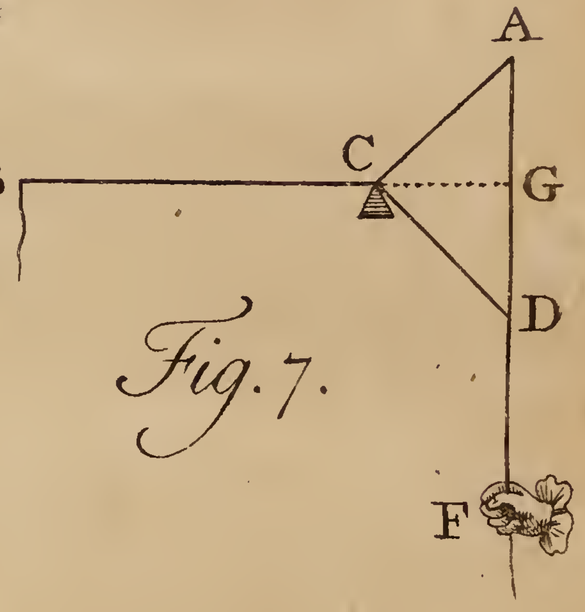
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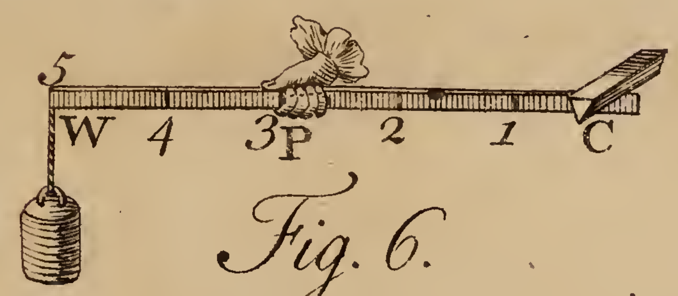
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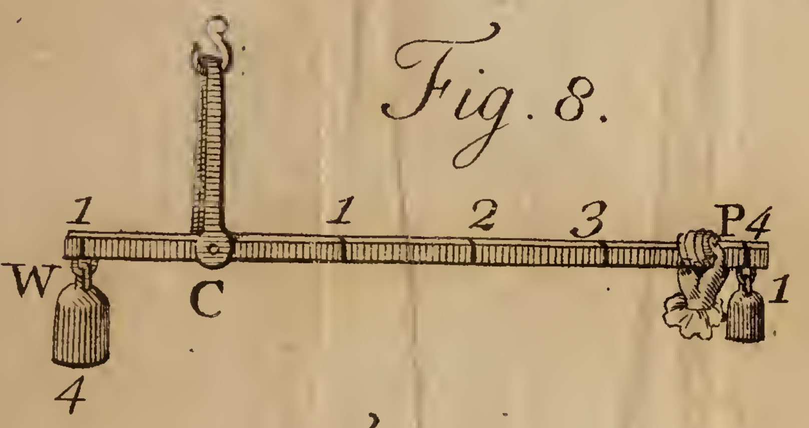
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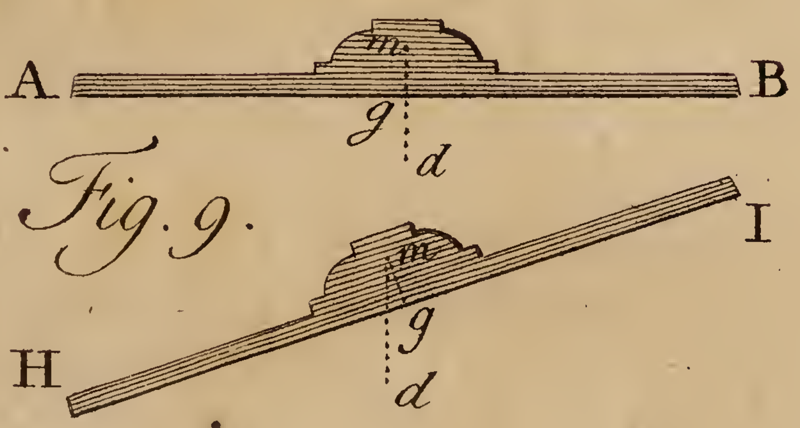
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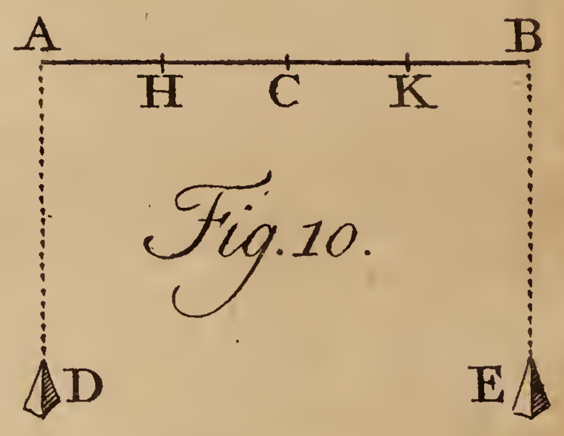


Fig. 10.

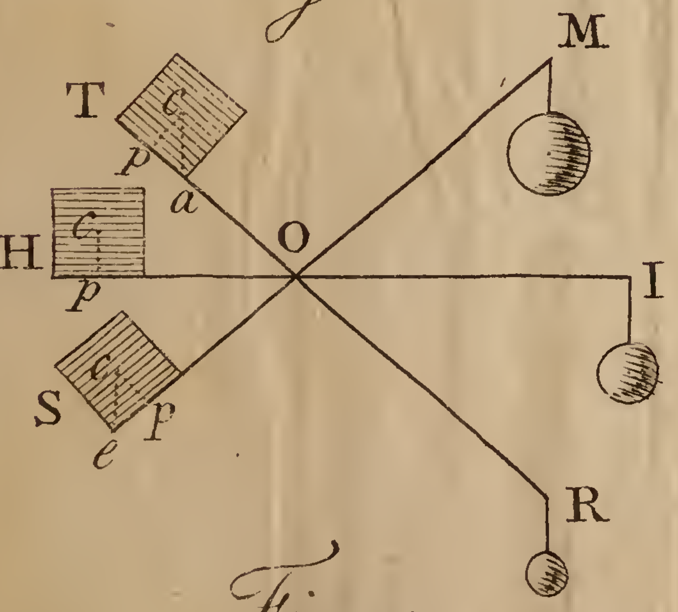


Fig. 11.

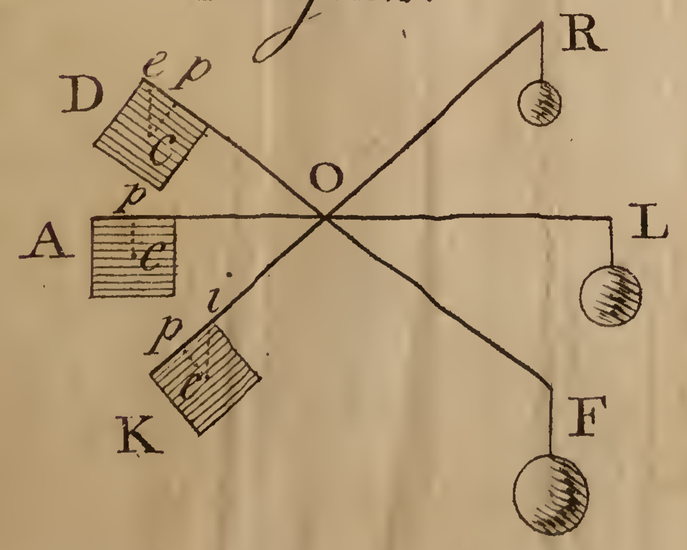


Fig. 12.

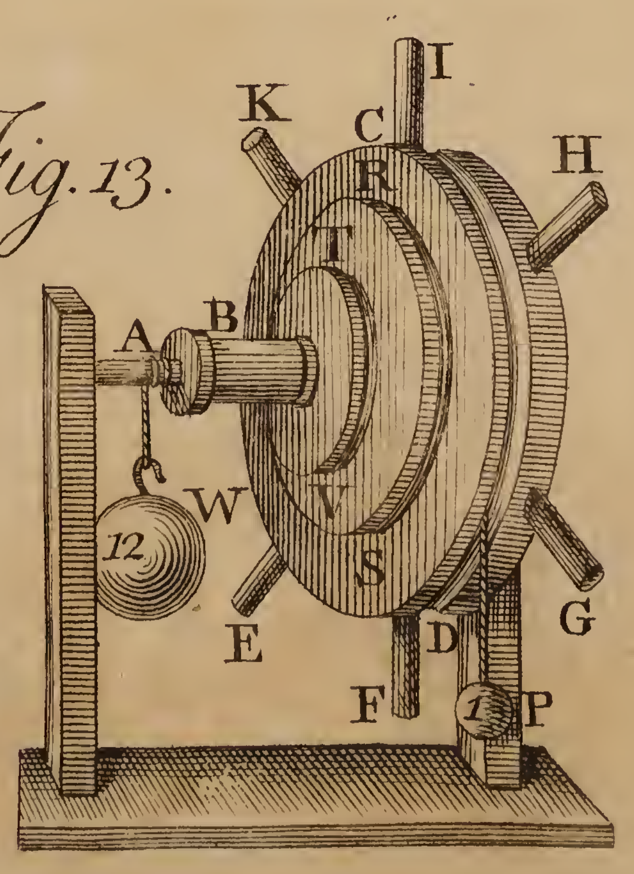


Fig. 13.

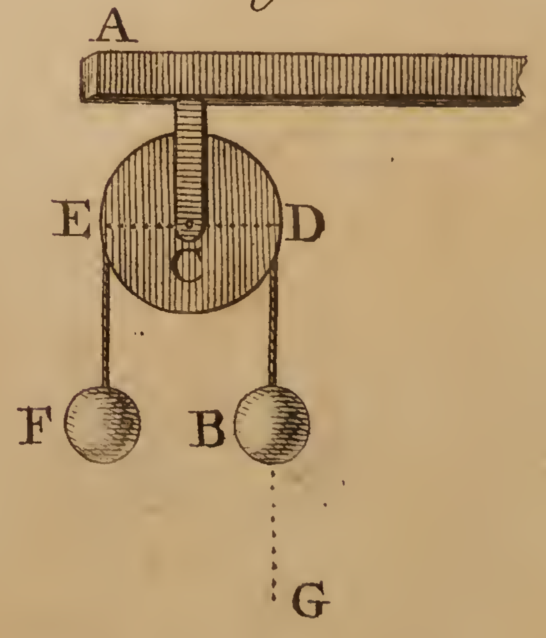


Fig. 14.

Gul. Stephens caelavit

whilſt he pulls at the other end. In this caſe his own weight is hung at that end of the rope, where his foot is; and his own ſtrength is applyed at the other end to raiſe this weight. Let us therefore enquire whether he could in theſe circumſtances pull himſelf up to the top of the room. We have ſeen that he could not raiſe a weight of 160 pounds. But though he weighs 160 pounds yet he will be able to raiſe himſelf. For here it muſt be obſerved that his whole weight cannot hang upon both ends of the rope at the ſame time. But whatever part of his weight he hangs at that end of the rope, where his hand is, by pulling at it; juſt ſo much he takes off from the other end, where his foot is in the nooſe. Thus for inſtance, ſuppoſe him to pull with a force equal to 80 pounds or half his own weight; he then hangs half his weight to that end of the rope where he pulls, and conſequently leaves but 80 pounds or half his weight in the nooſe at the other end. In this caſe his weight at one end, and his ſtrength, with which he pulls at the other, would juſt be equally balanced. But ſuppoſe him to pull with any force greater than half his weight he would raiſe himſelf from the ground. Thus if he pulls with a force equal to 90 pounds, which is 10 pounds more than half his weight; then he hangs 90 pounds of his weight at the end, where his hands are, by pulling at it; and conſequently leaves only 70 pounds in the other end, where his foot is. Therefore his ſtrength, or the weight he hangs at one end by pulling at it, will overbalance the weight, which he leaves at the other end: and upon this account the end, where his foot is, will riſe; and part of the rope will come over the pulley to that ſide, where his hand is. And if he keeps gathering up this rope as faſt as it comes over, he may raiſe himſelf up to the top of the room. From this inſtance the reader will eaſily underſtand what was before affirmed, that this will always be the caſe, if the man pulls with any force which is greater than half his own weight.

65. *When, beſides the upper pullies, there is another lower block of pullies, which will move as the weight moves: the velocity of the weight is to the velocity of the power as* 1 *to the number of ropes, that ſupport the lower block.*

The weight W Plat. III. fig. 1. is faſtened to a lower block with one pulley in it; and the block is ſupported by two ropes *fe*, and *dg*: that is, by two parts of the ſame rope, which has one end faſtened upon the beam A, and running acroſs the lower pulley is divided by it into two parts; and then running acroſs the upper pulley at *d* comes over to the ſide BP, where we ſuppoſe the power to act. We have reckoned only two ropes: for the third BP to which the power is applyed, does not ſupport the lower block.

Now in this inſtance the velocity of the weight is to that of the power as 1 to 2. For to draw the weight through any ſpace, as HW, each of the ropes *fe* and *dg* muſt be ſhortened by a quantity equal to HW; and what rope is drawn from this ſide of the fixed pulley at *d* will run over to the ſide BP, therefore the ſpace, which the power deſcribes, whilſt the weight is aſcending through HW, will be equal to BP or to twice HW. And, ſince the velocities, by propoſition 10, are as the ſpaces deſcribed in the ſame time, the velocity of the weight is to that of the power as 1 to 2, or as 1 to the number of ropes.

Univerſally; whatever is the number of ropes, by which the lower block is ſupported; when the weight is to be raiſed through any ſpace, each rope muſt be ſhortened by a quantity equal to that ſpace. And, ſince the ſum of all the parts taken from each rope runs over to the ſide of the power; it follows that, whilſt the weight is riſing through any given ſpace as 1, the power will deſcribe a ſpace which exceeds it as much as the ſum of all the parts exceeds one part, or as much as the number of ropes exceeds 1. Therefore the ſpace deſcribed by the weight is to the ſpace deſcribed by the power in the ſame time, or the velocity of the weight is to the velocity of the power, as 1 to the number of ropes.

The weight W, Plat. III. fig. 2. which hangs to the lower block, is ſupported by 5 ropes: the rope, to which P the power or the leſſer weight is applyed, is not to be reckoned into the number. Now to raiſe the lower block and the weight along with it 1 foot higher than it is, or one foot nearer to the upper block, theſe 5 ropes muſt be ſhortened 1 foot each. And, ſince the ſum of all theſe parts, or 5 feet of rope, runs over to the ſide where P is applyed; it follows that, whilſt the weight is riſing through one foot, the power deſcribes 5 feet. Therefore the velocity of the weight is to that of the power as the ſpaces deſcribed in an equal time, or is as 1 to 5, or as 1 to the number of ropes.

Theſe two blocks are made in an inconvenient form. For by placing the pullies one below another the blocks are made ſo long, that they will touch each other, before the weight comes near the hook, which the upper block hangs upon. To prevent this, the pullies are uſually placed ſo as to turn upon the ſame axis: and then a block for 3 pullies Plat. III. fig. 3. is no longer than a block for one. The weight W is here ſupported by 6 ropes. And ſince each muſt be ſhortened 1 foot in order to raiſe the weight 1 foot, and the power will in the mean time have 6 feet of rope run over to the ſide, where it is applyed, and conſequently will deſcribe 6 feet, whilſt the weight is deſcribing 1. Therefore the velocity of the weight is to that of the power as 1 to 6 or as 1 to the number of ropes.

66. *In pullies the power and weight will be in equilibrio, if the power is to the weight as* 1 *to the number of ropes that ſupport the lower block.*

For then, by propoſition 65, the power and weight are to each other as their velocities inverſely; therefore, by propoſition 13, their moments are equal.

Let W Plat. III. fig. 3. be 6 hundred weight, and P 1 hundred; then P is to W as 1 to the number of ropes. But P's velocity is to W's velocity as the number of ropes to 1, or as 6 to 1. Therefore P 1 hundred weight multiplyed into its own velocity or $100 \times 6 = 600$, and W 6 hundred weight multiplyed into its own velocity or $600 \times 1 = 600$.

From hence we may ſee that a weight acts with leſs velocity, and conſequently with leſs moment in proportion to the number of ropes that ſupport it. Therefore 1 hundred weight ſuſpended at a ſingle rope, which runs acroſs an upper pulley, will turn a jack as well as 6 hundred weight ſuſpended at a lower block ſupported by 6 ropes. But there is a good reaſon for preferring 6 hundred weight with tackle of ſix, (as it is called) or with a ſet of pullies having ſix ropes, to 1 hundred weight with a ſingle rope. For in the former caſe there is 6 times as much rope as there is in the latter; and conſequently the weight will be 6 times as long in going down; or the jack will go 6 times longer without winding up.

67. *The velocity of two equal reſiſtances, of which one acts perpendicularly upon one ſide of an iſoſceles wedge and the other perpendicularly upon the other ſide, is to the velocity of a power, which acts perpendicularly upon the back of it, as the baſe of the wedge is to the ſum of its ſides.*

Let ABC Plat. III. fig. 4. be an iſoſceles wedge, that is, a wedge, whoſe ſides AC and BC are equal to each other. And let PP be two equal weights, which are to be ſeparated from each other by driving the wedge between them. When the wedge is quite driven in by a power, ſuch as the ſtroke of a hammer, applyed perpendicularly to the back AB; the ſpace deſcribed by the reſiſtance, which theſe two weights make, will be to the ſpace deſcribed by the power in the ſame time, or, by propoſition 10, the velocity of the reſiſtance will be to the velocity of the power, as AB the baſe of the wedge is to AC added to CB the ſum of the ſides.

When the wedge is driven between the weights into the ſituation MON; one weight will have been removed from P to Q on one ſide, and the other from P to Q on the other ſide: ſo that CK perpendicular to one ſide is the ſpace deſcribed by one of them, and CL perpendicular to the other ſide is the ſpace deſcribed by the other of them. But ſince CK is equal to CL, the ſpace deſcribed by the whole reſiſtance is the ſame, whether one weight deſcribes CK, and the other CL, or both together deſcribe CL. Or, ſince one weight is only half the reſiſtance, and the other weight is the other half, CK added to CL is deſcribed by half the reſiſtance, CK by one weight which is only half the reſiſtance made by both of them and CL by the other. But CK is equal to CL. Therefore CK and CL together are equal to twice CL; and conſequently as it is only half the reſiſtance which in driving the wedge in deſcribes a ſpace equal to twice CL; the whole reſiſtance deſcribes but half this ſpace or once CL. In the mean time the power, which is applyed at D, having driven the wedge into the poſition MON, has moved from D to C or has deſcribed the ſpace DC. Therefore, taking DG perpendicular to BC and equal to CL, the ſpace deſcribed by the reſiſtance is to the ſpace deſcribed by the power in the ſame time, and conſequently, by propoſition 10, the velocity of the reſiſtance is to the velocity of the power, as DG to DC. But ſince DC the direction of the power is perpendicular to AB the back of the wedge, it follows that BDC is a rightangled triangle. And, ſince DG the direction in which the reſiſtance moves is perpendicular to BC, the triangle DGB is ſimilar to the triangle BDC, Eucl. b.VI. pr. 8. and DG is to DC as DB is to BC. Therefore the velocity of the reſiſtance is to the velocity of the power either as DG to DC, or which has juſt been proved to be the ſame thing, as DB half the baſe of the wedge to BC one of its ſides. But as DB half the baſe to BC one ſide, ſo is twice DB or AB the whole baſe to twice BC, or (ſince CB is equal to AC) to BC added to AC or the ſum of the ſides. Therefore the velocity of the reſiſtance is to the velocity of the power as the baſe of the wedge to the ſum of the ſides.

68. *If a power acts perpendicularly to the back of an iſoſceles wedge, and a reſiſtance is ſo divided that half of it may act perpendicularly to one ſide of the wedge and half of it perpendicularly to the other; they will be in equilibrio, if the power is to the reſiſtance, as the baſe of the wedge to the ſum of its ſides.*

For, by the laſt propoſition, the velocity of the reſiſtance is to the velocity of the power as the baſe of the wedge to the ſum of its ſides.

Therefore, if the power is to the reſiſtance as the baſe to the ſum of the ſides, the power and reſiſtance are to each other as their velocities inverſely; and, by propoſition 13, their moments are equal.

This propoſition may be proved independently of the foregoing one. Let ABC Plat. III. fig. 5. be an iſoſceles wedge, that is, a wedge, whoſe ſides AC and BC are equal to each other. A power applyed perpendicularly to the back of it acts in the direction GC. Let ED half the reſiſtance act in the direction EG perpendicular to the ſide AC; and *ed* the other half reſiſtance act in the direction *e*G perpendicular to the ſide BC. Now ſince EG and *e*G are oblique to GC, that is, ſince the reſiſtance acts obliquely to the power, it follows that the whole reſiſtance does not oppoſe itſelf to the power: for only that part of the reſiſtance, which is directly contrary to the powers direction, or to GC, oppoſes itſelf to the power, and prevents the motion of the wedge in that direction. Therefore the power will be in equilibrio with the reſiſtance, though it is not equal to the whole reſiſtance; provided it be equal to that part of it, which acts in a direction oppoſite to GC. To find out what part of the half reſiſtance ED is oppoſed to the power and prevents the wedge from moving in the direction GC, reſolve the oblique force ED into two others, of which EF is parallel to AB the back of the wedge and FD perpendicular to it. This is done, according to propoſition 20, by conſidering ED as the diagonal of a parallelogram, of which EF and FD are the two ſides. In the ſame manner the other half reſiſtance *ed* may be reſolved into *ef* parallel to AB, and *fd* perpendicular to it. Therefore the part of the reſiſtance, which by acting perpendicularly to the back of the wedge oppoſes itſelf to the power, is to the whole reſiſtance as FD + *fd* to ED + *ed*; or, ſince FD is equal to *fd*, and ED equal to *ed*, as twice FD to twice ED. But ſince the power will be in equilibrio with the reſiſtance, when it is equal to that part of the reſiſtance, which is oppoſed to it, there will be an equilibrium, when the power is to the whole reſiſtance as twice FD to twice ED, that is, as I ſhall prove preſently, as the baſe of the wedge to the ſum of its ſides. In the mean time it will be neceſſary to ſay ſomething concerning the two other parts of the half reſiſtance on each ſide EF and *ef*. Now theſe two parts being equal and in contrary directions deſtroy each other, and ſo neither oppoſe the power nor hinder the motion of the wedge in the direction GC.

But I am to prove that twice FD is to twice ED as the baſe AB to the two ſides AC and CB taken together. EFD and GDA are ſimilar triangles. For the angle EFD is a right one, becauſe FD is perpendi-

cular

cular to AB and EF is parallel to it. Eucl. b. I. pr. 29. And GDA is likewiſe a right angle, and therefore equal to EFD; becauſe EDG is perpendicular to AC by the ſuppoſition. AGE is likewiſe equal to GEF, becauſe AB and EF are parallel, and theſe two are alternate angles. Euc. b. I. pr. 29. From whence it follows, that, ſince the three angles in every triangle are only equal to two right ones, and two angles in the triangle EFD are reſpectively equal to two angles in the triangle ADG, the third angle EDF in one is equal to the third angle GAD in the other. Eucl. b. I. pr. 32. corol. 2. and the two triangles are ſimilar. But AGC is a rightangled triangle, and the line GD drawn from the right angle is perpendicular to AC: therefore the triangle ADG and conſequently the triangle DFE is ſimilar to the triangle AGC. Eucl. b.VI. pr. 8. And FD is to ED as AG to AC, or twice FD to twice ED as twice AG or AB the baſe of the wedge to twice AC or AC and BC together, which is the ſum of the ſides. Eucl. b.VI. pr. 4.

From this propoſition, when it is thus demonſtrated, the foregoing propoſition may be deduced as a corollary. For ſince the power and reſiſtance are in equilibrio, when the power is to the reſiſtance as the baſe of the wedge to the ſum of its ſides. And ſince, by propoſition 13, when they are in equilibrio, they are to each other as their velocities inverſely. It follows that the velocity of the reſiſtance is to the velocity of the power, as the power is to the reſiſtance, or as the baſe of the wedge to the ſum of its ſides.

69. *If in cleaving wood the power is to the reſiſtance of the wood, as the baſe of the wedge to the ſum of its ſides, they will be in equilibrio; and the power, if it be ever ſo little increaſed, will drive in the wedge.*

I find that one writer upon mechanics makes the proportion of the power to the reſiſtance of the wood to be leſs than this, and maintains that there will be an equilibrium, when the power is to the reſiſtance, as a quarter of the baſe to one ſide, or as half the baſe to the ſum of the ſides. He ſupports his opinion in this manner. If the wedge ACB Plat. III. fig. 6, is driven into the wood VCF, which is ſplit no farther than the point of the wedge; it is evident that the velocity of the wood at F and V muſt be greater than the velocity of it any where elſe between F and C or V and C: ſince the two pieces CF and CV turn round upon C as upon a joynt or center of motion. And indeed the velocity of each piece, or the velocity of the whole reſiſtance will in this caſe be only half what it would have been, if the lower parts had moved with the ſame velocity as the upper ones. For let us reckon from C to F, and imagine four points taken in CF at equal diſtances from each other C, M, R, and D; then let the velocity of C=0, the velocity of M=1, that of R=2, and that of D=3. Therefore the ſum of all the velocities, or the velocity of the whole piece =0+1+2+3=6. But if all theſe points had moved with the ſame velocity as D, that is, if each point C, M, R, D, had had 3 degrees of velocity, then the ſum of all theſe velocities or the velocity of the whole piece would have been =3+3+3+3=12. And univerſally, the velocity of a reſiſtance turning round upon a joynt as C may be repreſented by the ſum of a number of terms in arithmetical progreſſion beginning from 0 and ending with the greateſt velocity; whereas the velocity of a reſiſtance, which does not turn round in this manner, but has the ſame velocity at the bottom as at the top, may be repreſented by the ſum of an equal number of terms, each of which is equal to the laſt in the other progreſſion. But the former ſum will only be half the latter; as in the inſtance juſt produced, 6 is the half of 12. Therefore the velocity and conſequently the moment of the whole reſiſtance, when each of the pieces turns round upon a joynt, is but half what it would have been if the lower parts had moved with the ſame degree of velocity as the upper parts. Now were all the parts of the wood to have the ſame degree of velocity, the power, when there is an equilibrium, would be to the reſiſtance as AB to AC+BC that is as the baſe to the ſum of the ſides, or as GB to BC half the baſe to one ſide, by the laſt propoſition. Therefore when the moment of the reſiſtance is but the half of this, or when the reſiſtance turns round on a joynt, then half the power will be ſufficient to produce an equilibrium: that is, a power which is to the reſiſtance, as half the baſe of the wedge to the ſum of the ſides, or as a quarter of the baſe to one ſide.

In anſwer to this reaſoning, I ſhall firſt eſtabliſh the truth of the propoſition, and then ſhew what it was which miſled this writer.

Firſt, I am to ſhew that, if there is an equilibrium on a wedge, the power muſt be to the reſiſtance, as the baſe to the ſum of the ſides, or as half the baſe to one ſide, not only when the lower parts of the pieces FC and VC move with the ſame velocity, as the upper parts, but likewiſe, when they turn round on a joynt C at the bottom. The parts of the wood I, and Q, are not ſeparated by the whole wedge ACB, but by a ſmall wedge ICQ which is a part of the whole one. And at M the velocity of the reſiſtance, which the wood makes there, is to the velocity of the power, by which this ſmaller wedge is driven in, as NM to

N C or as N Q to QC, that is, ſince G C B and N C Q are ſimilar triangles, as GB to BC, or as half the baſe of the whole wedge to one ſide of it. Therefore to make an equilibrium at M, the power muſt be to the reſiſtance as N Q to Q C or as half the baſe to one ſide. So likewiſe the parts of the wood E and O are not ſeparated by the whole wedge ACB, but by a part of it ECO, which may be conſidered as a ſmall wedge by itſelf. And at R the velocity of the reſiſtance made there by the wood is to the velocity of the power conſidered as driving in the wedge ECO in the proportion of HR to HC or of HO to OC, that is, becauſe HCO and GCB are ſimilar triangles, as GB half the baſe of the whole wedge to BC one of its ſides. And laſtly the velocity of the reſiſtance at D is to the velocity of the power, which ſeparates that reſiſtance by driving in the whole wedge as GD to GC, or as GB to BC. But ſince the velocity of each part of the reſiſtance is to the velocity of the power by which that part is ſeparated as half the baſe of the wedge to one of its ſides; it follows, Eucl. b.V. pr. 12. that the velocity of the whole reſiſtance is to the velocity of the whole power in the ſame proportion, that is, as GB to BC half the baſe to one ſide, or as AB the baſe to AC and BC together, which is the ſum of the ſides. But there will be no equilibrium unleſs the power and the reſiſtance are to each other as their velocities inverſely, or unleſs the power is to the reſiſtance as half the baſe to one ſide or as the whole baſe to the ſum of the ſides.

The writers miſtake was owing to this. He had proved that the reſiſtance made by the two pieces FC and VC if they turn round upon a joynt at C has but half the moment of the ſame reſiſtance, ſuppoſing the two pieces were ſeparated as far at the bottom as they are at the top, and therefore he concludes that half the power will be ſufficient to make an equilibrium. Whereas the ſame power would be requiſite in both caſes. Becauſe the ſame power has but half the moment when the two pieces turn round upon a joynt at C that it would have, if the lower parts were to move as far aſunder as the upper parts do. For the velocity of the power, which ſeparates the wood at C=0, becauſe the point of the wedge is only contiguous to the wood at that place and does not enter it to any depth. The ſpace deſcribed by the power to drive in the wedge I C Q, which ſeparates the wood at I and Q is N C call this 1 and then the velocity of the power in that place = 1. The ſpace deſcribed by the power to drive in the wedge E C O, which ſeparates the wood at E and O is H C this is double N C, therefore the velocity of the power in that place = 2. In like manner, the ſpace deſcribed by the power, which drives in the whole wedge ACB and ſeparates the wood at the top, is GC or triple NC, therefore the velocity of the power at the top $= 3$. From whence it follows that the ſum of all the velocities or the whole velocity of the power reckoning from the bottom to the top $=0+1+2+3=6$. But if the lower parts of the wood had been ſeparated as far as the upper parts, then the effect of the power upon the wood at C would have been as great as if the whole wedge had been thruſt quite down from VF to C, that is the power would have acted in every part of the wood with the ſame velocity that it acts with to drive in the whole wedge, or the ſame that it acts with upon V and F or with the velocity $=3$. Therefore the ſum of all the velocities at the four places C, N, H, and G would have been $3+3+3+3=12$. And univerſally the velocity of a power, when it acts by means of a wedge on a reſiſtance, which turns round upon a joynt as C, may be repreſented by the ſum of a number of terms in arithmetical progreſſion, beginning from 0 and ending with the velocity with which the power acts to drive in the whole wedge: whereas the velocity of a power, if the two reſiſting bodies ſeparate as far aſunder at the bottom as at the top, may be repreſented by the ſum of an equal number of terms, each of which is equal to the laſt in the other progreſſion. But the former ſum will only be half the latter, as in the inſtance juſt produced 6 is the half of 12. Therefore the ſame power has but half the velocity and conſequently but half the moment, when the two reſiſting bodies turn round upon a joynt, that it would have, if the lower parts of them were to ſeparate as far aſunder as the upper parts do.

Now it is granted that to make an equilibrium, a power muſt be to a reſiſtance as half the baſe to one ſide, if the lower parts of the wood were to ſeparate as far as the upper ones. And though the moment of the reſiſtance is but half, when inſtead of this, they turn round as upon a joynt at C: yet the ſame power will be requiſite to make an equilibrium in both caſes. Becauſe, as in the latter caſe, the moment of the reſiſtance is but half what it is in the former; ſo the moment of the ſame power, when the parts of the wood turn round as upon a joynt, will be but half the moment it would have, if they ſeparated as far aſunder at the bottom as they do at the top. Therefore in cleaving wood with a wedge, there is an equilibrium, when the power is to the reſiſtance as half the baſe to one ſide of the wedge, or as the whole baſe to the ſum of the ſides.

70. *In a ſcrew the velocity of the weight or reſiſtance is to the velocity of the power, as the diſtance between two contiguous ſpiral rounds of the thread to the circumference, which the power deſcribes in one turn of the ſcrew.*

An inſide ſcrew is a cylinder of iron, braſs, or wood, or any other ſubſtance, with a ſpiral thread winding round it; which thread is made out of the ſame piece with the cylinder: G is the top of the cylinder Plat. III. fig. 7. AB*b* or C*d*D are parts of the ſpiral thread which winds round it. The outſide ſcrew, is a hollow cylinder in the plank or plate of metal K*k*, and in this hollow cylinder is another ſpiral thread like that of the inſide ſcrew; this is called the nut, and the thread of the inſide ſcrew turns in the cavities of the nut or outſide one. The power is commonly applyed to a bar or handle GF. If there is any weight to be raiſed, ſuch as the beam of a houſe which has ſunk, the piece IH with the whole machine being ſet under it, the ſcrews GG are turned round by the bars GF: the ends of the ſcrews bear againſt the plank ML; and as they turn round the plank K*k*, in which are the inſide ſcrews, riſes and the piece HI riſes with it, by which means the weight at I is raiſed. Now in one revolution of the ſcrew, the plank K*k*, and the piece HI, and conſequently the weight at I riſes through a ſpace equal to D*d*, the diſtance between two contiguous rounds of the ſpiral thread: for if the plank hung upon the lower round D at firſt, by turning the ſcrew once round, it will be made to hang on the next round *d*. In the mean time the power has deſcribed the circumference of a circle, whoſe ſemidiameter is GF. But the velocities are as the ſpaces deſcribed in the ſame time, by propoſition 10; therefore the velocity of the weight is to the velocity of the power as the diſtance between two contiguous rounds of the thread to the circumference, which the power deſcribes in one revolution of the ſcrew.

71. *A power and a weight or reſiſtance are in equilibrio by means of a ſcrew, if the power is to the weight or reſiſtance, as the diſtance between two contiguous rounds of the thread to the circumference, which the power deſcribes in one revolution of the ſcrew.*

For then, by the laſt propoſition, the power and weight are to each other as the velocity of the weight to the velocity of the power, that is, they are to each other as their velocities inverſely. Therefore, by propoſition 13, their moments will be equal.

72. *In compound machines there will be an equilibrium; if the power is to the weight, as the velocity of the weight is to that of the power.*

ABD Plat. III. fig. 8. is a compound lever: for it conſiſts of three levers. The velocity of the weight is to that of the power as 1 to 5, in the firſt of them, or A: as 1 to 4, in the ſecond of them, or B: and as 1 to 6, in the third of them, or D. By propoſition 49. Therefore, when all three are uſed together, the velocity of the weight is to that of the power as 1 to 5 and 1 to 4 and 1 to 6: that is, as $1\times1\times1=1$ to $5\times4\times6=120$. and conſequently if the power is to the weight as 1 to 120, by propoſition 14, their moments will be equal.

AB and FG are two wheels, CI and KH are their axles. Plat. III. fig. 9. The axle of the upper wheel has a pinion on it, the teeth of which, by taking the teeth in the circumference of the lower wheel, will turn it round. Let the circumference of the wheel AB be to the circumference of KH the axle of the other wheel FG as 8 to 1: and let the pinion have 8 teeth and the wheel FG have 40. Then the circumference AB, on which the power acts, being to the circumference of the axle KH, where the weight is ſuſpended, as 8 to 1; the velocity of the weight is to that of the power as 1 to 8, by propoſition 62. But ſince there are only 8 teeth in the pinion at C and 40 in the circumference FG, it follows, that whilſt the wheel AB and the pinion are turning round five times, the wheel FG will turn round but once. Therefore again the velocity of the weight ſuſpended at the axle KH will be to the velocity of the power acting at the wheel AB as 1 to 5. So that the velocity of the weight is to the velocity of the power, as 1 to 8 and as 1 to 5, or as $1\times1=1$ to $8\times5=40$. And if the power is to the weight as 1 to 40, they will be as their velocities inverſely, and conſequently, by propoſition 14, their moments will be equal.

In a machine compounded of four pullies with a ſeparate ſtring for each of them, Plat. III. fig. 10. a power M equal to 1 pound will ſupport a weight P equal to 16. For in each pulley uſed alone there would be an equilibrium, if the power was to the weight as 1 to 2. Therefore when all four are uſed together, the power muſt be to the weight in a ratio compounded of four times the ratio of 1 to 2, or as $1\times1\times1\times1=1$ to $2\times2\times2\times2=16$.

CHAP. VII.

Concerning the mutual action of bodies upon each other.

73. *All attraction is mutual.*

IF the earth attracts a ſtone, then, by the third law of motion, propoſition 15, equal quantities of motion will be produced in each of them; that is, they will approach each other with equal moments, or the ſtone will attract the earth juſt as much. For ſuppoſe the attraction on the ſide of the earth to be the greater, and conſequently that the quantity of motion produced by this attraction is greater in the ſtone than in the earth: then if there was any obſtacle, ſuch as a plate of iron, placed between the ſtone and the earth to prevent them from coming together, this plate muſt be more preſſed by the ſtone one way then it is by the earth the other way; and conſequently as the plate is acted upon by two unequal forces; the ſtone, the plate, and the earth would all begin to move in the direction in which the ſtone was moving, by propoſition 19, and this motion ſo produced would always continue; and would be conſtantly accelerated ſo long as the moment of the ſtone was greater than the moment of the earth. But this is abſurd and contrary to the firſt law of motion, propoſition 11: for it cannot be true unleſs a maſs of matter, ſuch as the ſtone, the plate, and the earth, can move itſelf, and when in motion accelerate itſelf. Therefore the plate could not be more preſſed one way by the ſtone than it is the other way by the earth: but the quantities of motion, and conſequently the attractions on each ſide, muſt be mutual and equal. Indeed the quantity of matter in the ſtone is ſo ſmall in reſpect of the earth, that, ſince, by propoſition 13, to make the moments equal, the velocity of the earth muſt be ſmall in proportion, we may conſider the whole velocity as belonging to the ſtone without any ſenſible error.

74. *The common center of gravity of two or more bodies is not made to change its ſtate of reſting or of moving uniformly in a right line by the mutual action of the bodies upon each other.*

The common center of gravity of two boats, is a point which divides the diſtance between them in the inverſe proportion of the quantity of matter. That is; the common center of gravity is a point taken between two unequal boats, as much nearer to the great one than it is to the little one as the quantity of matter in the former is greater than the quantity of matter in the latter, by propoſition 50. Now if a man in one boat pulls a rope, which is faſtened to the other, they will approach with velocities that are inverſely as their quantities of matter, by propoſitions 47, 15; that is, with velocities that are to each other as the reſpective diſtances of the boats from the common center of gravity. Since therefore the velocities produced on each ſide of this center by this mutual action are inverſely as the quantities of matter, it follows, by propoſition 14, that the quantities of motion on each ſide of it will be equal; that is, the two boats will approach it with equal moments. Or, if the boats were very near to each other, and the man in one of them was to puſh with a pole, or by any other means, againſt the other; then likewiſe by his action againſt the boat, which he puſhes, and the reaction of that againſt the boat in which he ſtands, the boats will be ſeparated from each other with velocities that are inverſely as their quantities of matter, by propoſition 16; that is, with velocities, that are to each other as the reſpective diſtances of the boats from the common center of gravity. Since therefore the velocities produced on each ſide of this center in this caſe of mutual action are likewiſe inverſely as the quantities of matter, it follows as before, that the quantities of motion on each ſide of it are equal; that is, the two boats will depart from it with equal moments. In all inſtances of impulſe, and colliſion, of attraction, and repulſion, or of mutual action of any ſort, the bodies which act upon each other will for the ſame reaſons either mutually approach to the common center of gravity, or mutually depart from it with equal moments: but whether they mutually approach to it or mutually depart from it, theſe equal moments on each ſide of the center are produced in contrary directions: and as a ſingle body or the center of gravity of a ſingle body would neither be accelerated nor retarded, nor be any ways made to change its preſent ſtate of motion or reſt, by equal moving forces acting upon it, or by equal changes of motion produced in contrary directions, by propoſition 19; ſo neither can theſe equal changes of motion produced in contrary directions on each ſide of the common center of gravity of two bodies accelerate or retard that center or any ways change its ſtate of motion or reſt,

Suppoſe there were four bodies, which mutually acted upon each other: then, if A and B were two of them, by what has been ſaid already, the common center of gravity would not change its ſtate of reſting or of moving uniformly in a right line on account of the mutual action of theſe two upon each other. And for the ſame reaſon, if C and D were the other two, their common center of gravity would not be made to change its ſtate by their mutual action. And laſtly, ſince the diſtance between theſe two centers is divided by the common center of gravity of all four in the inverſe proportion of the quantities of matter on each ſide of it, and ſince theſe two centers do not change their ſtate of reſting or of moving uniformly, therefore that center will not change its ſtate, upon account of the mutual action either of A and B or of C and D upon each other. For the ſame reaſon it would not change its ſtate upon account of the mutual action either of A and C or of B and D upon each other. And ſince the mutual action of all the four bodies may be thus reduced to the action of any two and two of them upon each other, as of A and B of C and D; or of A and C of B and D; or of C and B of A and D, &c; and ſince the common center of gravity of all four will not be made to change its ſtate by the mutual action of any two of theſe bodies upon each other: therefore neither will it change its ſtate by the mutual action of all four upon each other. The ſame ſort of proof may be applyed, whatever is the number of bodies, that act upon each other.

75. *The quantity of motion in the ſame direction is not altered by the mutual action of bodies upon each other.*

If two bodies move in the ſame direction, then the quantity of motion in the ſame direction is the ſum of their moments: but, if they move in contrary directions, then it is the difference of their moments. The former of theſe aſſertions is eaſily underſtood, and does not need any farther explanation. To explane the latter of them, it is neceſſary to obſerve that if motion in one direction is affirmative, motion in the contrary direction muſt be negative. Theſe ideas of affirmative and negative may perhaps be made more intelligible by conſidering them in the caſe of two lines drawn from the ſame point A, Plat. III. fig. 11. one in the direction A B the other in the contrary direction AC. Now I ſay if we look upon AB as an affirmative quantity, then AC will be a negative one, or the line AC in reſpect of a line drawn in the direction AB is leſs than nothing. For imagine the point B or extremity of the line or ſtring AB to be in motion, and that it runs in a ſtrait courſe from B to A. Imagine farther that, in this motion of the point B towards A, the line or ſtring BA runs up, that is, contracts itſelf, ſhortens, or decreaſes; then at the inſtant B gets to A and coincides with it, this ſtring will have no length or will have decreaſed to nothing. Since therefore a motion of the point B in the direction BA diminiſhes the line or ſhortens the ſtring BA; if we imagine it to move farther than A in the ſame direction, this farther motion of B will produce the ſame effect in the line B A that it did before, or will ſtill diminiſh it. But when B coincided with A, AB $=$ o: therefore, whilſt it moves farther than A towards C, AB being ſtill farther diminiſhed will become leſs than nothing. But the point B moving from A towards C will begin to draw the ſtring out again the other way, that is, will deſcribe or by its motion will generate the line AC; therefore though AC conſidered in itſelf is as much a poſitive quantity as AB is, yet conſidered in reſpect of AB it is leſs than nothing; that is, if a line AB drawn in one direction is looked upon as affirmative, then a line AC drawn in the oppoſite direction muſt be looked upon as negative.

From this inſtance of affirmative and negative lines the reader I hope will be able to underſtand, that, though a motion in the direction AC is in itſelf as much an affirmative motion as any can be, yet in reſpect of a motion in the direction AB it may be looked upon as leſs than nothing or negative.

I do not know whether the following caſe of motion, changing from affirmative to negative, will make this more intelligible. Suppoſe a body was moving in the direction AB, and a perſon was to move his hand againſt it in the contrary direction or BA. Then, if the impulſe of his hand was leſs than the moment of the body, he would deſtroy a part of the motion; and the body would continue to move in the direction AB with the remainder. If he was to increaſe the impulſe of his hand, then he would deſtroy a greater part of the bodys motion. And, if he made this impulſe equal to the moment, then he would deſtroy the whole motion or would ſtop the body. Suppoſe therefore he was to increaſe the impulſe of his hand in the direction BA ſtill farther, what effect muſt it produce in the bodys motion? The impulſe was ſufficient to ſtop the body before or to make its motion $=$ o: therefore, when it is farther increaſed, it will more than ſtop the body, or will make its motion leſs than nothing; that is, will change it from affirmative to negative. But if a body is moving in the direction AB, and any perſon impells it with his hand in the contrary direction or BA with a force greater than the bodys moment, he will turn it back again and make it move in the direction CA. Therefore, if a motion in the direction AB is affirmative, then a motion in the contrary direction, or AC, muſt be conſidered as negative. For the ſame reaſon, if a motion from B towards A is affirmative, then a motion in the contrary direction or from C towards A muſt be conſidered as negative. The reader may ſee this matter farther explaned in propoſition 49 of Optics.

From hence we may be able to underſtand why the quantity of motion in two bodies, when they move in contrary directions, is eſtimated by the difference of their moments. For if one body as X moves in the direction B A, and another as Z in the direction C A: then if we look upon the moment of X as affirmative, the motion of Z will be negative. Therefore the motion of both together muſt be the affirmative moment of X and the negative moment of Z together; that is, the moment of X with the moment of Z ſubſtracted from it, or the difference of their moments; for this is the method of joyning an affirmative and negative quantity together. Let the moment of $X = 20$, and the moment of $Z = 8$. Then, ſince the moment of Z is negative; the moment of both together is equal to the affirmative moment 20 with the negative one 8 added to it; that is, it is equal to $20 - 8 = 12$ or to the difference of their moments in contrary directions.

The reader by this time underſtands, that to prove the quantity of motion in the ſame direction not to be altered by the mutual action of two or more bodies upon each other, we need only ſhew, that the ſum of their moments in the ſame direction or the difference of their moments in contrary directions is not altered by this mutual action. When bodies impell or ſtrike upon each other, then, by the third law of motion, action and reaction being equal and in contrary direction, whatever change of motion is produced on one ſide, an equal change, but in a contrary direction, will be produced on the other ſide. Thus, if two bodies X and Z are moving the ſame way, and X overtakes Z, then X is ſaid to impell Z, and whatever change it makes in the moment of Z by the action of this impulſe, it will by the reaction of Z, have an equal change produced in its own moment in the contrary direction: that is, whatever moment X communicates to Z juſt ſo much it will loſe itſelf. From whence it follows, that the moments of X and Z taken together will be the ſame after the impulſe that they were before; ſince all the moment which X loſes is acquired by Z. But if X and Z are moving in contrary directions and ſtrike upon each other, then likewiſe equal changes will be made in the moments of both, that is, equal moments will be deſtroyed in both. But, if equal moments are taken from both, the difference of their moments is not altered by the ſtroke.

76. *If two perfectly hard bodies, which are void of elaſticity, are moving in the ſame direction; and one impells the other: or if they are moving in contrary directions, and ſtrike upon each other: the common velocity with which they will both move on together, after the impulſe or congreſs, is as the ſum of their moments in the ſame direction, or as the difference of their moments in contrary directions, divided by the ſum of their quantities of matter.*

When two bodies void of elaſticity, as 2 balls of clay, are moving in the ſame direction, and that which moves the faſter of the two overtakes the other; they will not fly off from each other, after the impulſe, but will continue moving in the ſame direction as one maſs, with a velocity, that will be the ſame in both of them. This, which is called the common velocity, may be thus found. The moment of each body before the impulſe being ſuppoſed known, add theſe moments together, and, by the laſt propoſition, this ſum is not altered by their mutual action upon each other. Therefore the moment after the impulſe, that is, the moment of the two bodies moving together as one maſs is known, for it is the ſum of their moments before it. The quantity of matter likewiſe is known, for it is the ſum of the quantities in both bodies. And ſince, by propoſition 13, the moment is a product ariſing from the quantity of matter multiplyed into the velocity, it follows, that if the moment is divided by the quantity of matter, the quotient will be the velocity.

Suppoſe a body, which we will call A, weighs 2 ounces, and another which we will call B, weighs 4 ounces. Let A be moving in any direction with 3 degrees of velocity, and let B move after it in the ſame direction with 6 degrees. If neither of the bodies are elaſtic, then, after B has overtaken A, they will move on together; and each of them will have the ſame velocity. To find out what this velocity will be, we muſt obſerve, that before the impulſe A containing a quantity of matter $=2$ was moving with 3 degrees of velocity and conſequently with a moment $=2\times3=6$. For the ſame reaſon B's moment before the impulſe was $4\times6=24$. Therefore the ſum of their moments in the ſame direction $=6+24=30$. This moment, by propoſition 75, continues the ſame or 30 after the impulſe, when both bodies A and B move on together, that is, when the quantity of matter $=A+B=2+4=6$. But if the moment 30 is divided by the quantity of matter 6, the quotient, which is the common velocity, or velocity of both bodies together, will be 5.

When two bodies void of elaſticity are moving in contrary directions and ſtrike upon each other; they will after the congreſs move on together as one maſs in that direction, in which the ſtronger of the two moved before it. The moment, when they move together, will, by the laſt propoſition, be equal to the difference of their moments before the congreſs. Therefore their moments before the congreſs being ſuppoſed known, the moment of them, when they move on as one maſs, is known likewiſe: and if this moment be divided by the quantity of matter in motion, that is, by the ſum of the quantities in both of them together, the quotient will be the common velocity after the congreſs.

Suppoſe A weighs 2 ounces and has 3 degrees of velocity, that is, ſuppoſe A's moment $=2\times 3=6$, if it meets B which weighs 4 ounces and has 6 degrees of velocity, or has a moment $=4\times 6=24$: then, as the difference of their moments in contrary directions is $24-6=18$, this, by the laſt propoſition, will be the moment, when, after the congreſs, if they are void of elaſticity, they both move together as one maſs. The moment therefore being 18, and the quantity of matter being $A+B$ or $2+4=6$ which is the ſum of the quantities of matter in both bodies; if 18 is divided by 6 the quotient or 3 will be the common velocity.

This propoſition may be applyed to find the common velocity after the ſtroke, when before it only one of the bodies is in motion and the other is at reſt. For if A moves alone and ſtrikes upon B at reſt, then A's moment may be conſidered either as the ſum of A's and B's moments in the ſame direction or as the difference of their moments in contrary directions.

Suppoſe A weighs 2 ounces and moves with 3 degrees of velocity or with a moment $=2\times 3=6$ and ſtrikes upon B, which weighs 4 ounces: then, becauſe B is at reſt, its moment $=0$, therefore the ſum of their moments is $6+0=6$, and the difference of them is $6-0=6$. From whence it follows, by what has been ſaid already, that if 6 or A's moment is divided by $A+B$ or $2+4$ which is the ſum of the quantities of matter in both bodies, then the quotient, or 1, will be the common velocity, or the velocity with which A and B will move on together after the ſtroke.

77. *The change of velocity which is made in any caſe of impulſe or congreſs, where the bodies are perfectly elaſtic, is double the change, which would be made, in like circumſtances, where the bodies are void of elaſticity.*

If an elaſtic body D Plat. III. fig. 12. is at reſt and another elaſtic body C, which is moving in the direction AB ſtrikes upon it, the ſides of both of them will be preſſed in by the ſtroke of impulſe, and C, by propoſition 75, will loſe velocity in the direction AB and D will acquire it. But the elaſticity of the bodies will make their ſides fly out again with a force equal to that which preſſed them inwards. This produces a ſecond ſtroke, which we may call the ſtroke of elaſticity. This ſtroke of elaſticity is equal to the former or to the ſtroke of impulſe, becauſe the parts of the bodies, which are preſſed in, fly out again with a force equal to the impulſe that preſſed them inwards; and it is likewiſe in the ſame direction; for by the ſtroke of impulſe C loſt velocity in the direction AB and D acquired it; and it is evident that the ſide of the body D flying out towards A will diminiſh C's velocity in the direction AB and the ſide of the body C flying out towards B will increaſe D's velocity in the ſame direction. Therefore, ſince in all caſes the ſtroke of elaſticity is equal to that of impulſe and both of them produce a change of velocity in the ſame direction, it follows that the change produced by both theſe ſtrokes together will be double the change produced by the ſtroke of impulſe alone. But where the ſtriking bodies are elaſtic, there is always a ſtroke of impulſe and another of elaſticity, and where they are not elaſtic, there is only a ſtroke of impulſe. Therefore the change of velocity produced in any caſe, where the ſtriking bodies are elaſtic, will be double the change produced in like circumſtances, where they are void of elaſticity.

This propoſition, though true in theory, cannot be applyed with exactneſs to any bodies, that we can make uſe of in trying whether it is true by experiments. Becauſe the demonſtration of it depends upon the perfect elaſticity of the ſtriking bodies. Whereas neither glaſs balls, nor ivory ones, nor any other body in nature is perfectly elaſtic. The ſides of many bodies when preſſed inwards will fly out again, but not with a force equal to that, with which they were preſſed inwards.

From this propoſition the two following rules may be deduced for determining the velocity and direction of elaſtic bodies, after they have ſtruck upon each other directly.

78. *If a body, that is void of elaſticity, would acquire velocity in any caſe of impulſe or congreſs; a body, that is perfectly elaſtic, will acquire twice as much in the ſame circumſtances.*

For the change of velocity made in elaſtic bodies is always double the change made in thoſe which are not ſo.

79. *If a body, that is void of elaſticity, would loſe velocity in any caſe of impulſe or congreſs; a body, that is perfectly elaſtic, will loſe twice as much in the ſame circumſtances.*

This rule is proved in the ſame manner as the foregoing one. Only here we muſt obſerve that the velocity, which a body is to loſe by this rule, will often be greater than the velocity, which it has. And in this caſe the difference between what it has before the ſtroke and what it is to loſe by the ſtroke is negative, or is to be taken in a direction contrary to that, in which the body was moving. Thus, if a body is moving in any direction with 3 degrees of velocity, and according to the rule ought by a ſtroke on another body to loſe 4 degrees; by this loſs it will have 1 degree of velocity leſs than nothing or 1 degree of negative velocity, or, by what has been ſaid already in explaning propoſition 75, it will be turned back in the contrary direction with 1 degree of velocity; that is, with the difference between 4 and 3, or between what it has and what it is to loſe.

In the propoſitions which follow upon this ſubject we will apply theſe rules to ſome particular caſes.

80. *If an elaſtic body* A *in motion ſtrikes upon another equal and elaſtic one* B *at reſt; then by the ſtroke* A *will communicate all its velocity to* B *and will reſt itſelf.*

Since A and B are equal, the quantity of matter in each of them may be called 1. Let A's velocity be called 6, and conſequently its moment will be $6\times1=6$, which is the product of its velocity multiplyed into its quantity of matter. Now if neither of the bodies had been elaſtic, the common velocity after the ſtroke, according to propoſition 76, might have been found by dividing 6 the moment by $A+B$ or by $1+1=2$ or by the ſum of the quantities of matter in both bodies together. For the quotient of $\frac{6}{2}$, or 3, would have been this common velocity. And, as A had 6 degrees of velocity before the ſtroke and has only 3 after it; A, when it is a body void of elaſticity, loſes 3 degrees of velocity. Therefore, by propoſition 79, A an elaſtic body will in the ſame circumſtances loſe double 3 degrees, or 6 degrees of velocity, which is all it has, and conſequently muſt reſt after the ſtroke. B likewiſe, when it is a body void of elaſticity, reſts before the ſtroke, but after the ſtroke moves with the common velocity 3; that is, it acquires 3 degrees of velocity. But, by propoſition 78, B an elaſtic body will in the ſame circumſtances acquire twice as much or 6 degrees, which is the ſame velocity that A had before the ſtroke.

81. *If there are ſix equal and elaſtic bodies, of which five as* B, C, D, E, F, *lie in a row contiguous to each other, and the other as* A *ſtrikes directly upon* B *the firſt in the row; then* A *and all the other bodies in the row will reſt, except* F *the laſt of them, and this will fly off with the ſame velocity, that* A *had when it ſtruck upon* B.

By the laſt propoſition, A will communicate all its velocity to B and will reſt itſelf. For the ſame reaſon B will communicate all its velocity to C and will reſt itſelf. And then C becomes the ſtriking body, and therefore will communicate all its velocity to the next or D and will reſt itſelf. The ſame may be ſaid of all the other bodies in the row, though there were ever ſo many of them, till we come to the laſt of them or F. For ſince there is no other body beyond F to which its velocity may be communicated, it will move off with the velocity received from the next before it or E, that is, as appears from what has been ſaid already, with the velocity, that A had when it ſtruck upon B.

82. *If there are ſix equal and elaſtic bodies, of which four only as* C, D, E, F, *lie in a row contiguous to each other, and two as* A *and* B *moving along cloſe together ſtrike upon* C *the firſt in the row; then* A *and* B *and all the other bodies in the row will reſt, except* E *and* F *the two laſt of them, and theſe will move off together with the ſame velocity that* A *and* B *had before the ſtroke.*

By the laſt propoſition, if B alone had ſtruck upon C then B and all the others in the row would have reſted except F. But when B has loſt all its velocity by the ſtroke, then A becomes the ſtriking body, and will communicate all its velocity along the row to the laſt body, which is then the body E. Thus B makes F fly off, and A makes E fly off: and becauſe A and B ſtrike together, E and F will fly off together.

83. *If an elaſtic body* A *ſtrikes upon another elaſtic body* B, *which is greater than itſelf: then the moment of the body* B *after the ſtroke, will be greater than the moment of* A *before it.*

Let A weigh 2 ounces and move with 3 degrees of velocity or a moment 6, and let B, which is at reſt, weigh 4 ounces. Then if theſe two bodies had been void of elaſticity, their common velocity after the ſtroke, by propoſition 76, (where this caſe is made one of the inſtances) would have been 1. Thus B, when void of elaſticity, acquires by the ſtroke 1 degree of velocity, and conſequently, by propoſition 78, B, if elaſtic, will acquire twice as much or 2 degrees. Therefore ſince B's quantity of matter $=4$, and its velocity after the ſtroke $=2$, its moment muſt $=2\times4=8$, which is the product of its velocity multiplyed into its quantity of matter. Whereas the moment of A before the ſtroke was only 6.

But notwithſtanding the moment 8, which is produced in B by the ſtroke, is greater than the moment 6 in A, which produced it; yet the mo-

moment of the 2 bodies conſidered together is exactly the ſame after the ſtroke that it was before it: as it ought to be, by propoſition 75. For, after the ſtroke, B moves along with the moment 8 in the ſame direction that A was moving before it. And let us conſider what in the mean while becomes of A. Now A had 3 degrees of velocity before the ſtroke, and, if void of elaſticity, would loſe 2 degrees by it, according to what was proved in this very inſtance under propoſition 76. But, by propoſition 79, A, if elaſtic, will loſe twice as much or 4 degrees. And, ſince it has but 3 degrees in all, it will be turned back again or reflected with the difference between 3 and 4 or with 1 degree of velocity: and ſince its quantity of matter $= 2$ its moment $= 1 \times 2 = 2$. But ſince after the ſtroke B moves in one direction, and A, which is reflected, moves in the contrary direction, A's moment is negative in reſpect of B's: and the moment of the 2 bodies taken together will be $8 - 2 = 6$ which is equal to the moment, that A had before the ſtroke.

I muſt therefore deſire the reader to remember, that though there is motion produced by the ſtroke, if we conſider B or the greater body alone, yet if both bodies are taken together there is no motion produced in the ſame direction that A was moving, but the moment in that direction is exactly the ſame after the ſtroke that it was before it.

84. *If an elaſtic body* A *overtakes another equal one* B, *which is moving ſlower than itſelf and in the ſame direction; after the ſtroke they will continue to move in the ſame direction with interchanged velocities.*

Let A have 8 degrees of velocity and B 4 degrees. Call A's quantity of matter 1; and B's, which is equal to it will be 1 likewiſe. Therefore A's moment $= 8 \times 1 = 8$; and B's $= 4 \times 1 = 4$. The ſum of their moments in the ſame direction is 8 added to $4 = 12$, this ſum divided by the ſum of their quantities of matter or by $A + B = 1 + 1 = 2$, gives 6 in the quotient. From whence it follows, by propoſition 76, that, if neither of theſe bodies had been elaſtic, they would have moved on together after the ſtroke with the common velocity 6. Therefore A, whoſe velocity before the ſtroke was 8, would have loſt 2 degrees; and B, whoſe velocity was 4 would have gained 2. But A, if it is perfectly elaſtic, will loſe twice as much velocity or 4 degrees: and B, if it is perfectly elaſtic, will gain twice as much or 4 degrees: by propoſitions 79. 78. And ſince A, which had 8 degrees of velocity at firſt, loſes

4 of them by the ſtroke; its velocity afterwards will be $8-4=4$. And ſince B, which had 4 degrees of velocity at firſt, gains 4 more; its velocity afterwards will be $4+4=8$. Thus before the ſtroke A's velocity was 8, and B's 4; and after the ſtroke A's is 4, and B's 8; that is, their velocities are interchanged.

85. *If an elaſtic body* A *meets and ſtrikes upon another equal one* B; *after the ſtroke they will be reflected with interchanged velocities.*

Let A have 8 degrees of velocity and B 4 degrees. Call A's quantity of matter 1, and then B's, which is equal to it will be 1. The moment of $A=8\times1=8$: and the moment of $B=4\times1=4$. The difference of their moments in contrary directions is $8-4=4$. Divide this difference 4 by $A+B=1+1=2$, or by the ſum of the quantities of matter, and $\frac{4}{2}=2$, which is the quotient, would be the common velocity, by propoſition 76, with which both the bodies, if neither of them had been elaſtic, would have moved on together after the ſtroke in the ſame direction that A was moving in before it. Thus A, if not elaſtic, would loſe 6 degrees of velocity; for it had 8 degrees before the ſtroke and would have only 2 after it. Therefore, by propoſition 79, A, if elaſtic, will loſe twice as much or 12 degrees. But it has no more than 8 degrees, and conſequently the difference between 8 and 12 or 4, that is, the difference between the velocity, which it has, and the velocity, which it is to loſe, is negative or is taken in the contrary direction. So that A will be reflected with the velocity 4. B moved with the velocity 4 to meet A before the ſtroke, and if neither body had been elaſtic, would have moved afterwards with the velocity 2 in the ſame direction with A. Therefore B, if it was not elaſtic, would loſe 4 degrees of velocity in one direction and acquire 2 degrees in the oppoſite direction. From hence it follows, by propoſition 78, 79, that B, if elaſtic, will loſe twice as much velocity or 8 degrees in one direction, and acquire twice as much or 4 degrees in the oppoſite direction. But ſince it is to loſe 8 degrees in the direction in which it moved before the ſtroke and has no more than 4 degrees to loſe, the difference between 8 and 4, or $8-4=4$ is to be taken in the oppoſite direction: and theſe 4 degrees added to the other 4 which it acquires in that oppoſite direction, will make the velocity with which it is reflected $=4+4=8$. Thus before the ſtroke A's velocity was 8 and B's 4; and when they are reflected A's is 4 and B's 8: that is, the velocities are interchanged.

86. *If two equal and elaſtic bodies* A *and* B, *which are moving in contrary directions, meet and ſtrike upon each other with equal velocities; each of them will be reflected with the ſame velocity that it had before the ſtroke.*

This may be conſidered as one caſe of the laſt propoſition: for if the two bodies had equal velocities at firſt, they will likewiſe have equal velocities after they have exchanged. Or otherwiſe; the bodies, if they had not been elaſtic, would have loſt all their velocities, becauſe both of them ſtrike with equal moments, and theſe equal and contrary forces would have deſtroyed each other, and the two bodies would have reſted after the ſtroke. But A, as it is ſuppoſed elaſtic, will loſe twice as much velocity, by propoſition 79, that is, double what it has, or will gain as much negative velocity as is equal to the difference between the velocity, which it has, and the double of that velocity. This difference is plainly equal to the velocity which it has. The ſame may be ſaid of the body B. Therefore the negative velocity, or velocity with which each body is reflected, is equal to that, with which it moved before the ſtroke.

87. *To determine the direction, in which two elaſtic bodies will be reflected, when they ſtrike upon each other obliquely.*

Let two equal and elaſtic bodies P and Q move in the reſpective directions PA and Q*a* and ſtrike upon each other. Theſe oblique motions PA and Q*a* may each of them be reſolved into two others, by propoſition 20, in ſuch a manner, that one part of PA, as PB or CA, ſhall be parallel to one part of Q*a* as Q*b* or *ca*; and the other part of PA as PC or BA ſhall be directly oppoſite to the other part of Q*a*, as Q*c* or *ba*. Now the two parallel motions CA and *ca* will not be altered by the ſtroke. For, if the bodies had moved in theſe directions only, they would never have approached each other, and conſequently could never have ſtruck upon each other. Therefore as no part of the ſtroke is made by either of theſe motions, they can neither of them be at all affected by it. And if we take AE equal to CA and in the ſame direction; and take *ae* equal to *ca* and in the ſame direction; from what has been ſaid it follows, that after the ſtroke the parallel motion of P will be AE and that of Q will be *ae*. The oppoſite motions BA and *ba* will have their velocities interchanged, by propoſition 85. Therefore taking AD equal to *ba*, and *ad* equal to BA, AD will be the motion, with which P will be reflected, and *a d* will be that, with which Q will be reflected. Since therefore after the ſtroke P is carried on with the motion AE and the motion AD, make theſe the two ſides of a parallelogram, and, by propoſition 16, the diagonal A *p* will be the line in which the body P will be reflected. In like manner, ſince the two motions of Q after the ſtroke are *ae* and *a d*, make theſe the ſides of a parallelogram and draw the diagonal *aq*, then, by propoſition 16, *aq* is the line in which the body Q will be reflected.

From this inſtance the reader will underſtand, that, in all other caſes of the oblique congreſs of two elaſtic bodies, their motions may each of them be reſolved into a parallel one and an oppoſite one, that the parallel motion of each will be the ſame after the ſtroke that it was before it, and that the other motion of each after the ſtroke may be found by the foregoing rules relating to the direct congreſs of bodies. And when the two motions of each body after the ſtroke are thus found, then by making theſe the ſides of a parallelogram and drawing the diagonal, the direction in which the bodies will be reflected is determined: for they will always be reflected in that diagonal.

88. *If an elaſtic body ſtrikes perpendicularly upon an immoveable obſtacle, it will be reflected in the ſame line that it deſcribed, and with the ſame velocity that it had, before the ſtroke.*

If an elaſtic body moves in the direction E B Plat. III. fig. 14. and ſtrikes perpendicularly on the immoveable elaſtic obſtacle DBF; the body will be reflected in the line BE with the ſame velocity that it had before the ſtroke. For if the body and obſtacle had been void of elaſticity then by the ſtroke it would have loſt all its velocity. Therefore, by propoſition 79, becauſe they are elaſtic, it will loſe twice the velocity that it has, or will loſe its own velocity, and acquire a negative one that is equal to it, that is, will be reflected or be made to move in a contrary direction with the ſame velocity that it had before the ſtroke.

89. *When a body ſtrikes obliquely on an immoveable obſtacle, the angle contained between the line, which the body deſcribes and a line, that is perpendicular to the obſtacle at the point where the body ſtrikes it, is called the angle of incidence.*

Thus Plat. III. fig. 14. if AB is the line, which the body deſcribes, and BE a perpendicular to the obſtacle DBF at the point B where the body ſtrikes it, then the angle of incidence is the angle A B E. The reader will obſerve that A B D is the complement of the angle of incidence to a right one.

90. *When a body, after ſtriking obliquely on an immoveable obſtacle is reflected, the angle contained between the line, in which it is reflected, and a line, which is perpendicular to the obſtacle at the point of reflection, is called the angle of reflection.*

Thus Plat. III. fig. 14. the angle of reflection is EBC contained between BC the line which the reflected body deſcribes and EB a perpendicular to the obſtacle at B the point of reflection. CBF is the complement of reflection.

In theſe two definitions of the angles of incidence and reflection, I have copyed from Sir Iſaac Newtons definitions of them in his optics. The mechanical writers commonly call A B D the angle of incidence, and CBF the angle of reflection.

91. *When an elaſtic body ſtrikes obliquely on an immoveable and elaſtic obſtacle, and is reflected from it: the angle of reflection is always equal to the angle of incidence.*

In order to prove this, ſuppoſing the body to ſtrike the obſtacle in the direction AB, Plat. III. fig. 14. let us firſt trace out the line in which it will be reflected. The oblique motion AB of the incident body may be reſolved into two others, by propoſition 20, one of them parallel to the obſtacle as AE equal to DB, the other perpendicular to it, as AD equal to EB. Of theſe two motions the parallel one DB will not be altered by the ſtroke. For if the body had moved in this parallel direction only, it would never have approached the obſtacle, and conſequently could never have ſtruck upon it. Therefore as no part of the ſtroke is made by the parallel motion AE or DB, it is not affected by the ſtroke but will be the ſame after it that it was before. And, if BF is taken equal to DB, from what has been ſaid it follows that B F will be the direction and velocity of the body after the ſtroke. The perpendicular motion, which before the ſtroke is AD or EB, will, by propoſition 88, be the ſame after it, only in a contrary direction as BE. Therefore after the ſtroke the body being carryed on by the two motions B F and B E will, by propoſition 16, deſcribe BC the diagonal of a parallelogram, of which BF and BE are the ſides. Thus BC is proved to be the line, in which the body will be reflected.

And from what has been ſaid in tracing out this line BC, in which the body will be reflected, we may ſhew that the angle of reflection EBC is equal to the angle of incidence ABE. For AE and DB are equal; becauſe they are oppoſite ſides of a parallelogram. Euc. book I. prop. 34. And for the ſame reaſon EC and BF are equal. But DB and BF are equal by the conſtruction, becauſe one is the parallel motion before the ſtroke and the other is the ſame motion after it. Therefore AE becauſe it is equal to DB and EC becauſe it is equal to BF are likewiſe equal to each other. The ſide EB is common to the two triangles AEB and CEB. And, ſince AC is parallel to DBF and EB is perpendicular to it by the conſtruction, therefore EB is perpendicular to AC, and the angles AEB and CEB are equal. But, becauſe AE, EB, in one of theſe triangles are equal to CE, EB, in the other and the angles AEB and CEB contained between theſe equal ſides are likewiſe equal; it follows, Euc. b. I. prop. 4. that the other two angles in one triangle are each of them reſpectively equal to the other two angles in the other triangle, which are ſubtended by equal ſides. Therefore EBC the angle of reflection is equal to ABE the angle of incidence.

92. *If a body, which is in motion, ſtrikes upon another body either in motion or at reſt; the quantity of the ſtroke is proportional to the moment, which is loſt by the ſtriking body,*

If one of the bodies has a greater moment than the other, then that which has the greater moment, or if their moments are equal, then either of them, may be conſidered as the ſtriking body.

Now the quantity of the ſtroke is the force impreſſed by the ſtriking body upon the other. But ſince, by the third law of motion, propoſition 15, reaction is equal to action, whatever force is impreſſed on the body, which is ſtruck, will by its reaction be deſtroyed in the ſtriking body. Therefore the motion loſt by the ſtriking body is proportional to the force impreſſed by it on the other, or to the quantity of the ſtroke. From hence it follows, that when a moving body ſtrikes perpendicularly upon an immoveable obſtacle; the quantity of the ſtroke will be as the ſtriking body's moment; becauſe its whole moment is loſt by the ſtroke.

93. *If a given body ſtrikes upon another given body at reſt, the quantity of the ſtroke is always proportional to the velocity of the ſtriking body.*

This follows from the laſt propoſition, becauſe the moment loſt is always proportional to this velocity. Call one of the bodies A and let it be ſuppoſed to weigh 2 ounces: call the other B, and let its weight be 4 ounces: we may conſider this propoſition in two caſes.

Firſt, if B is the body at reſt, then the quantity of the ſtroke or moment loſt will always be as A's velocity. If A's velocity before the ſtroke $= 3$, and conſequently its moment $= 3 \times 2 = 6$; then ſuppoſing it void of elaſticity, it appears by propoſition 76, that its velocity after the ſtroke $= 1$ and conſequently its moment $= 1 \times 2 = 2$. Therefore the moment loſt by the ſtroke $= 6 - 2 = 4$. By the ſame 76 propoſition it may be proved that, if A's velocity before the ſtroke had been 6 and its moment $= 6 \times 2 = 12$, then its velocity after the ſtroke would $= 2$ and its moment $= 2 \times 2 = 4$. Therefore the moment loſt by the ſtroke $= 12 - 4 = 8$. But ſince as 4 is to 8 ſo is 3 to 6, that is, ſince the moment loſt is as A's velocity, it follows by the laſt propoſition, that the quantity of the ſtroke is as A's velocity.

The ſecond caſe of this propoſition is, that when the velocity is given the quantity of the ſtroke will be the ſame whether A ſtrikes upon B or B upon A. Thus if the given velocity is called 3, the quantity of the ſtroke will be the ſame whether A with the velocity 3 and conſequently with the moment $3 \times 2 = 6$ ſtrikes B at reſt, or B with the ſame velocity 3 and conſequently with the moment $3 \times 4 = 12$ ſtrikes A at reſt. It has juſt been proved that if A ſtrikes B the moment loſt $= 4$. And, by propoſition 76, it may be ſhewn that, if B weighing 4 ounces, and moving with 3 degrees of velocity, that is, with a moment $= 12$, ſtrikes A at reſt, the common velocity after the ſtroke, ſuppoſing both of them void of elaſticity, will be 2. Therefore B's velocity being this common velocity or 2 and its quantity of matter 4, its moment after the ſtroke will be $2 \times 4 = 8$. So that the moment loſt by the ſtroke $= 12 - 8 = 4$. And ſince the moment loſt is 4 whether A ſtrikes with a given velocity upon B at reſt, or B ſtrikes with the ſame velocity upon A at reſt: it follows, by the laſt propoſition, that the quantity of the ſtroke will be the ſame.

We may by aſſigning a reaſon for this laſt caſe, which ſeems a paradox, give a general proof of the propoſition. When A is the moving body the moment is 6, and when B is the moving body the moment is 12: and yet B will not hit double the blow that A does, but the quantity of the ſtroke would in either caſe be the ſame. For we muſt remember that the quantity of the ſtroke does not depend upon the moment of the ſtriking body only, but likewiſe upon the reſiſtance of the body that is ſtruck. Thus if we hold our hand ſteady againſt any blow, we ſhall feel more of it, than if we had made no ſuch reſiſtance, but had let our hand yield to it. And in all other caſes the more the body, which is hit, bears up againſt the blow, ſo much the greater ſtroke it will ſuſtain; and if it makes no reſiſtance at all, the blow will be nothing. Now, where the reſiſtance is given, the quantity of the ſtroke is as the moment of the ſtriking body, and where this moment is given, it is as the reſiſtance of the body that is ſtruck. Therefore in all caſes it is in a ratio compounded of them both or as the moment multiplyed into the reſiſtance. The reſiſtance of the body, which is ſtruck, being, where other circumſtances are alike, as the quantity of matter contained in it, when A ſtrikes B with 3 degrees of velocity the quantity of the ſtroke is as $6 \times 4 = 24$, or as A's moment into B's quantity of matter. And when B ſtrikes A it is as $12 \times 2 = 24$ or as B's moment into A's quantity of matter.

Therefore univerſally, where the quantity of matter in two bodies, of which one moves and the other reſts, is given, the quantity of the ſtroke will be as a product ariſing from one body multiplyed into the other and then multiplyed again into the velocity with which the ſtriking body moves. But this product will always be as this velocity, becauſe the quantity of matter in both is given. Therefore the quantity of the ſtroke will likewiſe be as the velocity.

Upon this principle it is that thoſe, who call themſelves ſtrong men, will let a ſmith cut a bar of iron aſunder upon an anvil, which is laid upon their breaſt. The only difficulty in this is to bear the weight of the anvil: for the force with which the hammer ſtrikes it, though it is ſufficient to drive a chiſſel through the iron, will make the anvil give a very inconſiderable blow to their breaſt, who ſupport it. They lay their back hollow from the ground, that they may have room to yield to the blow; and then the quantity of the ſtroke will be proportional to the velocity, with which the anvil is driven down upon their breaſt. One blow of the hammer upon their naked breaſt without the anvil would kill them. But though the hammer ſtrikes the anvil with an equal force, and this force communicated to the anvil ſtrikes their breaſt; they will feel little or no inconvenience from it. Indeed there are two reaſons why a blow of the hammer immediately upon their breaſt ſhould hurt them more, than the ſame blow would when communicated firſt to the anvil. The firſt reaſon is becauſe the whole force of the hammer ſtrikes upon a very ſmall part of the breaſt; whereas if by ſtriking the anvil it was to make the anvil hit as great a blow, yet this would reach over the whole breaſt and conſequently would be leſs upon each part, than if all the force had ſtruck one ſingle part. But the other reaſon, why a blow immediately from the hammer would hurt them ſo much more than the ſame blow upon the anvil, is, that the anvil moves with a much leſs velocity. For ſuppoſe the hammer to weigh 1 pound and the anvil to weigh 100 pounds. The hammer by ſtriking the anvil communicates its motion to the anvil; ſo that the moment of one is equal to that of the other; but the anvil will have only a hundreth part of the hammers velocity, and conſequently will ſtrike a leſs blow. From hence it is evident, that the heavier the anvil is in proportion to the hammer, the leſs will be the velocity of the anvil and therefore ſo much leſs the ſtroke upon the breaſt of a man, which ſupports it. The greater quantity of matter there is in the anvil, ſo much the more difficult it will be to ſupport it, but when it is ſupported, this will make the blow the eaſier: and if the quantity of matter in the anvil was infinite, the velocity communicated by the ſtroke of the hammer would be nothing; and if a man could bear the weight of ſuch an anvil as that, he would feel no blow at all, whilſt the iron was cut aſunder upon it.

94. *If a given body in motion ſtrikes directly upon another given body, which is moving ſlower and in the ſame direction; the quantity of the ſtroke is the ſame, as if the body, which moves ſlower, had reſted, and the other had ſtruck upon it with the difference of their velocities.*

The quantity of the ſtroke is, by the laſt propoſition, as the ſtriking velocity, that is, as the velocity with which the two bodies approach each other. But when both of them are moving in the ſame direction they approach only with the difference of their velocities; and ſo they do when one of them reſts and the other moves with this difference. Therefore the quantity of the ſtroke will be the ſame in both caſes.

Call one body A and let it be ſuppoſed to weigh 2 ounces; call the other B and let its weight be 4 ounces. Then, if A's velocity is 3, its moment will be $3 \times 2 = 6$, and if B moves after it in the ſame direction with the velocity 6, its moment will be $6 \times 4 = 24$. And if they are both void of elaſticity, the caſe will be ſo much the more ſimple and, by propoſition 76, their common velocity after the ſtroke will be 5. Therefore B's moment after the ſtroke will be $5 \times 4 = 20$: and as its moment before was 24, it muſt have loſt a moment $= 24 - 20 = 4$. But ſuppoſe A had been at reſt and B had ſtruck upon it with the difference of their velocities, which is in the caſe propoſed $6 - 3 = 3$. Then B's moment before the ſtroke is $3 \times 4 = 12$. And when this moment is divided by $A + B$, or $2 + 4 = 6$, or by the ſum of the quantity of matter in the two bodies, the quotient is 2, which, by propoſition 76, is the common velocity after the ſtroke. Therefore B's moment after the ſtroke will be 2 × 4 = 8, and, as its moment before was 12, it muſt have loſt a moment = 12 − 8 = 4. And ſince 4 is the moment loſt whether A and B are both in motion in the ſame direction, or A reſts and B ſtrikes upon it with the difference of their velocities; it follows, by propoſition 92, that the quantity of the ſtroke is the ſame in both caſes.

95. *If two bodies, which are moving in contrary directions, ſtrike upon each other; the quantity of the ſtroke is the ſame, as if one of them had reſted and the other had ſtruck upon it with the ſum of their velocities.*

The quantity of the ſtroke is, by propoſition 93, as the ſtriking velocity, that is, as the velocity with which the two bodies approach each other. But when both of them move in contrary directions they approach with the ſum of their velocities; and ſo they do, when one of them reſts and the other moves with this ſum. Therefore the quantity of the ſtroke will be the ſame in both caſes.

Call one body A, and let it be ſuppoſed to weigh 2 ounces; call the other B, and let its weight be 4 ounces. Then if A's velocity is 3, its moment will be 3 × 2 = 6. If B moves in a contrary direction with the velocity 6, its moment will be 6 × 4 = 24. And, if they are both void of elaſticity, their common velocity after the ſtroke, by propoſition 76, will be 3. Therefore B's moment after the ſtroke will be 3 × 4 = 12: and, as its moment before the ſtroke was 24, it muſt have loſt a moment = 24 − 12 = 12. But ſuppoſe A had been at reſt and B had ſtruck upon it with the ſum of their velocities which in the caſe propoſed is 6 + 3 = 9. Then B's moment before the ſtroke is 9 × 4 = 36. And when this moment is divided by A + B or 2 + 4 = 6, or by the ſum of the quantities of matter in the two bodies, the quotient is 6, which, by propoſition 76, is the common velocity after the ſtroke. Therefore B's moment after the ſtroke will be 6 × 4 = 24, and as its moment before was 36 it muſt have loſt a moment = 36 − 24 = 12. And ſince 12 is the moment loſt whether A and B are both moving in contrary directions, or A reſts and B ſtrikes upon it with the ſum of their velocities, it follows, by propoſition 92, that the quantity of the ſtroke is the ſame in both caſes.

96. *The motion of bodies amongſt themſelves is the ſame whether the place, in which they are all included, reſts or moves uniformly in a right line.*

The motion of two or more balls on a billiard table on board a ſhip, is the ſame in all reſpects and particularly as to the ſtrokes upon each other, whether the ſhip and the table in it are at reſt, or whether they move on uniformly in a right line. For, by the two laſt propoſitions, the quantity of the ſtroke is always as the difference of the ſtriking velocities in the ſame direction or as the ſum of them in contrary directions. And ſince the uniform motion of the table, or place, in which all of the balls are included, is common to all of them, it will not alter the difference of their velocities in the ſame direction or the ſum of them in contrary directions. Therefore the uniform motion of the table will not make their ſtrokes upon each other different from what they would have been if the table had reſted.

We may apply this propoſition more particularly to another caſe; where it is of more importance. If the earth turns round its axis from weſt to eaſt once in every day; then if you diſcharge a bullet from weſt to eaſt againſt a wall, the bullet and wall are both moving in the ſame direction; but if you diſcharge it from eaſt to weſt they are moving in contrary directions. It is therefore enquired, why the ſtroke of the bullet with the ſum of the velocities, as in the latter caſe, ſhould not be greater than it is in the former, when it is made only with their difference. To this I anſwer that the difference of the bullets and walls velocities in the former caſe is exactly the ſame as what is here called the ſum of their velocities in the latter; and either of them is the ſame as the velocity with which the bullet would have ſtruck the wall ſuppoſing the earth and the wall had been at reſt. Therefore, by the three foregoing propoſitions, the quantity of the ſtroke will be the ſame in all the three caſes. Suppoſe that the velocity with which the earth turns round its axis is to the velocity, with which the bullet is diſcharged as 10 to 1: for though this is far from being the true proportion, it will illuſtrate the caſe as well as any other would. Then if the wall had been at reſt the ſtriking velocity of the bullet upon it would be $=1$. But if the wall and earth are ſuppoſed to move from weſt to eaſt with the velocity 10; then it muſt be remembered that the bullet before it is diſcharged is moving along with the earth in the ſame direction and with the ſame velocity; and conſequently when it is diſcharged from weſt to eaſt there is 1 degree of velocity added to it in this direction ſo that it moves with a velocity $=10+1=11$. Now as the wall moves with the velocity 10, and the bullet in the ſame direction with the velocity 11; the ſtriking velocity will be the difference between them or $11-10=1$. But if the bullet is diſcharged from eaſt to weſt, then becauſe, before it was diſcharged it was moving along with the earth from weſt to eaſt with 10 degrees of velocity, and by being diſcharged has only 1 degree of velocity given to it in the contrary direction, though in reſpect of the earths ſurface and the things upon the ſurface it appears to move from eaſt to weſt, yet in abſolute ſpace it will ſtill continue to move from weſt to eaſt with the difference between the velocities 10 and 1 or with the velocity 9. So that what is called in the enquiry the ſum of the bullets and walls velocity is properly the difference. For after the bullet is diſcharged it moves from weſt to eaſt with the velocity 9 and the wall moves in the ſame direction with the velocity 10. Therefore the ſtriking velocity will be $10-9=1$. And ſince the ſtriking velocity is 1, whether the wall is at reſt and the bullet is diſcharged againſt it with the velocity 1; or the wall is in motion and the bullet is diſcharged againſt it either from weſt to eaſt or from eaſt to weſt with the ſame velocity; it follows, by propoſitions 93. 94. 95. that the blow, which the bullet hits upon the wall, will be the ſame in all the three caſes.

97. *The quantity of an oblique ſtroke, where the moment is given, is to the quantity of a direct one, as the coſine of incidence is to the radius.*

Thus Plat. III. fig. 14. if a given body ſtrikes the bar DBF obliquely in the direction AB with a velocity repreſented by the ſame line AB, the blow will be leſs than if the body had ſtruck it directly or perpendicularly with the ſame velocity, as much leſs as AD the coſine, or ſine of the complement, of incidence is leſs than AB, which is called the radius. For, by a conſequence deduced from propoſition 92, the ſtroke, when a body ſtrikes perpendicularly with the velocity AB, is as the moment, that is, if the quantity of matter in the body is given, as the velocity. But, when the ſame body ſtrikes with the ſame velocity obliquely or in the direction AB, then this oblique motion may be reſolved into two others, of which one as AE will be parallel to the bar and the other as AD perpendicular to it. Now it is evident that the parallel part of this motion would never bring the body to the bar or make it ſtrike upon it. Therefore the whole ſtroke is made with the perpendicular part of the motion or AD. And, becauſe the oblique ſtroke is given only with the moment AD, but the direct or perpendicular ſtroke would be given with the whole moment AB, it follows that the oblique ſtroke is to the direct one as AD to AB, or as the coſine of incidence to the radius.

98. *The friction of the parts of a machine againſt each other will in time deſtroy any motion, that has been communicated to it.*

Amongſt the other mutual actions of bodies upon each other we reckon friction. No machine can have the ſurface of its parts made exactly ſmooth by poliſhing, though their roughneſs may be made too ſmall for our eyes to diſcover it. And as a body by ſtriking directly upon another, that is either at reſt or in motion, will loſe velocity, ſo, as the parts of a machine rub againſt each other, when the prominence of one part, though ever ſo ſmall, ſtrikes upon the prominence of another part, the motion of the parts will by this means be diminiſhed, and by frequent ſtrokes of the ſame ſort will in time be entirely deſtroyed.

From hence it follows, that no machine can be ſo contrived as to have a perpetual motion, or always to preſerve the motion once impreſſed upon it. For the motion communicated will by the friction of the machine be conſtantly decreaſing, and muſt end at laſt.

To prevent this, in all the contrivances for a perpetual motion, the uſual attempt has been to find out a way of reparing the motion which is loſt by friction. And as bodies move themſelves by the force of gravity; and as motion is likewiſe generated by the ſpring of ſuch bodies as are elaſtic; the way of reparing the motion, as it decays by friction muſt be by the application of one or other of theſe two properties of matter. The motion of ſome clocks is kept up by the force of gravity: but then this motion is not a perpetual one; for the clock ſtops as ſoon as the weights are down. Other clocks are kept in motion by elaſtic ſprings, but this motion ceaſes likewiſe, when the ſpring by having expanded itſelf is grown too weak to repare it any longer. And indeed neither of theſe properties can be ſo applyed as to make a perpetual motion. This I will endeavour to ſhew, from one or two of the principal contrivances, in the two following propoſitions.

99. *The force of gravity cannot produce a perpetual motion.*

The general reaſon for this has been already given. For ſince gravity produces motion in bodies only whilſt they can deſcend; the motion of a machine, which is loſt by the friction of its parts, can be repared by a weight no longer than till the weight is down, or is come to the ground and is prevented by it from deſcending any longer.

In order to remove this difficulty machines have been contrived with more weights than one, of which ſome are to be conſtantly aſcending, whilſt the others are deſcending. Where we may obſerve that the deſcending

PLATE III.

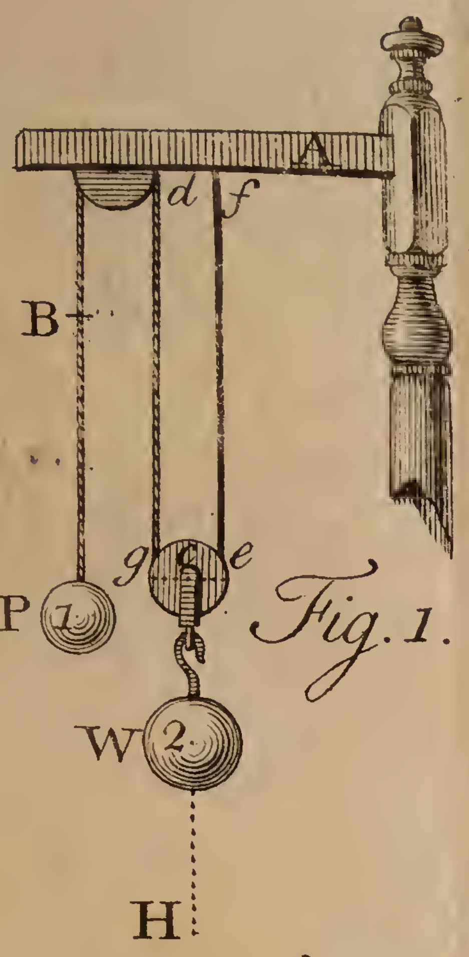


Fig. 1.

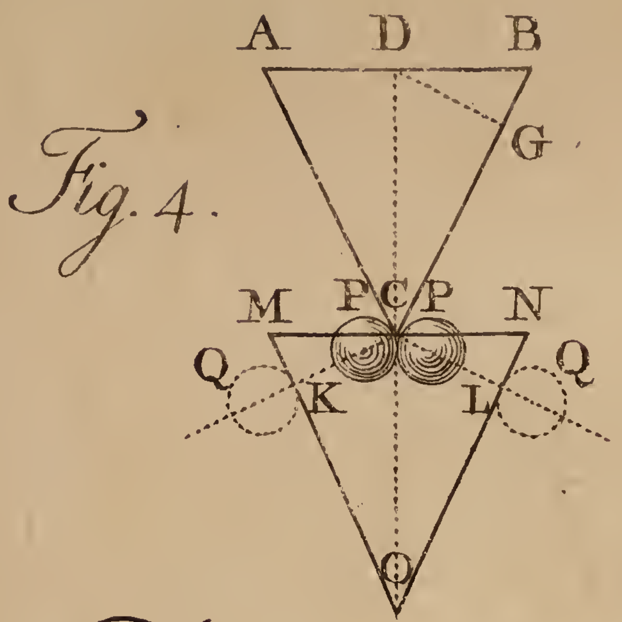


Fig. 4.

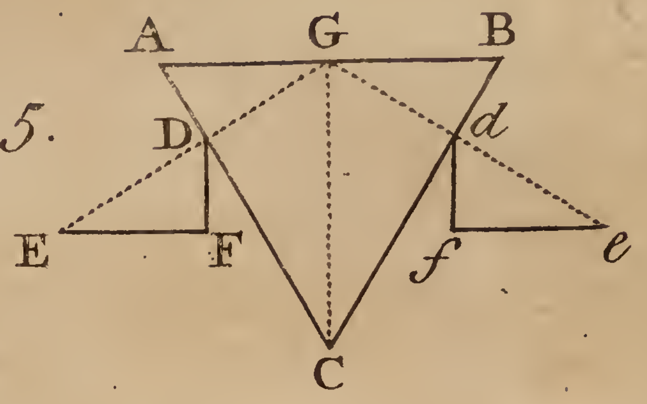


Fig. 5.

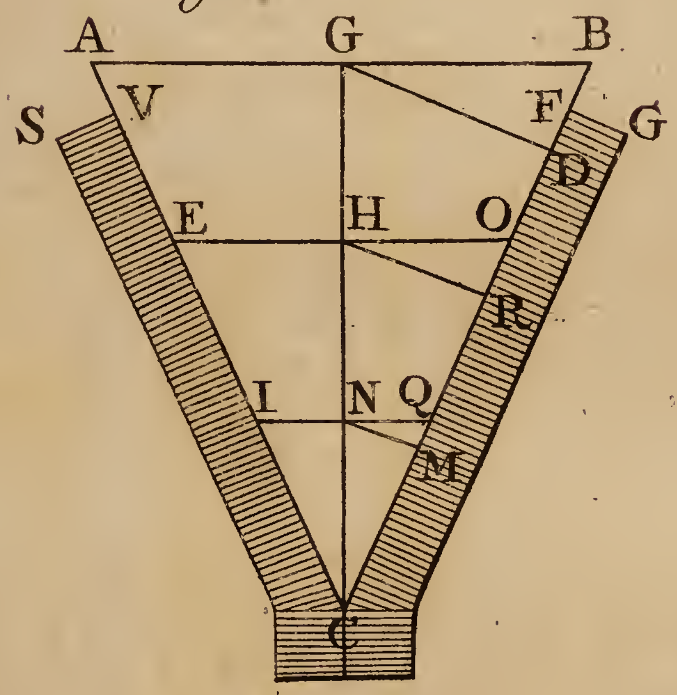


Fig. 6.

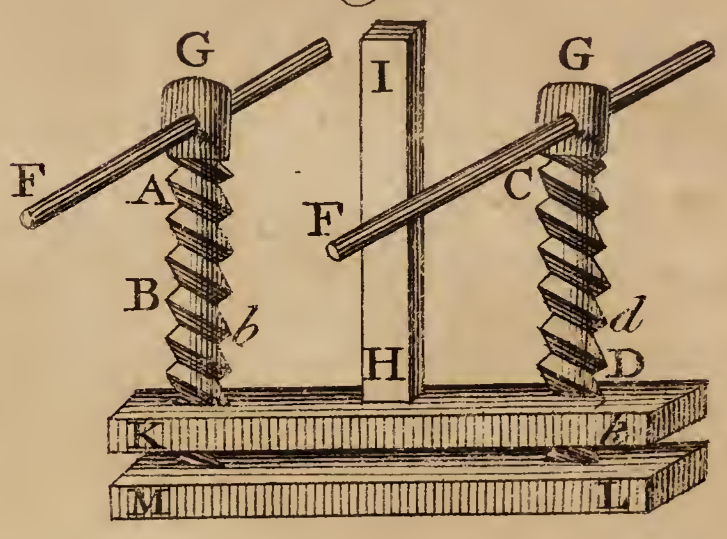


Fig. 7.

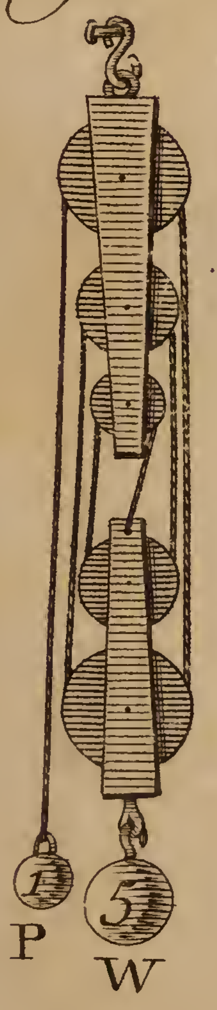


Fig. 2.

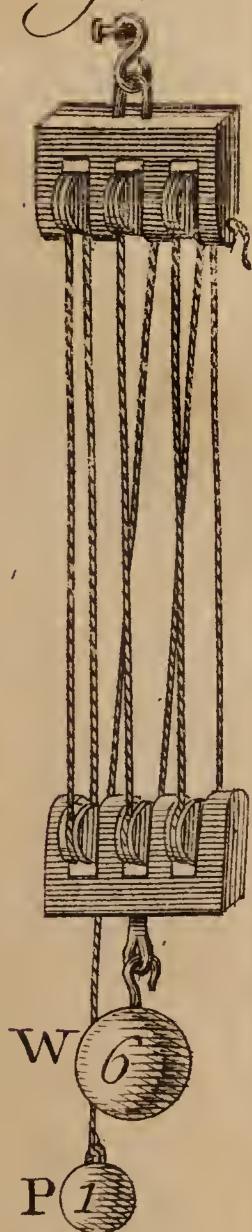


Fig. 3.

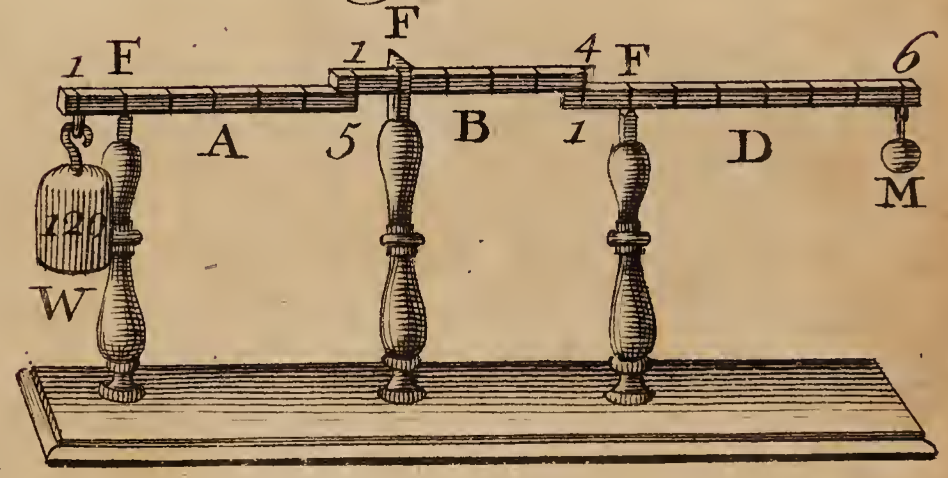


Fig. 8.

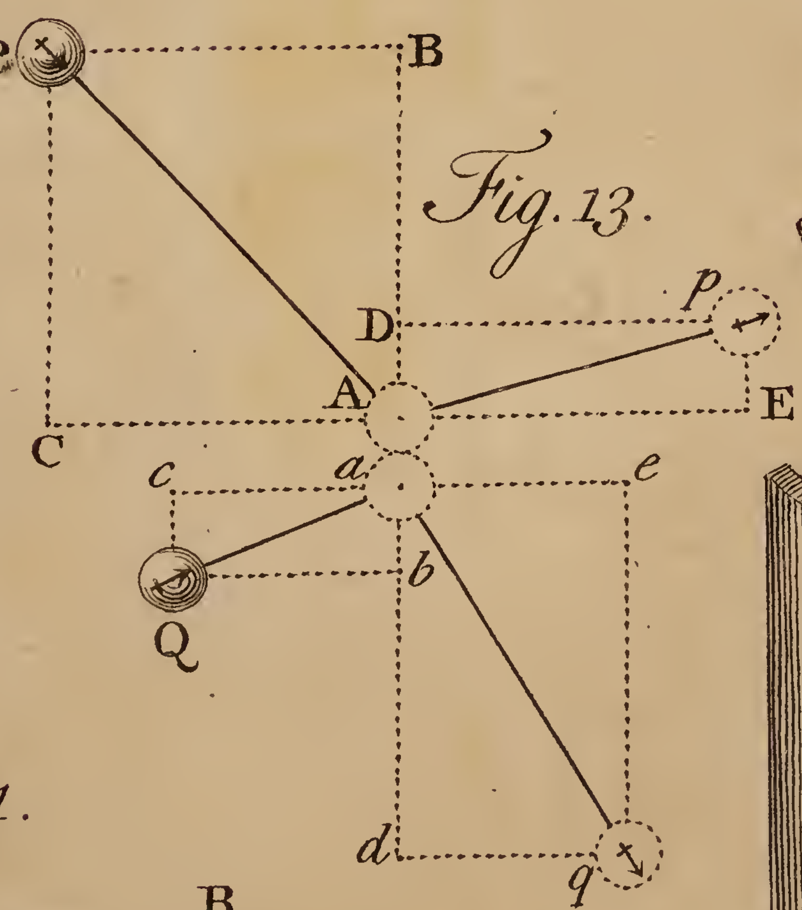


Fig. 13.

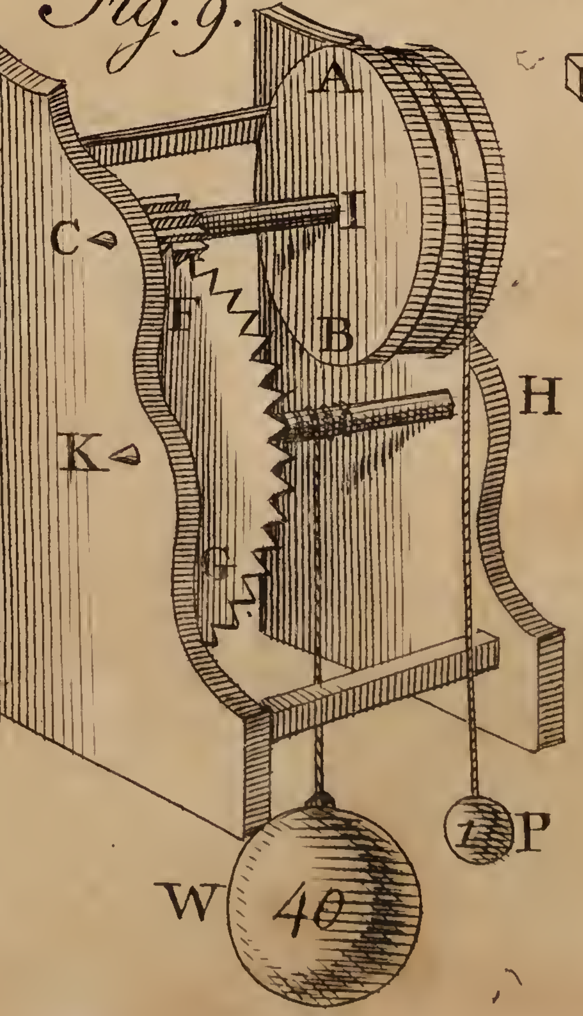


Fig. 9.

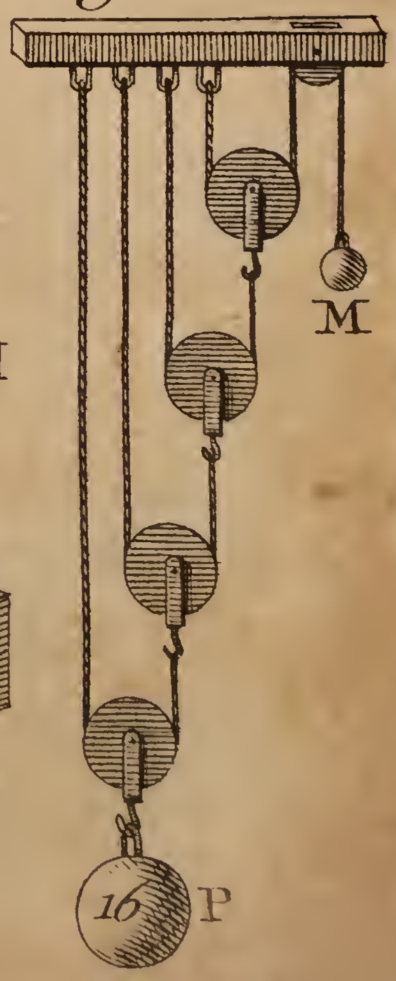


Fig. 10.

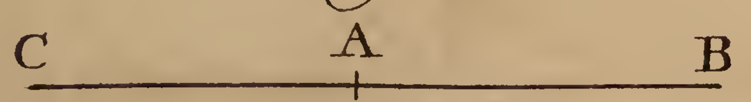


Fig. 11.

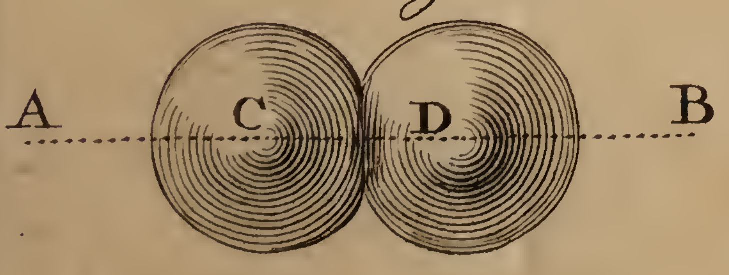


Fig. 12.

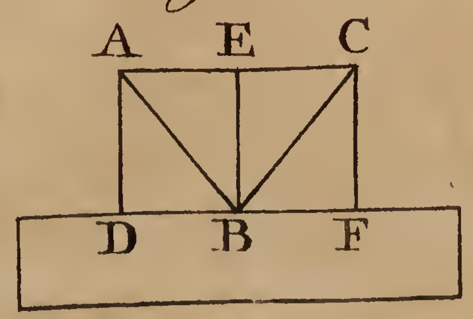


Fig. 14.

ſcending weights are applyed to two purpoſes; they are to repare the motion in the machine, and likewiſe to raiſe the other weights which were down as low as they could go. And theſe weights, when they are raiſed, are to deſcend again and in their turn to anſwer the ſame purpoſes. For this end ſome one or more of the mechanical powers is made uſe of, by the aſſiſtance of which the deſcending weights may be made to move always with a greater velocity than the aſcending ones: and then, if the quantities of matter in the weights be equal, their moments will be as their velocities, and conſequently the deſcending weights may by this means have a moment given them ſo much greater than the aſcending ones have, as to be ſufficient to anſwer the two purpoſes already mentioned.

It would be endleſs to examine all the inventions of this ſort. The principle, upon which they depend, is applyed in the neateſt manner in the wheel deſcribed by Deſaguliers: and by examining the principle in this one inſtance, we ſhall ſee the fallacy of it, and how little hopes of ſucceſs there is in any other application of it. The wheel Plat. IV. fig. 1. has two rims or circumferences D*e*A B and *r h i f*. The ſpace between the two rims is divided into cells by the ſpokes *q* A, *f t*, *p* D, *h e*, *&c.* which are bent in ſuch a manner as to cauſe weights placed looſe in the cells to deſcend on the ſide A *s t* B from A to B, which is the loweſt point, on the outer rim of the wheel, and to aſcend again from B on the other ſide of the wheel B D in the line B *n m p r*, which comes nearer the center C and touches the inner rim. Thus the weights on the deſcending ſide of the wheel being farther from the center of motion than thoſe on the aſcending ſide, it is imagined that the moment of the former will always be greater than the moment of the latter. The weight A in particular deſcends to B in the arc A *s t* B which is a quarter of a circle; but it riſes from B to D in the curve B *n m p r*, which is leſs than a quarter of a circle. And, ſince the velocity is as the ſpace deſcribed in a given time, the velocity, and conſequently the moment of A, when it deſcends, is greater than its moment, when it aſcends. This will be the caſe of every weight in its turn: therefore the deſcending weights will always have a greater moment than the aſcending one.

The fallacy is this. The velocity with which the weight A deſcends to B is eſtimated by the line A *s t* B, whereas the velocity of all bodies in motion is to be eſtimated in the line of their direction, by the ſecond law of motion. For it is only in the line of direction that any force acts ſo as to communicate motion to a body: and it is in the ſame line only that the body in motion acts, when it communicates motion and is itſelf conſidered as a moving force. From hence it follows, that the velocity with which a weight deſcends, if the weight is conſidered as a moving force, muſt be eſtimated in the line of its direction, that is, in a line drawn from its centèr of gravity to the center of the earth. Or, the velocity of a weight is proportional to the approach of it in any given time to the earths center. In like manner the velocity, with which the weight aſcends, is to be eſtimated in the line of its direction, and in any given time it is as the diſtance to which it is raiſed from the earths center in that time. From hence it appears, that, though the weight A deſcends to B in the curve A*st*B, yet the velocity, with which it deſcends, being as its approach to the earths center, is as AG = CB. And though the ſame weight aſcends in the leſſer curve B*nmpr*, yet the velocity, with which it aſcends, being as the diſtance to which it is raiſed from the earths center in that time, is as E*r* = CB = AG. Therefore the deſcending and aſcending velocities are equal, and conſequently the moment of the deſcending weights will be no greater than the moment of the aſcending ones.

I do not know whether this will be made more intelligible by diſtinguiſhing between the weights velocity whilſt it deſcends, and the velocity with which it deſcends. This may appear to the reader too nice a diſtinction, but I hope he will underſtand my meaning, if he obſerves, that whilſt A deſcends, it does not move directly downwards in the line AG, but moves partly downwards and partly in a horizontal direction, and by both theſe motions together is carryed to B. Since therefore, whilſt it deſcends, it is likewiſe carryed horizontally, I think it is evident that it does not deſcend with its whole velocity. Therefore its velocity whilſt it deſcends, is different from the velocity with which it deſcends, as different as the whole is from its part. Or otherwiſe; whilſt A deſcends to B in the line A*st*B, we may conſider its motion as reſolved into the two ſides of a parallelogram AG and AC = GB. Where the velocity with which it deſcends is AG, and the other part of its velocity AC or GB carries it horizontally. In like manner we may diſtinguiſh between the velocity which the weight has whilſt it aſcends, and the velocity with which it aſcends. For, if, as before, the weight aſcends in the curve B*nmpr*, we may conſider its motion as reſolved into the two ſides of a parallelogram BC = E*r* and BE; where the velocity with which it aſcends is E*r*, and the other part of its velocity BE carries it horizontally. Now the deſcending velocity AG is equal to the aſcending velocity E*r*: and therefore the moment with which the weight deſcends, will be equal to the moment with which it aſcends. The horizontal velocity GB of the deſcending weight is greater indeed than the horizontal

zontal velocity BE of the aſcending one. But, I ſuppoſe it is unneceſſary to prove that this cannot poſſibly contribute any thing towards turning the wheel, or towards making the moment on the deſcending ſide greater than the moment on the aſcending ſide. For, I imagine, the reader will eaſily ſee that a motion in the direction GBE or AC*r* will not contribute in the leaſt towards making the point A deſcend, or towards turning the line ACD round upon the center of motion.

100. *The force of elaſticity cannot produce a perpetual motion.*

Elaſtic ſprings, as has been obſerved already, by expanding themſelves grow weak; and, when they are quite expanded, become entirely unable to produce any motion at all. But in propoſition 83 it was ſhewn that if an elaſtic ball A ſtrikes upon another B, which is greater than itſelf; the moment of B after the ſtroke will be greater than that of A was before it. In like manner, if B was to ſtrike upon C another elaſtic body greater than itſelf, the motion might ſtill be increaſed. And thus by placing more elaſtic bodies in the row, of which each ſhould be greater than the laſt before it, the motion might be increaſed in what proportion we pleaſed. Now why might not motion produced in this manner be applyed to repare what is loſt in a machine by the friction of its parts, and to make the motion of the machine a perpetual one? I intended to ſupply the reader with an anſwer to this queſtion when I deſired him to remember in the inſtance of two bodies, in propoſition 83, that though there is motion produced if we conſider the greater body alone, yet if both bodies are taken together there is no motion produced in the ſame direction; but the moment in the ſame direction is exactly the ſame after the ſtroke, that it was before it. This would be the caſe were there ever ſo many bodies. But a machine can only be moved by moving forces in the ſame direction. For equal moving forces in contrary directions would deſtroy each other and produce no effect on the machine, and the effect produced by unequal ones in contrary directions will be as their difference. Since therefore the moving force, by which the motion of the machine ſhould be repared, is the moment of theſe elaſtic bodies, and ſince their moment in the ſame direction is the ſame after the ſtroke that it was before; it follows, that no ſuch moment is produced by the ſtroke as can keep up the motion of the machine and make it perpetual. For inſtance, ſuppoſe the machine conſiſts of four wheels; call them *c*, *d*, *e*, and *f*. Let *c* be the firſt mover, and let *f* be the laſt. Now if *c* begins to move with 8 degrees of moment and by the friction 2 degrees are loſt when the motion comes to the wheel *f*;

then *f* can return only 6 degrees of motion on *c* the firſt mover. And thus the motion, as it decays in every round, will at laſt be entirely loſt, unleſs ſome method could be found for reparing it. Suppoſe therefore that there were two elaſtic balls A and B, and that A is to B as 2 to 4: then, by propoſition 83, if A ſtrikes B with a moment 6, B will gain by the ſtroke a moment 8. Now if *f* the laſt wheel, which has only the moment 6, was to ſtrike upon A, and A to ſtrike upon B, and B to ſtrike upon the firſt wheel *c*, then this firſt mover would have the ſame motion 8 communicated to it, with which it began to move. And as the two elaſtic bodies would always repare the 2 degrees of motion that are loſt by friction in conveying the motion from *c* to *f*, why might not this contrivance be ſo applyed as to make the machine a perpetual motion? I anſwer, becauſe, though B ſtrikes *c* with 8 degrees of motion, yet A will at the ſame time be reflected with 2 degrees againſt *f*, and ſtriking *f* the laſt mover by reflection will ſtop the machine with a force 2, whilſt B is moving it with a force 8. Therefore by the action of both bodies together, of B impelling *c* with the moment 8, and of A reflected back upon *f* with the moment 2, the machine will be moved only with the difference or 8 — 2 = 6. So that if *c* began to move at firſt with the moment 8, and this moment by being carried round through *d* and *e* to *f* was reduced to 6, the two elaſtic balls A and B will not repare the moment loſt; but the machine will have only 6 degrees of moment, notwithſtanding the ſeeming increaſe of motion by A's ſtriking upon B.

CHAP. VIII.

Of the perpendicular aſcent and deſcent of heavy bodies.

100. *Motion is ſaid to be accelerated, if its velocity continually increaſes; and to be uniformly accelerated, if its velocity increaſes equally in equal times.*

101. *Motion is ſaid to be retarded, if its velocity continually decreaſes; and to be uniformly retarded, if its velocity decreaſes equally in equal times.*

102. *When heavy bodies fall perpendicularly, near the earths ſurface, they are uniformly accelerated by the force of gravity.*

THOSE, who are learning philoſophy, having been taught beforehand, that the force of gravity increaſes as a body approaches to the earths center, are apt to look upon this as the reaſon why a heavy body is accelerated in falling, and imagine that it moves faſter at the end of the fall than it did at the beginning of it, only becauſe the force of gravity, which acts upon it, is conſtantly increaſing. But this is ſo far from being the true reaſon; that, though a force, which is greater at the end of the fall than at the beginning, would accelerate a body; yet it would not accelerate it uniformly. Upon this account, I reſtrained the propoſition to bodies falling near the earths ſurface: for if they fall from inconſiderable heights, the force of gravity is the ſame through the whole line of deſcent; it is no greater when the bodies come to the ground than when they firſt begin to fall thither. The ſemidiameter of the earth is about 4000 miles long, and the greateſt heights from whence we can let a body fall, bear a ſmall proportion to this length. Suppoſe the body let fall from ſome eminence which is 100 yards from the earths ſurface; then at firſt it is 4000 miles + 100 yards from the center, and when it comes to the ground it is 4000 miles from thence. But this difference is ſo inconſiderable in reſpect of the whole that we may look upon the force of gravity as the ſame from the beginning to the end of the fall: and the continued action of ſuch a force as this will produce an uniform acceleration.

For if we conceive the whole time of deſcent to be divided into infinitely ſmall inſtants, which are equal to each other; then the continued force of gravity by acting equally upon the body in each of thoſe inſtants, will communicate equal velocities to it in each of them; that is, the force of gravity will increaſe its velocity equally in equal times or will accelerate it uniformly. To explane this a little farther, let us ſuppoſe that one ſingle impulſe of gravity by acting upon the body in the firſt inſtant of its deſcent communicated to it 1 degree of velocity, but that after this impulſe the force of gravity ceaſed entirely and acted no more upon the body: then, according to the firſt law of motion, the body once moved downwards by this ſingle impulſe would continue to move uniformly in the ſame direction, or to fall without being accelerated or retarded. Thus without any ſecond impulſe of gravity the velocity of the body in the ſecond inſtant of deſcent, would be the ſame that it was in the firſt. But the force of gravity does not ceaſe acting after one impulſe: it acts as ſtrongly, therefore produces as great a velocity, in the ſecond inſtant as it did in the firſt. Therefore the velocity produced by this ſecond impulſe being added to that which the body would have had without it, will make the velocity in the ſecond inſtant double what it was in the firſt. In like manner, if at the end of the ſecond inſtant, the force of gravity was to ceaſe, then the body would continue to fall uniformly with this double velocity. But in the third inſtant another impulſe equal to either of the former acts upon it, and produces a third degree of velocity, which added to what the body had before makes the velocity in this inſtant triple what it was in the firſt. An equal impulſe in the fourth inſtant, will add an equal degree of velocity and make the whole velocity 4 times what it was in the firſt. And by the ſame means in the fifth inſtant, the velocity will be increaſed ſo as to be 5 times what the firſt impulſe of gravity had produced. And thus, in every inſtant 1 degree of velocity being added to what the body had before, equal velocities are produced in equal times, or the body is uniformly accelerated.

103. *The velocity acquired by heavy bodies falling perpendicularly is as the time in which it is acquired.*

This is a neceſſary conſequence of the foregoing propoſition. For if equal velocities are produced in equal times, then in a double time a double velocity will be produced, and in a triple time a triple one: that is an uniformly accelerated velocity increaſes in the ſame proportion with the time, or is directly as the time, in which it is produced.

104. *When a body in a given time has fallen from a certain height, and has acquired any degree of velocity: this laſt acquired velocity, if it was to continue uniform, would carry the body in the ſame time through a ſpace equal to twice the height from whence it fell to acquire it.*

For if we call the laſt acquired velocity V, then $\frac{1}{2}$V will be the middle velocity between the firſt and the laſt. Now a body moving uniformly with half the velocity acquired by falling from a certain height, or with the middle velocity between the leaſt and the greateſt, in a time equal to what it has ſpent in falling, will move juſt as far as it has fallen. But ſince $\frac{1}{2}$V would carry it through a ſpace equal to the height from whence it has fallen, twice this velocity or V, that is the laſt acquired velocity, will carry it in the ſame time through twice the height from whence it fell to acquire it.

Or otherwiſe. Let AB Plat. IV. fig. 2. repreſent the whole time of deſcent, and let BC perpendicular to AB repreſent the velocity acquired in that time, and ſuppoſe the time AB divided into infinitely ſmall inſtants, ſuch as *ei*, *im*, *mp*, *po*, &c. then, to find what velocity a falling body will have at the inſtant *ei*, complete the triangle ABC and draw *ef* parallel to BC. The velocity BC is acquired in the whole time

AB, and the velocity that we are ſeeking is acquired in a part of that time A*e*. But, by the foregoing propoſition, the velocities acquired in different times are to each other as the times themſelves, that is, as AB the whole time of deſcent is to A*e*, ſo is BC the laſt acquired velocity, to the velocity acquired in the time A*e*. And ſince the triangles ABC and A*ef* having one angle at A common and the angles at *e* and B right ones are equiangular, Eucl. b. I. pr. 32. corol. 2. the ſides of theſe triangles are proportional, Eucl. b.VI. pr. 4. or as AB is to A*e* ſo is BC to *ef*. Therefore *ef* is the velocity of the body at the inſtant *ei*, or the velocity acquired by falling for the time A*e*. For the ſame reaſon the velocity at the inſtant *im*, which is acquired in the time A*i*, is proportional to the line *ik*, that is drawn parallel to BC or to the baſe of the triangle ABC: *mn* likewiſe is the velocity in the inſtant *mp* acquired in the time A*m*. And univerſally, when the whole time of deſcent is repreſented by AB the height of a rightangled triangle and the velocity by BC the baſe of it; then a line drawn parallel to the baſe will be as the velocity acquired in ſuch a part of the whole time as is repreſented by the part of AB, which lies between the point A and that line. Now though the body falls with an accelerated motion, ſo that the velocity is conſtantly increaſing and is greater in any inſtant than it was in the preceding inſtant; yet we may ſuppoſe it to continue the ſame during the time of one ſingle inſtant, eſpecially if we take the inſtant infinitely ſhort. Therefore let *ef* repreſent the velocity at the beginning of the inſtant *ei*, and becauſe *ei* is infinitely ſhort a line equal to *ef* will likewiſe repreſent the velocity at the end of it. And becauſe the ſpaces deſcribed, where the velocity is uniform, are, by propoſition 10, as the times and velocities conjointly, it follows that the ſpace deſcribed in the time *ei* with the velocity *ef* will be as the area of a rectangle, the ſides of which are *ei* and *ef*. For the ſame reaſon the ſpace deſcribed in the inſtant *im* with the velocity *ik* will be as the rectangle *mk*. And the ſpace deſcribed in the inſtant *mp* with the velocity *mn* will be as the rectangle *pn*. But if we ſuppoſe the whole time AB divided into inſtants in the ſame manner, then the ſum of all the rectangles, to which the ſpaces deſcribed in each of theſe inſtants are proportional, will fill up the whole triangle ABC; except only the ſmall triangular ſpaces about the line AC. But theſe ſmall ſpaces may be neglected, ſince they ariſe from the conſtruction of the figure not agreeing exactly with the ſuppoſitions that were made. For we ſuppoſed *ei* to be an infinitely ſhort inſtant, and the triangular ſpace *fk* aroſe from not making *ei* infinitely ſhort, ſince if *e* and *i* had been cloſe together, *f* and *k* would have been ſo too, and then there would have been no ſuch ſmall triangle at *fk*. And as we may account for all theſe other ſmall triangular ſpaces in the ſame manner; the ſum of the ſpaces deſcribed in the ſeveral inſtants of the whole time AB to acquire the velocity BC is proportional to the area of the triangle ABC. Therefore univerſally, the lines deſcribed by heavy bodies falling perpendicularly are to each other as the areas of rightangled triangles whoſe heights are as the times of deſcent and whoſe baſes are as the laſt acquired velocities. Thus Plat. IV. fig. 3. the ſpace or line deſcribed in the time AF to acquire the velocity FG, is to the line deſcribed in the time AB to acquire the velocity BC as the area AFG is to the area ABC. But to return to the propoſition and to fig. 2. If a heavy body is falling for the time AB and acquires the velocity BC, then the ſpace deſcribed is as the area of the triangle ABC. But if with the laſt acquired velocity BC, it was to move on uniformly for an equal time AB then, by propoſition 10, the ſpace deſcribed would be as the area of the rectangular parallelogram ABCD. But the parallelogram ABCD is double the triangle ABC upon the ſame baſe BC and between the ſame parallels AD and BC. Eucl. b. I. pr. 41. Therefore a body by falling from a certain height in a given time acquires a degree of velocity, which, if it was to continue uniform, would carry the body in the ſame time through a ſpace equal to twice the height from whence it fell to acquire this velocity.

Near the ſurface of the earth a heavy body deſcends through $15\frac{1}{12}$ French feet or $16\frac{1}{10}$ Engliſh feet in 1 ſecond of time. For the ſake of expreſſing this fractional quantity $16\frac{1}{10}$ feet by an integer, ſuppoſe it $16\frac{1}{2}$ feet, then E the ſpace through which a heavy body falls in one ſecond of time will be equal to an Engliſh meaſure that is called a rod or pole. Therefore, by what has been proved, the velocity that is acquired by a heavy body in falling through 1 pole, or for 1 ſecond of time, would carry it uniformly through 2 poles in an equal time.

The ſame thing may be expreſſed arithmetically. If a body is in motion with 6 degrees of velocity, and moves on uniformly for 7 inſtants, then by propoſition 10, the ſpace deſcribed will be as $6\times7=42$, that is, the ſpace deſcribed will be as the velocity multiplyed into the time, or as the ſum of all the velocities, which the body had in each inſtant. But if the body from being at reſt had been uniformly accelerated for 7 inſtants then its velocity in the ſeveral inſtants reckoning from the beginning of the fall would have been 0. 1. 2. 3. 4. 5. 6. and the ſpace deſcribed in the 7 inſtants being as the ſum of all theſe velocities would be as $0+1+2+3+4+5+6=21$. Whereas we have ſeen already that the laſt velocity, or 6, will in 7 inſtants carry the body through a ſpace as 42, which is double 21, or double the height from whence the body fell to acquire this velocity. And univerſally, the ſpace which a body deſcribes with a velocity uniformly accelerated is as the ſum of a number of terms in arithmetical progreſſion, beginning from 0 and ending with the greateſt velocity; and the ſpace which it would deſcribe in the ſame time with the laſt velocity continuing uniform is as the ſum of an equal number of terms, each of which is equal to the laſt in that progreſſion. And ſince this latter ſum will always be double the former, the ſpace deſcribed with an uniform velocity in a given time will always be double the height from whence the body fell to acquire that velocity.

105. *The ſpaces deſcribed in unequal times by a heavy body falling perpendicularly are, if we reckon from the beginning of the fall, as the ſquares of the times.*

Thus, if a body falls through 1 pole in 1 ſecond of time, then the number of poles, that it will fall through in 2 ſeconds, is to the number it falls through in 1 as 2^q to 1^q or as 4 to 1. For the ſame reaſon in 3 ſeconds, it will fall through 3^q or 9 poles, in 4 through 4^q or 16, &c.

For let AB Plat. IV. fig. 3. repreſent the whole time of a bodys deſcent, and AF a part of that time; let BC repreſent the laſt acquired velocity, and FG the velocity acquired in the time AF. Then, by what was ſaid in proving the foregoing propoſition, the ſpace deſcribed in the time AF is to the ſpace deſcribed in the time AB as the area of the triangle AFG is to the area of the triangle ABC. But ſince theſe triangles are equiangular, the area AFG is to the area ABC as the ſquare of AF to the ſquare of AB, Eucl. b.VI. pr.19. that is, the ſpaces deſcribed being as AFG to ABC are as the ſquare of the time AF to the ſquare of the time AB.

The ſpaces deſcribed by bodies in motion are as the time multiplyed into the velocity, or, calling the time T and the velocity V, as $T\times V = TV$. But in uniformly accelerated motions V is as T, by propoſition 49. Therefore TV or the ſpace deſcribed is as TT the ſquare of the time.

I cannot leave this propoſition without applying it to one caſe, where it is of ſingular uſe; and that is to determine how far an heavy body, which is as far from the earth as the moon is, would be made to fall towards the earth in 1 minute. A heavy body near the earths ſurface falls through 1 pole in 1 ſecond of time. Therefore, by this propoſition, in

1 minute, or 60 ſeconds, it will fall through the ſquare of 60 or through 3600 poles. Now the diſtance of the body from the earths center, when it is near the ſurface, is only 1 ſemidiameter of the earth; but if it was as far off as the moon is, then the diſtance from the center would be 60 ſemidiameters. But, by propoſition 23, the force of gravity at different diſtances from the center is as the ſquares of thoſe diſtances reciprocally. Therefore, if we call the force of gravity at the earth's ſurface 1, where the diſtance is 1 ſemidiameter, then at the diſtance of the moon or 60 ſemidiameters this force will be as much leſs as $60^q=3600$ is greater than $1^q=1$, that is the force of gravity at the earths ſurface is to the ſame force at the diſtance of the moon as 1 to $\frac{1}{3600}$. And conſequently the force of gravity, which at the earths ſurface makes a heavy body fall though 3600 poles in a minute, will, at a diſtance equal to that of the moon from the earth, make a heavy body fall through $\frac{1}{3600}$ of this, or through 1 pole in a minute. So that a heavy body, which falls 1 pole in a ſecond of time near the earths ſurface, would take up a minute to fall 1 pole, if it was as far off as the moon.

106. *The ſpaces deſcribed by a heavy body falling perpendicularly are, if we reckon from the beginning of the fall, as the ſquares of the laſt acquired velocities.*

The ſpaces deſcribed to acquire the velocities GF and CB Plat. IV. fig. 3. are to each other as the triangle AGF to the triangle ACB. But theſe triangles being equiangular are to each other as the ſquare of GF to the ſquare of CB, Eucl. b. VI. pr. 19. that is, the ſpaces deſcribed are as the ſquares of the velocities which are acquired in deſcribing them.

The ſpaces deſcribed by bodies in motion are as the time multiplyed into the velocity; or, calling the time T, and the velocity V, as $T\times V=TV$. But in uniformly accelerated motions V is as T, therefore TV or the ſpace deſcribed is as VV the ſquare of the laſt acquired velocity.

If we can by any means find out what proportion the velocities, which two bodies have acquired in falling, bear to each other, then by this propoſition we may determine the difference of the heights from whence they have fallen. For ſuppoſe their laſt acquired velocities to be in the proportion of 2 to 6, then the ſpaces deſcribed are as 2^q to 6^q or as 4 to 36.

107. *The time of deſcent from unequal heights, and alſo the acquired velocities, are as the ſquare roots or in a ſubduplicate ratio of thoſe heights.*

This is a neceſſary conſequence of the two laſt propoſitions. For ſince the ſpaces deſcribed are as the ſquares of the times; therefore the times of deſcent muſt be as the ſquare roots of the ſpaces, Thus, calling the ſpace deſcribed S, and the time of deſcent T, ſince TT is as S, T will be as $\sqrt{S}$ or as the ſquare root of the ſpace deſcribed.

And ſince the ſpaces deſcribed are as the ſquares of the laſt acquired velocities, therefore thoſe velocities are as the ſquare roots of the ſpaces. Thus, calling the velocity V, ſince VV is as S, V will be as $\sqrt{S}$.

Suppoſe a body to fall 1 pole in a ſecond, I aſk how long it will be in falling 25 poles. The anſwer, by this propoſition, is 5 ſeconds. For the time of deſcent through 25 poles is to the time of deſcent through 1 as the ſquare root of 25 to the ſquare root of 1; or as 5 to 1.

If two bodies, which contain unequal quantities of matter, are let fall from heights, which are as thoſe quantities inverſely; then, ſince by this propoſition, the velocities acquired will be as the ſquare roots of thoſe heights, it follows, from what we have here ſuppoſed, that the velocities will be as the ſquare roots of the quantities of matter inverſely. Let the body A weigh 9 ounces and the body B 25. Then, if A falls from a height of 25 feet, and B from a height of 9 feet, the ſpaces which they deſcribe in falling, will be as their quantities of matter inverted. Now, by this propoſition, the velocity which A acquires in falling through 25 feet is to the velocity which B acquires in falling through 9, as 5 to 3. But the ſquare root of A's quantity of matter, or 9 ounces, is to the ſquare root of B's, or 25 ounces, as 3 to 5. Therefore the laſt acquired velocities being to each other as 5 to 3 are as the ſquare roots of the quantities of matter inverted.

108. *The ſpaces which a heavy body falls through in equal parts of time, when thoſe parts are taken ſeparately, are, if we reckon from the beginning of the fall, as the uneven numbers* 1. 3. 5. 7. &c.

For ſince, by propoſition 105, a heavy body falls 4 poles in 2 ſeconds, therefore, as it falls 1 pole of the 4 in the firſt ſecond, it muſt fall 3 in the next. Since it falls 9 poles in 3 ſeconds, 1 of them in the firſt, and 3 in the next, therefore it muſt fall the remaining 5 in the third. Since it falls 16 poles in 4 ſeconds, 1 of them in the firſt, 3 in the next, and 5 in the third, it muſt fall the remaining 7 in the fourth.

109. *The motion of a heavy body, when it is thrown perpendicularly upwards, is uniformly retarded.*

When a body is falling the impulſes of gravity act in the ſame direction in which the body moves, and therefore by communicating equal degrees of velocity in equal times, they produce an uniform acceleration. But when the body is thrown upwards theſe impulſes act in a direction contrary to the bodys motion, and therefore by deſtroying equal degrees of velocity in equal times, they will retard it uniformly.

110. *Whatever is the height to which a body aſcends when thrown perpendicularly upwards, it would have aſcended to the ſame height in half the time, if the firſt velocity had continued uniform.*

For as the laſt velocity, when the body has been uniformly accelerated, would in a given time, if this velocity was afterwards to continue uniform, carry the body through twice the ſpace, which was deſcribed to acquire it; ſo for the ſame reaſons, if the firſt velocity of an aſcending body was to continue uniform, it would carry the body, in a given time, to twice the height to which it will aſcend, when that velocity is uniformly retarded. Therefore this firſt velocity, which would carry the body to double the height in the ſame time, muſt, by propoſition 9, carry it to the ſame height in half the time.

111. *The heights, to which heavy bodies aſcend, when they are thrown perpendicularly upwards, are to each other as the ſquares of the times ſpent in aſcending.*

For, as in uniformly accelerated motions, ſo in uniformly retarded ones, the ſpaces deſcribed are to each other as the areas of rightangled triangles, whoſe heights are the times of aſcent and whoſe baſes are the greateſt velocity. The only difference is, that in uniformly accelerated motions this greateſt velocity is the laſt, or that, which is acquired in falling; and, in uniformly retarded ones, it is the firſt, or that, with which the body is thrown upwards.

Let the velocities, with which two bodies are thrown perpendicularly upwards, be to each other as GF to CB, Plat. IV. fig. 3, then, becauſe theſe velocities are uniformly retarded, that is, becauſe they decreaſe as the times of aſcent increaſe, it follows that the time in which the velocity GF is loſt will be to the time in which CB is loſt as GF is to CB, or as AF to AB. And the ſpace deſcribed by the body whoſe velocity is GF, will be to the ſpace deſcribed by that whoſe velocity is CB as the area AGF to the area ACB or as the ſquare of AF to the ſquare of AB, Eucl. b.VI. pr. 19. that is, the ſpaces deſcribed will be as the ſquares of the times, in which they are deſcribed.

The ſpaces deſcribed by bodies in motion are as the time multiplyed into the velocity, or calling the time T and the velocity V, the ſpaces de-

deſcribed are as T × V = TV. But in uniformly retarded motions V or the velocity loſt in aſcending is always as T or as the time of aſcent, therefore TV, the ſpace deſcribed is as TT or as the ſquare of the time.

112. *When heavy bodies are thrown perpendicularly upwards the heights, to which they aſcend, will be as the ſquares of their firſt velocities.*

After what has been ſaid already the reader cannot want a diſtinct proof of this propoſition, he need only be reminded, that when the velocities are as GF to CB, Plat. IV. fig. 3. the heights are as the areas AGF, ACB; that is, as the ſquare of GF to the ſquare of CB. Or, that ſince the ſpaces deſcribed are as TV, and in uniformly retarded motions V is as T, therefore TV the ſpaces deſcribed are as VV the ſquares of the velocities loſt in aſcending.

From this propoſition it follows, that whatever is the height to which any force will throw a bullet; a double force communicating twice the velocity to the ſame bullet would throw it to 4 times that height, and a triple force would throw it to 9 times that height. Becauſe in all caſes the height increaſes as the ſquare of the velocity with which the bullet is thrown.

113. *The times of aſcent, as alſo the firſt velocities, are as the ſquare roots of the heights to which the bodies riſe.*

This is a neceſſary conſequence of the two laſt propoſitions; and needs no other ſort of proof than what the reader will remember was made uſe of in proving propoſition 107.

114. *A heavy body, when thrown perpendicularly upwards, aſcends to ſuch a height that in falling back again it will acquire a velocity equal to that, with which it was thrown.*

For the firſt velocity is as the ſquare root of the line of aſcent, by the laſt propoſition; and the velocity, which the body acquires in falling back again, is as the ſquare root of the line of deſcent, by propoſition 107. But ſince the body falls in the ſame line through which it had riſen, the line of deſcent is equal to the line of aſcent. Therefore their ſquare roots are equal; and conſequently the velocities, which are as thoſe roots, that is, the velocity acquired in falling, and the velocity, with which the body was thrown at firſt, muſt be equal too.

The ſame impulſes of gravity, which retard the motion of the body in the line of aſcent, accelerate it in the ſame manner in the line of deſcent. Therefore they will communicate in one caſe as much velocity as they deſtroy in the other, or the ſame velocity, that was loſt in aſcending, will be acquired again in deſcending.

115. *When a heavy body has fallen from any height, if its direction could be inverted without changing the velocity that it has acquired; this velocity would make it riſe to the ſame height from whence it fell. And the time of aſcent would be equal to the time of deſcent.*

For the line of deſcent is as the ſquare of the laſt acquired velocity, by propoſition 106, and when this acquired velocity, by inverting the bodies direction, becomes the firſt velocity of aſcent, then the line of aſcent will be as the ſquare of the ſame velocity, by propoſition 113. And ſince the ſame ſquare is equal to itſelf the line of aſcent will be equal to the line of deſcent; or the body will riſe to the ſame height from whence it fell.

The ſame impulſes of gravity, that accelerated the body in its deſcent, will retard it, when it is turned back again and made to riſe with the laſt acquired velocity. Therefore the velocity, which was communicated by theſe impulſes whilſt the body deſcended, will not be loſt in aſcending but by an equal number of contrary impulſes, that is, not till an equal time has paſſed, and the body has riſen to the ſame height from whence it fell.

The time of aſcent is, by propoſition 113, as the ſquare root of the line of aſcent, and the time of deſcent as the ſquare root of the line of deſcent; and ſince by the former part of this propoſition the body riſes to the ſame height from whence it fell, that is, ſince the line of aſcent is equal to the line of deſcent; therefore the ſquare roots of thoſe lines will be equal or the time of aſcent will be equal to the time of deſcent.

From this propoſition it follows, that, becauſe the ſpaces through which a body deſcends in equal parts of time, when taken ſeparately, are as the uneven numbers 1, 3, 5, 7, &c. by propoſition 108; the ſpaces through which a heavy body thrown perpendicularly upwards aſcends in equal parts of time, when taken ſeparately, are, if we reckon from the inſtant of projection, as the uneven numbers taken backwards. For the velocity in the line of aſcent, and conſequently the ſpace run through in a given time, is every where the ſame as in the line of deſcent. So that if a body was 5 ſeconds in falling, the ſpaces run through in each ſecond taken ſeparately, would be as 1, 3, 5, 7, 9. And if it was thrown upwards with ſuch a force as would make it 5 ſeconds in riſing to its greateſt height, the ſpaces run through in each ſecond would be as 9, 7, 5, 3, 1.

116: *When the times are known, which heavy bodies have ſpent in falling, the heights from whence they fell may be found.*

For if they fall 1 pole in 1 ſecond of time, then, by propoſition 105, they fall 4 poles in 2 ſeconds, 9 poles in 3 ſeconds. And univerſally the number of poles, through which they have fallen, is always equal to the ſquare of the number of ſeconds, which they have ſpent in falling.

The reader is deſired to obſerve, that in this propoſition and the following one, I ſuppoſe the bodies to meet with no reſiſtance. So that neither of them can be applyed with exactneſs, to meaſure either the heights from whence bodies fall, or the heights to which they riſe in ſuch a reſiſting fluid as the air is.

117. *Suppoſing a body, that is thrown perpendicularly upwards, to move in a medium void of reſiſtance, the height, to which it aſcends may be determined,*

The time ſpent from its firſt ſetting out to the moment it returns to the place it was thrown from, is the time of aſcent and deſcent together. But as the time of aſcent is equal to the time of deſcent; half of this is equal to the time of deſcent alone. And, when this is known, the height from whence the body has fallen, which is likewiſe the height that it aſcended to, may be determined by the laſt propoſition. If the time of aſcent and deſcent together is 6 ſeconds: then the time of deſcent alone or the half of this is 3 ſeconds. Therefore, by the laſt propoſition, the height from whence it has fallen; or the height to which it had aſcended, is 9 poles.

118. *The moment of a given body, which is uniformly accelerated, is not as the ſquare of the laſt acquired velocity.*

The foreign philoſophers differ from Sir Iſaac Newton in the manner of eſtimating the moments of bodies. His rule has been explaned already, when I ſhewed that the moment is as the quantity of matter, when the velocity is given; as the velocity, when the quantity of matter is given; and univerſally in a ratio compounded of the quantity of matter and velocity. They on the other hand maintain, that the moment is indeed as the quantity of matter, when the velocity is given, but as the ſquare of the velocity, when the quantity of matter is given, and therefore univerſally in a ratio compounded of the quantity of matter and ſquare of the velocity. One of the arguments, which they make uſe of to prove this theory of the moment will be beſt underſtood now we are ſpeaking of uniformly accelerated motions.

They ſay that if you put a body in motion and give it any degree of velocity by the action of ſome external force ſuch as the preſſure of your hand againſt it; then in order to double this degree of velocity you muſt double the force of your hand. For if your hand was to keep preſſing the body only with the ſame force that it did at firſt, they would both keep moving on together in the ſame direction and with the ſame velocity, but by doubling the force you will double the bodys velocity. In like manner to triple the velocity of the body, your hand muſt preſs it with a triple force; to give a fourth degree of velocity, you muſt preſs it with four times the force that you made uſe of to give it one degree; and univerſally the force with which you preſs the body muſt always be as the velocity, which you communicate to it. In the triangle ABC Plat. IV. fig, 2. let *ef*, drawn parallel to the baſe BC, repreſent any degree of velocity communicated to a body by the preſſure of your hand againſt it: then to increaſe this velocity and make it *ik*, the force of your hand muſt be increaſed in the proportion of *ik* to *ef*: to give it the velocity *mn*, the preſſure muſt be as *mn*, to give it the velocity *pq*, the force acting upon it muſt be as *pq*; and to give it the laſt velocity BC the moving force of your hand muſt be as BC. Therefore if in the time AB the body, by the preſſure of your hand againſt it, is uniformly accelerated and acquires at laſt the velocity BC, the ſum of the moving forces, which have acted upon it in that time, will be as the ſum of innumerable lines drawn from A to B in the triangle ABC parallel to the baſe of it, in the ſame manner that *ef*, *ik*, *mn*, &c. are drawn. But all theſe lines together will fill up the area of the triangle ABC. Therefore the ſum of all the moving forces are as the area of the triangle, whoſe height is the time of acceleration, and whoſe baſe is the laſt acquired velocity.

Suppoſe therefore that one body in the time AF, Plat. IV. fig. 2. acquires the velocity GF, and that another body in the time AB acquires the velocity BC, then, by what has been ſaid, the ſum of the moving forces that have acted upon the former of them, is to the ſum of the moving forces that have acted upon the latter; as the area AGF to the area ACB; that is, ſince the triangles are equiangular, as the ſquare of GF to the ſquare of CB. Eucl. book VI. prop. 19. Therefore the moving force or rather the ſum of the moving forces, by which a body is accelerated, is as the ſquare of the laſt acquired velocity. And, ſince the quantity of motion or moment is, by the ſecond law of motion, as the moving force, the moment muſt likewiſe be as the ſquare of the laſt acquired velocity.

What has been already ſaid concerning the acceleration of a body by the force of gravity might ſerve to confute this whole argument. For when the firſt impulſe of gravity has communicated one degree of velocity to it, a ſecond impulſe, which is only equal to the former, will double the velocity, and a third equal impulſe will triple it. So that in the caſe of this acceleration we find that to double the velocity, which a body has, a double force need not be made uſe of, nor a triple one to triple it: neither is the ſum of the accelerating forces as the ſquares of the laſt velocity, but as the velocity itſelf.

But the defenders of the opinion, againſt which I am diſputing, have an anſwer ready for this objection. They diſtinguiſh between an internal force and an external one. Gravity, they ſay, is an internal force; it acts upon bodies in motion, juſt as it does upon bodies at reſt: and therefore as one impulſe would produce one degree of velocity; ſo the ſecond impulſe, though only equal to the former, will have the ſame effect upon the body in motion as the firſt had upon it before it began to move, and will produce a ſecond degree of velocity; and a third equal impulſe will for the ſame reaſon produce a third degree. But an external force does not act upon a body in motion and at reſt in the ſame manner. A force which preſſes on the outſide of a body may produce one degree of velocity, but if it continues to preſs with the ſame force, then inſtead of doubling or tripling the bodys velocity, it will move on cloſe to the body without altering its motion at all. To double the velocity you muſt preſs with twice the force, and to triple it the force itſelf muſt be tripled.

This miſtake in the caſe of an external force, ſuch as our hand is, has ariſen from not diſtinguiſhing between the force with which our hand moves, and the force with which it ſtrikes or impells the body. For though the ſum of the moving forces in our hand is as the ſquare of the velocity laſt acquired by the body in motion, yet the moment of the body will not be in this proportion, unleſs the ſum of the forces, with which the hand has impelled it, is likewiſe as the ſquare of this velocity. But it is eaſy to ſhew, that the ſum of theſe impulſes, or forces acting upon the body and communicating motion to it, are in a ſimple ratio of the velocity, which it acquires. For ſuppoſe with a force or velocity of your hand as 1, you communicate 1 degree of velocity to the body impelled by it : then to double this velocity in the body you double the force, or, which is the ſame thing, you double the velocity of your hand. But though your hand has the moving force 2, yet the moving force with which it acts upon the body will not be 2. Becauſe they both move in the ſame direction the body with the velocity 1 your hand with the velocity 2 ; and conſequently the quantity of the ſtroke, or moving force which acts upon the body, will, by propoſition 94, be as the difference of theſe velocities, or as 2—1=1. In like manner, when you would triple the bodys velocity you triple the velocity or moving force of your hand ; but as the body is already moving in the ſame direction with 2 degrees of velocity, the quantity of the ſtroke, the impulſe, or moving force, will be as the difference of the velocities in the ſame direction or as 3—2=1. If when the body is moving with 3 degrees of velocity you want to communicate a fourth, you muſt move your hand againſt it with 4 degrees of velocity. But here again though your hand has the velocity and conſequently the moving force 4, yet it will act upon the body with only the force 1. Becauſe the body and your hand are moving in the ſame direction one with the velocity 3 the other with the velocity 4 ; therefore, by propoſition 94, the quantity of the ſtroke of your hand againſt the body is as the difference between 4 and 3, or as 4—3=1. Thus each impulſe or moving force of your hand againſt the body is only 1, whatever may be the moving force in your hand itſelf ; and as each of theſe impulſes communicates 1 degree of velocity ; the number of impulſes, that is, the ſum of the moving forces which have acted upon the body, will be proportional to the number of degrees of velocity communicated. And conſequently the moment or motion produced by theſe forces will be as the ſum of them or as the velocity of the body.

CHAP. IX.

Of the deſcent of heavy bodies upon inclined planes.

119. *A power will ſupport a weight upon an inclined plane, if the power is to the weight, as the height of the plane is to the length of it.*

IF a power at R, Plat. IV. fig. 4. that acts upon a weight B in the direction BR parallel to the inclined plane AC, ſupports the weight and keeps it at reſt ; then the power R is to the weight B as AD the perpendicular height of the plane to AC the length of it.

The weight upon the inclined plane is acted upon by three forces: The firſt is its own gravity in the line of direction BE. The ſecond is the force of the inclined plane, which, by throwing it out of the line of direction or preventing it from falling in the line BE, makes it fall in the line BC. This force acts in the direction BH perpendicular to the plane, as all bodies, by the ſecond law of motion, act in lines perpendicular to their ſurfaces. The third force is the power R, which acts in the direction BR. But, by propoſition 21, if a body is acted upon by three forces at the ſame time and reſts between them, then theſe forces are to each other as the three ſides of a triangle reſpectively parallel to the directions in which they act. In the triangle GBF the ſide BF is parallel to BH or to the direction in which the plane acts, the ſide BG is parallel to BR or to the direction in which the power acts, and the ſide GF is parallel to BE or to the line of direction in which the gravity of the body acts. Therefore ſince theſe three forces are to each other reſpectively as the three ſides of this triangle, the power is to the weight as GB to GF. We muſt therefore prove that GB is to GF as AD the height of the plane to AC the length of it. Now BE is perpendicular to DC, becauſe DC is a horizontal plane and BE is the line of perpendicular deſcent. AD is likewiſe perpendicular to DC, becauſe AD is the perpendicular height of the plane. From whence it follows that AD and BE are parallel. Eucl. b. XI. pr. 6. And as GF, by the conſtruction, is parallel to BE, it is likewiſe parallel to AD. Eucl. b. I. pr. 30. Therefore CA, croſſing GF and AD, will make the external angle CGF equal to CAD the internal oppoſite one on the ſame ſide. Eucl. b. I. pr. 29. Again ADC is a right angle becauſe AD, by the conſtruction, is perpendicular to DC and GBF is a right one, becauſe HF, by the conſtruction, is perpendicular to AC. Therefore in the triangle GBF there are two angles reſpectively equal to two angles in the triangle ADC, from whence it follows, that the third angle in each will be equal, Eucl. b. I. pr. 32. corol. 2. and the ſides about the equal angles will be proportional, Eucl. b. VI. pr. 4. GB to GF, or the power to the weight, as AD to AC, or as the height of the plane to the length of it.

We may prove the ſame propoſition in an eaſier manner. For when the weight is drawn quite up to the top of the plane from C to A, the power in its line of direction muſt have deſcribed a ſpace equal to CA; but the weight in the mean time, though it has paſſed over CA, has deſcribed in its line of direction no more than a ſpace which is equal to AD, or to the diſtance through which it has riſen from the earths center. But the velocity of the power and weight are to each other as the ſpaces which they have deſcribed in the ſame time in their reſpective lines of direction. Therefore the velocity of the power is to the velocity of the weight as CA the length of the plane to DA the height of it. And conſequently, if the power is to the weight as DA to CA, or as the height of the plane to the length of it, then they will be to each other as their velocities inverſely, and by propoſition 14, their moments will be equal, or the power will be ſufficient to ſupport the weight.

120. *The accelerating force of a bodys gravity, when it deſcends upon an inclined plane, is to the accelerating force of its gravity, when it falls perpendicularly, or to the whole force of its gravity, as the height of the plane is to the length of it.*

The force wherewith a body endeavours to deſcend, or the accelerating force of its gravity upon an inclined plane is equal to the power that is able to ſupport it. But, by the laſt propoſition, this power is to the whole weight of the body, as the height of the plane is to the length of it. Therefore the force, with which it endeavours to deſcend on an inclined plane, is to the whole weight in the ſame proportion.

121. *Heavy bodies deſcend upon inclined planes with an uniformly accelerated motion.*

The whole accelerating force of a bodys gravity, or the force wherewith it endeavours to deſcend perpendicularly, is always the ſame; and the accelerating force of gravity upon every part of the plane, whilſt the body is deſcending from A to C, has the ſame proportion to the whole force. For, by the laſt propoſition, the gravity upon an inclined plane is to the whole gravity as the height of the plane is to the length of it. Since therefore the whole gravity is given and the gravity upon the plane has always the ſame proportion to it, it follows that the gravity upon the plane muſt be given too. But the conſtant action of an uniform force, ſuch as the gravity of a body upon an inclined plane has been proved to be, will produce equal degrees of velocity in equal times, that is, will cauſe an uniformly accelerated motion.

From hence it follows that all the propoſitions, which have been already proved in relation to the uniformly accelerated motion of a body falling perpendicularly, are true likewiſe of a body falling down an inclined plane. Therefore, without repeating either the propoſitions or the demonſtrations, I ſhall refer the reader to the foregoing chapter. But muſt deſire him to remember that the accelerating force upon an inclined plane is leſs than the accelerating force of perpendicular deſcent, in the proportion of the planes height to its length.

122. *The velocity, which a heavy body acquires in any given time upon an inclined plane, is to the velocity, which it would acquire in the ſame time by perpendicular deſcent, as the height of the plane is to its length.*

In an uniformly accelerated motion the velocities produced in equal times are as the cauſes, which produce them, that is, as the accelerating forces. But, by propoſition 120, the accelerating force upon an inclined plane is to the accelerating force in perpendicular deſcent as the height of the plane to the length of it. Therefore the velocities produced in equal times are in the ſame proportion.

123. *The ſpace, which a heavy body deſcribes in a given time upon an inclined plane, is to the ſpace, which it would deſcribe in the ſame time by falling perpendicularly, as the height of the plane is to the length of it.*

The ſpaces deſcribed in a given time are as the velocities, by propoſition 10. And the velocities produced in a given time in two bodies, one of which falls upon an inclined plane and the other perpendicularly, are, by the foregoing propoſition, as the height of the plane to the length of it. This therefore will likewiſe be the proportion of the ſpaces, which they deſcribe in the ſame time.

Let AB, Plat. IV. fig. 5. be an inclined plane and ſuppoſe a heavy body was to fall perpendicularly from A to C, or through a ſpace equal to the height of the plane. How far would another heavy body fall in the mean time upon the plane from A towards B? The ſpaces deſcribed in the ſame time will be to each other as the height of the plane to the length of it. Therefore the ſpace ſought is to AC, or to the ſpace deſcribed by the body which falls perpendicularly, as the ſame line AC is to AB, or as the height of the plane is to the length of it. Now the angle ACB is always a right one; becauſe CB is an horizontal plane, and AC is the perpendicular height of the inclined one. And, if from the angle C in the rightangled triangle ACB you draw the line CD perpendicular to AB, then ADC is ſimilar to ACB; Eucl. b.VI. pr. 8. and AD is to AC as AC to AB. Eucl. b.VI. pr. 4. Therefore AD is the ſpace ſought. So that, if the two bodies were let fall from A at the ſame inſtant, when one had fallen perpendicularly to C, the other would have fallen upon the plane no farther than D.

124. *An heavy body takes up the ſame time in falling either perpendicularly through the diameter of a circle, or upon an inclined plane through any chord of it.*

A heavy body will fall through AD, Plat.IV. fig. 6. or AE, or any other chord of the circle ADBH in the ſame time, that it would fall perpendicularly through the diameter AB.

For if from B, the loweſt point in the diameter, a line be drawn to D, the loweſt point of the chord; then the angle, which this line makes with the chord, being in a ſemicircle will always be a right one. Eucl. b. III. pr. 31. Upon which account from the anſwer to a queſtion, that we were led to by the laſt propoſition, it follows, that, conſidering the chord AD as part of an inclined plane AC, the chord will always be to the diameter as AB, which is the height of the plane, to AC, which is the length of it. Therefore, by the laſt propoſition, AD the chord and AB the diameter will be deſcribed in the ſame time.

In like manner, if the chord AE is continued to G, it might be proved, that a body will fall in the ſame time either through the chord AE, or through the diameter AB. And if we draw a chord HB parallel to AD; then, becauſe they are parallel, they are equally inclined: therefore the accelerating force of gravity, when a body is to deſcend upon HB, will be the ſame as when it is to deſcend upon AD. And, becauſe they are drawn from oppoſite ends of the diameter and are parallel, therefore they are at equal diſtances from the center, and HB is equal to AD. Eucl. b. III. pr. 14. But ſince the two inclined planes AD and HB are equal to each other and the accelerating force of gravity is the ſame upon both, it follows, that a heavy body will take up an equal time to deſcribe either of them, and that the time of oblique deſcent through HB as well as through AD is equal to the time of perpendicular deſcent through the diameter. The ſame may be proved of the chord FB, which is parallel to AE; and of any chord, that can be drawn in the circle ADBH.

125. *The time in which a body deſcribes an inclined plane is to the time in which it would fall perpendicularly from the ſame height, as the length of the plane is to the height of it.*

The time that a body takes up to run from A to B, Plat. IV. fig. 5. is juſt as much longer than the time it would take up to fall perpendicularly from A to C as AB is longer than AC.

To prove this we muſt remember, what has already been taken notice of, that AD is to AC as AC is to AB, or that AD, AC, and AB are in continued geometrical progreſſion. From whence it follows, Eucl. b.V.

b. V. def. 10. that the firſt of theſe quantities AD is to the third AB as the ſquare of the firſt, or AD^q, is to the ſquare of the ſecond, or to AD^q. The reader may illuſtrate this concluſion by trying it in numbers. 3. 6. 12. are in continued geometrical progreſſion for 3 is to 6 as 6 is to 12. And 3 the firſt number is to 12 the third, as 9 the ſquare of the firſt is to 36 the ſquare of the ſecond. For as three is contained 4 times in 12 ſo is 9 in 36.

This being premiſed, let us firſt enquire what proportion the time of deſcent through a part of the plane AD bears to the time of deſcent through the whole plane AB. Now, by propoſition 121, the motion of a body deſcending upon an inclined plane is uniformly accelerated; and, by propoſition 105, the ſpaces deſcribed by uniformly accelerated motions are as the ſquares of the times, in which they are deſcribed. Therefore as AD is to AB ſo is the ſquare of the time in which AD is deſcribed to the ſquare of the time in which AB is deſcribed. So that if we can find any two quantities, whoſe ſquares are to each other as AD to AB, theſe two quantities are as the times of deſcent through AD and AB. But by what was premiſed AD^q is to AC^q as AD to AB. Therefore the time of deſcent through AD is to the time of deſcent through AB as AD to AC. And by what has been ſhewn already AD is to AC as AC to AB. From whence it follows that the time of deſcent through AD is to the time of deſcent through AB as AC the height of the plane is to AB the length of it. And, ſince, by propoſition 123, a heavy body will take up the ſame time to deſcend either through AD upon the inclined plane or through AC perpendicularly, we may conclude that the time of perpendicular deſcent through AC is to the time of oblique deſcent along the inclined plane AB, as AC the height of it is to AB the length of it.

126. *The velocity, which a body acquires in falling from a given height is the ſame; whether it has fallen perpendicularly or along a plane inclined in any manner whatſoever.*

The velocity acquired by a heavy body is the ſame when it has fallen from the point A to the line CB, Plat. IV. fig. 5. whether it has fallen perpendicularly through AC or along the inclined plane AB. The velocity produced in a given time is as the accelerating force that produces it, by propoſitions 12. 31, and the velocity produced by a given accelerating force is as the time of producing it, by propoſition 103. Therefore univerſally the velocity is as the accelerating force and time conjointly; and conſequently if the accelerating force is inverſely as the time then the velocity produced will be a given quantity. But the accelerating force upon an inclined plane is to the accelerating force in perpendicular deſcent, by propoſition 120, as the height of the plane to the length of it, and the times of deſcent, by the laſt propoſition, are to each other as the length to the height. Therefore the accelerating forces are inverſely as the times, and the velocities produced will be equal.

This may be otherwiſe expreſſed. The force of gravity, which acts upon a body deſcending perpendicularly from A to C is greater than the force which acts upon it when it deſcends from A to B. But then as much as this latter force is leſs than the former, juſt ſo much longer it acts, and therefore in the end both will produce an equal effect, that is, an equal velocity in the body.

From what has been ſaid it follows that a body by falling from the line AK, Plat. IV. fig. 7. into the line CB, that is, by falling from a given perpendicular height, will acquire the ſame velocity, whether it falls through the plane AD, or AB, or KB, ſince in falling through any one of them it acquires the ſame velocity as in falling through AC, which is the perpendicular height of them all.

127. *Heavy bodies by falling from a given perpendicular height acquire the ſame velocity, whether they fall through one plane, or through more, which are contiguous to each other.*

The perpendicular diſtance of the line HE, Plat IV. fig. 8. from the line DF is HD: and a body will acquire the ſame velocity in falling through this diſtance either along the inclined plane ED or along the three contiguous ones AB, BC, CD. For firſt the body acquires the ſame velocity in falling to B either through GB or AB, by the laſt propoſition. Therefore ſecondly, the velocity which it has, when it comes to C, will be the ſame whether it fell to C through the two planes AB and BC or through the one continued plane GBC. Becauſe by the firſt ſtep the velocity at B will be the ſame in both caſes, and the line BC, which it deſcends through afterwards is the ſame in both. But thirdly, the body acquires the ſame velocity by falling to C either through EC or GC, by the laſt propoſition, that is either through EC or through AB and BC by the ſecond ſtep of this demonſtration. Therefore fourthly, the velocity which it has when it comes to D will be the ſame whether it fell to D through the one continued plane ECD, or through the two contiguous ones GC and CD, or through the three contiguous ones AB, BC, and CD, Becauſe by the third ſtep the velocity at C is the ſame in all the three caſes, and the line CD, through which it deſcends afterwards, is the ſame in all of them.

128. *If two heavy bodies deſcend upon two or more planes, which are ſimilarly inclined and proportional; the time taken up to run down thoſe planes will be as the ſquare roots of their lengths.*

Let one heavy body deſcend upon the two inclined planes AB, BC; and the other upon DE, EF. By the planes being ſimilarly inclined is meant that AB and DE, as likewiſe BC and EF make equal angles ABT and DES with the horizontal planes BT and ES or that AB is juſt as much inclined as DE is, and likewiſe BC juſt as much as EF. From hence it follows, that ABG and DEH are equi-angular triangles. For, ſince AB and DE make equal angles ABT and DES with the horizon, they will likewiſe make equal angles at A and D with AG and DH, two lines that are parallel to the horizontal planes BT and ES, Euc. b. I. prop. 29. becauſe the angle at A is the complement of ABT and the angle at D is the complement of DES to two right ones. And, ſince BC and EF make equal angles BCR and EFP with the horizon, they will likewiſe, if they are continued to G and H, make equal angles at G and H with AG and DH two lines that are parallel to the horizontal planes CR and FP, Euc. b. I. prop. 29. becauſe the angle at G is the alternate of that at C, and the angle at H is the alternate of that at F. But, ſince there are two angles in the triangle AGB reſpectively equal to two angles in the triangle DHE, and the three angles of every triangle are only equal to two right ones, it follows Euc. b. I. prop. 32. coroll. 2. that the third angle at B in one of them is equal to the third angle at E in the other.

By the planes being proportional is meant, that AB bears the ſame proportion to DE that BC bears to EF. The propoſition affirms, that when two planes AB and BC are ſimilarly inclined and proportional to two other planes DE and EF, then the time, which a heavy body takes up to deſcend along ABC is to the time, which any other heavy body takes up to deſcend along DEF as the ſquare root of ABC is to the ſquare root of DEF.

The reader will remember that when I uſe the word root in the following demonſtration, I always mean ſquare root. From the planes being proportional, and from the ſimilarity of the triangles ABG and DEH, which is a conſequence of the planes being ſimilarly inclined we have the eight following proportions. Firſt, AB is to DE as BC is to EF; becauſe the planes are proportional. Secondly, AB is to DE as GB to HE; becauſe the triangles, of which theſe are the ſides, are equi-angular. Euc. b. VI. prop. 4. Thirdly, Becauſe AB is to DE as BC to EF; therefore AB is likewiſe to DE as AB added to BC is to DE added to EF, or as ABC is to DEF. Euc. b. V. prop. 12. Fourthly, Becauſe AB is to DE as GB to HE, and as BC to EF; therefore AB is likewiſe to DE as GB added to BC is to HE added to EF, that is as GC to HF. Euc. b. V. prop. 12. The other proportions are deduced from theſe. For fifthly, ſince by proportion the firſt, AB is to DE as BC is to EF; therefore the root of AB is to the root of DE as the root of BC to the root of EF. Sixthly for the ſame reaſon, by proportion the ſecond, the root of AB is to the root of DE as the root of GB to the root of HE. Seventhly, By proportion third, the root of AB is to the root of DE as the root of ABC to the root of DEF. And eighthly, by proportion the fourth, the root of AB is to the root of DE as the root of GC is to the root of HF.

Now as AB and DE are two planes equally inclined, the accelerating force of gravity will, by propoſition 120, be the ſame upon both of them; and two heavy bodies will deſcend upon them, as if they were different parts of the ſame plane. Therefore by propoſitions 121. 107. The time of deſcent along AB is to the time of deſcent along DE as the root of AB to the root of DE. For the ſame reaſon, the time of deſcent of one heavy body along GC is to the time of deſcent of any other heavy body along HF, as the root of GC to the root of HF, that is, by proportion eighth, as the root of AB to the root of DE. And again for the ſame reaſon, the time of deſcent along GB is to the time of deſcent along HE as the root of GB to the root of HE, that is, by proportion ſixth, as the root of AB to the root of DE. But, ſince the time of deſcent along the whole plane GC is to the time of deſcent along the whole plane HF at the root of AB to the root of DE; and ſince the time of deſcent along the part GB is likewiſe to the time of deſcent along the part HE as the root of AB to the root of DE; it follows that the time of deſcent along the remainder BC is to the time of deſcent along the remainder EF, as the root of AB to the root of DE. Euc. b. V. prop. 19. This is the caſe when one of the bodies has deſcended to B along GB, and the other to E along HE. But, by the laſt propoſition, the velocity acquired at B and E and likewiſe the deſcent afterwards along BC and EF will be the ſame, whether the bodies deſcended firſt down GB and HE or down AB and DE. From hence it follows, ſince the time of deſcent down AB is to the time of deſcent down DE as the root of AB to the root of DE, and ſince afterwards the time of deſcent down BC is likewiſe to the time of deſcent down EF as the root of AB to the root of DE, therefore the time of deſcent down AB added to BC or ABC is to the time of deſcent down DE added to EF or DEF as the root of AB to the root of DE, Euc. b. V. prop. 12. But by proportion the ſeventh as the root of AB to the root of DE ſo the root of ABC to the root of DEF. Therefore the time of deſcent along ABC is to the time of deſcent along DEF, as the root of ABC to the root of DEF.

129. *The time of deſcent of a heavy body along ſimilar curves, which are in ſimilar poſition, is in a ſubduplicate proportion, or as the ſquare roots, of their lengths.*

Let AB and DE, Plat. IV. fig. 10. be two ſimilar curves, ſuppoſe them for inſtance to be two ſimilar arcs of two circles; and let them be in ſimilar poſitions or equally inclined to the horizon. Now every curve may be conſidered as conſiſting of innumerable infinitely ſhort right lines: Therefore AB and DE may be conſidered as conſiſting of innumerable infinitely ſhort inclined planes. But becauſe AB and DE are in ſimilar poſition, all the planes in the whole curve AB and all thoſe in the whole curve DE are ſimilarly inclined. And becauſe AB and DE are ſimilar arcs therefore they conſiſt of an equal number of planes, each of which planes are proportional to the reſpective arcs AB and DE, or to their reſpective wholes of which they are ſimilar parts, and conſequently all theſe parts are proportional amongſt themſelves. Since therefore the times of deſcent of heavy bodies along ſimilar and proportional inclined planes are as the ſquare roots of their length, by propoſition 128, it follows, that the time of deſcent along BA is to the time of deſcent along ED, as the ſquare root of BA to the ſquare root of ED.

130. *When a heavy body has deſcended along one or more inclined planes; if its direction could be changed, without changing the velocity that it has acquired; this velocity would make it riſe to the ſame height from whence it fell; and the time of aſcent upon the ſame planes would be equal to the time of deſcent.*

When a body has fallen down the planes ABCD Plat. IV. fig. 8. if it could be turned back without deſtroying any part of the velocity, which it acquired in deſcending; this velocity, with which the body would begin to riſe from D, would be ſufficient to carry it up the planes DCBA to the ſame height from whence it fell. The force of gravity to accelerate a body in its deſcent and to retard it in its aſcent is always the ſame, when the body deſcends and aſcends upon the ſame plane. For the whole force of gravity in perpendicular deſcent and aſcent is always the ſame: and the force of gravity upon a given inclined plane always bears a given proportion to the whole force, becauſe, by propoſition 120, it is always as the height of the plane is to its length. From hence it follows that the ſame impulſes of gravity, which accelerate the body in its deſcent along ABCD, will retard it again in its aſcent along DCBA, when it is turned back again, and begins to riſe with the laſt acquired velocity. Therefore this propoſition may be proved in the ſame manner as propoſition 112, for the velocity which was communicated by the impulſes of gravity whilſt the body deſcended, will not be loſt in aſcending but by an equal number of contrary impulſes, that is, not till it has riſen to the ſame height from whence it fell, and has aſcended for the ſame time that it ſpent in deſcending.

Let AB Plat. IV. fig. 11. be a plate of braſs, exactly poliſhed and bent into the arc of a circle. This may be conſidered as innumerable inclined planes, that are contiguous to each other. And, by this propoſition, a braſs ball by rolling down them will have acquired ſuch a velocity when it comes to B, as is ſufficient to carry it up again to the ſame height from whence it fell. Therefore if juſt ſuch another plate BC was to receive the ball when it came to B, the acquired velocity would make it riſe to C, ſuppoſing A and C to be at the ſame perpendicular diſtance from B. Now when the ball was riſen to C and had loſt all its velocity, there being nothing to ſtop it, its own weight would make it fall back again to B, and in deſcending it would, as before, acquire ſuch a velocity as would carry it up to A the place from whence it fell at firſt. And thus the ball would continue alternately deſcending and aſcending along the curve ABC. For in deſcending either from A or from C to B it always acquires velocity enough to carry it up to the ſame height again on the oppoſite ſide.

The reader muſt obſerve, that we here ſuppoſe that the ball is not reſiſted by the air, in which it moves, and that none of its motion is loſt by friction againſt the planes ABC. For unleſs theſe two cauſes are ſuppoſed away, they will conſtantly retard the ball, and ſo ſtop that motion at laſt, which would otherwiſe have been a perpetual one.

131. *The ſquares of the times, in which equal ſpaces are run through, are inverſely as the accelerating forces.*

If two bodies deſcend upon two planes differently inclined; or if two bodies, one at the poles where the force of gravity is greater, the other at the equator, where it is leſs, deſcend perpendicularly, or upon planes equally inclined; or in any other caſe, where the accelerating forces are different, by which the bodies are put in motion; then the ſquares of the times, in which equal ſpaces are deſcribed by them, will be inverſely as the accelerating forces. Call the ſpace deſcribed S, the accelerating force A, and the ſquare of the time T^q. Now, from propoſition 105 and the manner of proving it, we ſee, that where the accelerating force is given the ſpace deſcribed is as the ſquare of the time in which it is deſcribed; that is, where A is given S is as T^q. And, ſince the velocities produced in a given time are as the cauſe which produces them, or as the accelerating forces; it follows, by propoſition 10, that where the ſquare of the time, or, which amounts to the ſame, where the time itſelf is given the ſpace deſcribed is as the accelerating force, that is, where T^q is given S is as A. Therefore univerſally where neither A nor T^q are given, S will be in a ratio compounded of both, as A directly and as T^q directly, or as $T^q \times A = T^qA$. But if T^qA is as S, then T^qA divided by A will be as S divided by A. That is, $\frac{T^qA}{A} = T^q$ is as $\frac{S}{A}$. But becauſe by the terms of the propoſition we are ſpeaking of equal ſpaces, therefore S is a given quantity: and all fractions, whoſe numerators are given, are inverſely as their denominators: therefore $\frac{S}{A}$ is inverſely as A. But T^q is as $\frac{S}{A}$, therefore, where S is given, T^q is inverſely as A. That is, where the ſpaces deſcribed are equal, the ſquares of the times, in which they are deſcribed, are inverſely as the accelerating forces.

CHAP. X.

Concerning pendulums.

132. *A pendulum is a heavy body hanging to a ſmall cord or wire, which is moveable upon a center.*

133. *The oſcillations of a pendulum are produced by the force of gravity.*

FOR if the ball D Plat. IV. fig. 12. is drawn up to A and let fall from thence, its own weight will make it deſcend, and the ſtring CD will prevent it from going to any greater diſtance from C the center of motion; therefore it will deſcend in the arc of a circle AED keeping always at the ſame diſtance from C. But in deſcending to D the body acquires velocity, and with the acquired velocity, by propoſition 130, it will riſe on the oppoſite ſide to B the ſame height from whence it fell. In falling back from B to D it will, as before, acquire a velocity ſufficient to make it riſe to A: and thus it will alternately by deſcending acquire velocity and by aſcending loſe it, that is, it will continue to oſcillate or vibrate backwards and forwards, till the reſiſtance of the air and the friction of the ſtring upon the center of motion ſtop it.

Thus the vibration of a pendulum is performed in the ſame manner and by the ſame laws, as the motion of a heavy body deſcending along the innumerable inclined planes AB, Plat. IV. fig. 11. and aſcending along the innumerable inclined planes BC. And the ſame cauſes which prevent one of theſe motions from being a perpetual one, prevent the other from being ſo.

A pendulum may indeed be made to vibrate in a vacuum, and then the reſiſtance of the air will be removed. But ſtill there will be the friction of the ſtring upon the center of motion. Or ſuppoſe the ſtring was to be tyed faſt to the center, ſo that there would be no friction; yet ſtill there would be ſomething equivalent to it. For then the pendulum could not vibrate without bending and unbending the ſtring. And it will be eaſy to ſee that this will deſtroy the motion of the pendulum, by imagining the ſtring to be a wire ſo ſtiff that the weight of the ball could not unbend it. For in this caſe if the ball was at A it muſt continue there, ſince the ſtiffneſs of the wire would prevent it from deſcending. Now if the ſtring is leſs ſtiff it will be a leſs hindrance to its motion, but ſtill let it be ever ſo flexible, it will be ſome hindrance every time it is to be bent or unbent: and this hindrance will in time deſtroy the motion of the pendulum as effectually as a friction upon the center.

134. *If a pendulum was to vibrate in the chords of a circle, inſtead of vibrating in the arcs of it; the time of one oſcillation would be equal to the time, that a heavy body would take up to fall through* 8 *times the length of the pendulum.*

If a pendulum was to deſcend in the right line or chord AD Plat. IV. fig. 12. inſtead of deſcending in the arc AD, and was to riſe in the right line or chord DB inſtead of the arc DB, the time of deſcent and aſcent together or of one oſcillation from A to B would be equal to the time, which a heavy body would ſpend in falling perpendicularly through 8 times CD or 8 times the length of the pendulum. For, whilſt the pendulum is deſcending along AD a chord of a circle, a heavy body would fall perpendicularly through the diameter of the ſame circle, by propoſition 124. And, ſince the time of the pendulums aſcent along DB is equal to the time of its deſcent along AD, by propoſition 121. 130. therefore the time of deſcent and aſcent together, or the time of an oſcillation, is double the time of deſcent alone, that is, the time of deſcent is to the time of an oſcillation as 1 to 2. Now in the time of the pendulums deſcent a heavy body would fall perpendicularly through 1 diameter;

ameter, and the ſpaces deſcribed by a body falling perpendicularly, are to each other as the ſquares of the times, by propoſition 105. Therefore the ſpace deſcribed by a body falling perpendicularly in the time of deſcent alone is to the ſpace deſcribed by it in the time of deſcent and aſcent together, or in the time of an oſcillation, as the ſquare of 1 to the ſquare of 2, that is as 1 to 4. But the body would fall perpendicularly through 1 diameter in the time of deſcent alone; and conſequently in the time of an oſcillation it muſt fall through 4 diameters of the circle in which AD and DB are two chords. Now the length of the pendulum CD is only a ſemidiameter of that circle, and 4 diameters = 8 ſemidiameters. Therefore, whilſt the pendulum is performing an oſcillation upon the chords AD and DB, a heavy body would fall perpendicularly through eight times the length of the pendulum.

135. *The vibrations of the ſame pendulum in unequal arcs are performed nearly and as to ſenſe in the ſame time, provided the arcs are very ſmall.*

The ſame pendulum, whether it vibrates in the arc ADB or in the leſſer arc EDF, will perform its vibrations nearly and as to ſenſe in the ſame time, provided thoſe arcs are very ſmall. By the laſt propoſition, the time of vibration along the two chords AD and DB would be equal to the time taken up by a heavy body to fall perpendicularly through 8 times the length of the pendulum. And for the ſame reaſon the time of vibration along the two chords ED and DF would be equal to the time taken up by a heavy body to fall perpendicularly through 8 times the length of the pendulum. Therefore ſince we ſuppoſe it to be the ſame pendulum, which vibrates either in the chords AD and DB or in the chords ED and DF, 8 times the length and conſequently the time of perpendicular deſcent of a heavy body through 8 times its length is a given quantity. From whence it follows that the time of vibration in the chords AD and DB is equal to the time of vibration along the chords ED and DF. Now very ſmall arcs and their chords, for inſtance the arcs and chords ED and DF, differ but little either in length or declivity. Therefore vibrations either in chords or in very ſmall arcs will be performed nearly in the ſame manner. And as the times of vibration in the unequal chords AD and DB, ED and DF have been proved to be equal, it follows that the times of vibration in the unequal arcs AD and DB, ED and DF, will be nearly and as to ſenſe equal.

This is the reaſon why clocks with long pendulums ſhould go more exact for any long courſe of time than thoſe with ſhort ones. For where a pendulum vibrates in an arc of a given length, this arc will be ſo much leſs in proportion to the whole circle, as the circle is larger of which that arc is a part. But ſince circles are as their ſemidiameters, the circle will be larger as the pendulum, which is the ſemidiameter of it, is longer. From whence it follows that where pendulums vibrate in arcs of a given length, this arc is ſo much the leſs in proportion, as the pendulum is longer. Or where two pendulums of different lengths ſwing through an equal ſpace the arc in which the longer pendulum vibrates, though equal in length to that in which the ſhorter vibrates, may be looked upon as a ſmaller arc, ſince it is leſs in proportion to the whole circle of which it is a part. Now when a clock is clean and the parts of it move more freely, upon that account the force, with which the ſwing-wheel acts upon the pallets of the pendulum, will be ſo much the greater; and when it is foul and the parts are clogged this force will be ſo much the leſs. For this reaſon the pendulum of a clock, when it is clean, being thrown out more forcibly will ſwing farther or vibrate in a longer arc, than when the clock is foul. And unleſs the arcs, when they are the longeſt, are very ſmall in proportion to the whole circle of which they are parts, a pendulum, will take up more time to vibrate in a long arc than in a ſhort one. But the longer the time is which a pendulum takes up to vibrate ſo much the ſlower the clock will move. Therefore unleſs a pendulum vibrates in very ſmall arcs, the clock, if it goes right, when it is clean, will go too faſt when it is foul. But, where other circumſtances are equal, long pendulums vibrate in ſmaller arcs than ſhort ones do. Therefore a clock with a long pendulum will go more exactly alike, when it is clean and when it is foul, than a clock with a ſhort pendulum.

136. *The vibrations of the ſame pendulum in unequal arcs of a cycloid are performed in the ſame time.*

If the circle BS, Plat IV. fig. 13. is rolled along the right line BH like a wheel upon a plane ſurface, till it has gone once round, or till the point *b* which touched that line at B, when the circle began to move, is brought round and touches it again at H, ſo that the generating circle as it is called, is got to HT. The curve BGH which this point *b* has deſcribed in one revolution of the generating circle is called a cycloid.

To make a pendulum vibrate in a cycloid the ſtring of it AF, Plat V. fig. 1. muſt be pliable, and it muſt ſwing between two plates ACB, and AED, which are ſemicycloids: by which means the ſtring conſtantly applying itſelf to the plate towards which the ball moves, and bending itſelf into the form of the plate, will make the ball P, that is faſtened to it, vibrate in the curve BFD.

For the uſe of thoſe, who are not much acquainted with the mathematics, it will be neceſſary to mention two properties of the cycloid. Firſt. If a tangent CF is drawn to any point C in the cycloid, Plate IV. fig. 13. and a right line CE is drawn from the ſame point perpendicular to GA the axis of the cycloid, and from D, where the right line interſects the generating circle, a chord DG is drawn in the generating circle to G the vertex of the cycloid. Then CF the tangent is parallel to DG the chord. Secondly. If from any points H and K the lines HM and KR in a cycloid ACD Plat V. fig. 2. are drawn perpendicular to EC its axis, and the chords GC and LC are drawn in the generating circle as before. Then the arc HC in the cycloid is double the chord GC, and the arc KC is likewiſe double the chord LC.

If GC and LC were two inclined planes, in the ſame time that one heavy body fell down GC any other heavy body would fall down LC. For the time of deſcent down either of them, though they are of different lengths, muſt be the ſame; becauſe as they are chords of the ſame circle the time of deſcent down either of them is equal to the time of perpendicular deſcent down EC the diameter of that circle, by propoſition 124. But when two unequal ſpaces are deſcribed in the ſame time, the velocities of the moving bodies muſt be as thoſe ſpaces, the body which falls down GC muſt move as much faſter than the body which falls down LC as GC is longer than LC, by propoſition 10. And, ſince the cauſe is proportional to the effect, the forces which produce thoſe velocities, that is, the accelerating forces of gravity muſt be as the velocities themſelves; or the accelerating force of gravity on the plane GC is to the accelerating force of gravity on the plane LC as GC to LC. And ſince twice GC is to twice LC as GC to LC, theſe accelerating forces are to each other as twice GC to twice LC, that is, by the ſecond property of the cycloid, as HC to KC. Thus we have proved that the force of gravity, that acts upon a body to make it deſcend along GC is as much greater than the force of gravity, that acts upon a body to make it deſcend along LC, either as GC is greater than LC or as HC is greater than KC. But ſince, by the firſt property of the cycloid a tangent at H would be parallel to GC and a tangent at K would be parallel to LC, therefore the point H and the plane GC, becauſe they are parallel to each other, are equally inclined, and ſo likewiſe are the point K and the plane LC. But ſince H and GC, as likewiſe K and LC are planes equally inclined, the accelerating force which acts upon a heavy body at H will be the ſame as that which acts upon a heavy body at G: and likewiſe the accelerating force upon a heavy body at K will be the ſame as upon a heavy body at L. Therefore the force of gravity, that acts upon a heavy body at H is to the force of gravity that acts upon another heavy body at K as HC to KC. The ſame might be proved in every point of one bodys deſcent from H to C down the arc HC of the cycloid, and likewiſe in every point of the other bodys deſcent from K to C, down the arc KC, that is, it might be proved in the ſame manner that the accelerating force is always as the bodys diſtance from C. But the velocities produced are as the cauſes which produce them, that is as the accelerating forces. Therefore the velocity of the body which deſcends along HC is to the velocity of the body which deſcends along KC as HC is to KC. And ſince their velocities are as the ſpaces to be deſcribed in deſcending, that is, ſince as much longer as HC is than KC, ſo much faſter the body moves which is to deſcribe HC, than the body which is to deſcribe KC; it follows that one of theſe two bodies will deſcribe HC and the other KC in the ſame time.

In the ſame manner it might be ſhewn, that the time of aſcent will be the ſame either through CB = CH or through CF = CK. For the force of gravity, which retards the body in its aſcent will always be as the diſtance from the point C. Therefore the time of a vibration, or the time of aſcent and deſcent together, through HCB is equal to the time of a vibration through KCH.

137. *The times in which pendulums perform their vibrations, where the accelerating force of gravity is given, are as the ſquare roots of their lengths.*

Suppoſe two pendulums AB and CD, Plat. V. fig. 3. to vibrate in ſimilar arcs of circles EF and GH, that is, in arcs, which bear the ſame proportion to each other that the circles do, of which they are parts. Now the circumferences of all circles are to each other as their diameters or as their ſemidiameters, and ſimilar arcs are in the ſame proportion; therefore the arc EF is to the arc GH as AB the ſemidiameter of one circle or length of one pendulum, to CD the ſemidiameter of the other circle or length of the other pendulum. But if EF and GH are ſimilar arcs their halves or EB and GD muſt be ſimilar too, and they are likewiſe in ſimilar poſition. Therefore, by propoſition 129, the time of deſcent along EB is to the time of deſcent along GD as the ſquare root of EB to the ſquare root of GD, that is, ſince EB is to GD as AB to CD, as the ſquare root of AB to the ſquare root of CD. But the time of deſcent is half the time of a vibration. Therefore ſince the whole times of vibration muſt be in the ſame proportion with their halves, the time of a vibration in the arc EF is to the time of a vibration in the arc GH, in a ſubduplicate ratio of the lengths of the pendulums, that is, as the ſquare root of AB to the ſquare root of CD.

But, if AB vibrates in a cycloid, then, by the laſt propoſition, the time of vibration will not be altered, whether the arc of vibration is longer or ſhorter than EF. So likewiſe if CD vibrates in a cycloid the time of vibration is not altered, whether the arc of vibration is longer or ſhorter than GH. Therefore in a cycloid the times of vibration, and conſequently the proportion which the times of vibration bear to each other, will be the ſame, though EF and GH by lengthening or by ſhortening either of them are ſo altered as not to be ſimilar. From whence it follows, that when pendulums vibrate in cycloids, whether the arcs of their vibration are ſimilar or not, the times of vibration will be as the ſquare roots of their lengths. So that if the lengths of the pendulums are as 16 to 9, the times of vibration will be as the ſquare roots of thoſe lengths or as 4 to 3.

From hence we may find how long a branch is which hangs down from the roof of a church, and conſequently, by meaſuring from the ball of the branch to the floor and adding this to the length of the branch, we may find how high the church is. For ſuppoſe the branch, when you ſwing it, to vibrate once in 3 ſeconds, then, ſince the time of vibration is as the ſquare root of a pendulums length, it follows, that the lengths of pendulums are as the ſquares of their times of vibration, and that the length of the branch is to the length of a pendulum, which performs a vibration in 1 ſecond, as the ſquare of 3 to the ſquare of 1, or, as 9 to 1. Now the length of a pendulum which vibrates ſeconds, being 39 inches and 2 tenths, this multiplyed by 9 will give 352 inches and 8 tenths for the length of the branch. And ſo univerſally, if the length of a pendulum, which vibrates ſeconds, is multiplyed by the ſquare of the number of ſeconds contained in the time of one vibration of the branch, the product will be the length of the branch.

138. *The times, in which a pendulum of a given length performs its vibrations, are inverſely as the ſquare roots of the accelerating forces.*

If the pendulum vibrates in the cycloid ACD. Plat. V, fig. 2. the time of a vibration either in the arc HCB or KCF, or any other arc, will be inverſely as the ſquare root of the accelerating force of gravity, which produces the motion of the pendulum. The reader will remember, that hanging a different weight to the pendulum does not alter the accelerating force. For ſince, by propoſition 26, bodies of different weights fall with the ſame velocity, the accelerating force, which is proportional to the velocity produced in a given time, is no greater when the ball of a pendulum weighs 10 pounds than when it weighs only 1 pound. The moving force of gravity is indeed different, for this is as the quantity of motion produced in a given time, that is, ſince the velocity is given, as the quantity of matter: and conſequently the moving force of gravity in one ball would be 10 times what it is in the other, though the accelerating force is the ſame. The accelerating force can then only be greater or leſs, when heavy bodies in the ſame circumſtances are made by the force of gravity to deſcend with a greater or a leſs velocity. Thus, as will be proved in its proper place, a heavy body acquires a greater velocity in a given time, when it falls perpendicularly at either of the poles than when it falls perpendicularly at the equator. Therefore the accelerating force of gravity at the poles is greater than at the equator. And the times of vibration of the ſame pendulum at the poles and at the equator will be inverſely as the ſquare roots of thoſe accelerating forces. The ſquare of the time of deſcent along the given ſpace HC, Plat. V. fig. 2. is inverſely as the accelerating force, by propoſition 131; and conſequently the time of deſcent is inverſely as the ſquare root of the accelerating force. But the time of an oſcillation in the arc HB is double the time of deſcent along HC. Therefore the time of an oſcillation in the arc HB is likewiſe inverſely as the ſquare root of the accelerating force. And ſince the time of vibration of the ſame pendulum in the arc KF or in any other arc of a cycloid is equal to the time of vibration in the arc HB, it follows that the time of vibration of the ſame pendulum in any arc of a cycloid is inverſely as the ſquare root of the accelerating force.

139. *If the lengths of two pendulums are to each other as the accelerating forces, which act upon them; they will perform their vibrations in the ſame time.*

Call the length of one pendulum L and the length of the other l. Call the accelerating force, which acts upon the former A, and the accelerating force which acts upon the latter a. Then by the ſuppoſition L is to l as A is to a. Now, by propoſition 137, 138, the times of vibration are as the ſquare roots of the lengths directly, and as the ſquare roots of the accelerating forces inverſely. But L divided by A, is to l divided by a, or $\frac{L}{A}$ is to $\frac{l}{a}$, as L to l directly and as A to a inverſely. Therefore the times of vibration are as the ſquare root of $\frac{L}{A}$ to the ſquare root of $\frac{l}{a}$. And, ſince L is to l, as A is to a, the firſt quantity, or L, divided by the third, or by A, is equal to the ſecond, or l, divided by the fourth, or by a, that is, $\frac{L}{A}$ is equal to $\frac{l}{a}$: therefore the ſquare roots of $\frac{L}{A}$ and $\frac{l}{a}$ are equal; and conſequently the times of vibration are equal.

140. *The center of oſcillation is that point of a pendulum on each ſide of which the quantities of motion are equal, or in which all the gravity of the pendulum might be collected without altering the time of its vibrations.*

It is evident that this center of oſcillation muſt be different from the center of gravity. For if we imagine a plane perpendicular to the ſtring CD, Plat. IV. fig. 12. to paſs through the center of the ball D and to divide it into two equal parts: then, as the ball oſcillates upon the center of motion C, the lower half of it being farther from C muſt in the time of one vibration deſcribe a greater arc than the upper half. Therefore the velocity of the lower half, and conſequently the quantity of motion, muſt be greater than that of the upper half. From whence it follows that the quantities of motion on each ſide the center of gravity being unequal, this cannot be the center of oſcillation. But the center of oſcillation muſt be farther from C than the center of gravity is; ſo that a plane paſſing through it may divide the ball D into two unequal parts in ſuch a manner as to make the greater quantity of matter above it compenſate for the greater velocity below it, that the quantity of motion on each ſide may be the ſame.

141. *The center of oſcillation is likewiſe the center of percuſſion.*

Any weapon will ſtrike the greateſt blow, when the ſtroke is given with that part of it, where the whole motion acts: now this is the part on each ſide of which the quantities of motion are equal, or the common center of oſcillation of the weapon and our arm. For if an obſtacle ſtops this center, the whole motion is deſtroyed at once, and conſequently the whole force of that motion is ſpent upon the obſtacle. That end of a cane, which is fartheſt from our ſhoulder, will not ſtrike the greateſt blow, becauſe, it doth not ſtrike with the accumulated weight of the cane and our arm, ſince this weight all acts in their common center of gravity. Neither will this common center of gravity ſtrike the greateſt blow, becauſe there are other parts which move with a greater velocity. But it will be the center of oſcillation, which is a point between both ſo ſituated that the greater quantity of matter on one ſide of it makes amends for the greater velocity on the other ſide. If we ſtrike an obſtacle with any part of the cane, which is farther from our ſhoulder than the center of oſcillation, as if we ſtrike with the end of it; then the whole motion is not ſpent in the blow: but the part of the cane between us and the obſtacle will continue to move after the end is ſtopped by the obſtacle; therefore the cane will be bent and may be broken. In like manner, if we ſtrike with any part of the cane, which is nearer to us than the center of oſcillation, ſo that this center is beyond the obſtacle, then the whole motion is not ſpent in the blow, but the part of the cane beyond the obſtacle will continue to move, after the part we ſtruck with is ſtopped by the obſtacle, and in this caſe too the cane will be bent and will be in danger of breaking. But if we ſtrike with the center of oſcillation, there will be no danger of breaking the cane, ſince the whole motion of it is ſpent in the blow upon the obſtacle, and is ſtopped by it at once.

CHAP. XI.

Of the motion of projectiles.

142. *The random of a projectile is the horizontal diſtance to which a heavy body is thrown.*

143. *Heavy bodies, when they are thrown either obliquely or horizontally, deſcribe a curve.*

LET a heavy body be thrown in the direction AV, Plat. V. fig. 4. then, by propoſition 11, or the firſt law of motion, if no other force acted upon the body but the projectile force, or force with which it is thrown, it muſt continue to move on uniformly in the right line ABCV. But becauſe the body is a heavy one, the force of gravity will draw it out of the line AV or will make it conſtantly fall from the line of its direction towards the earths center. Suppoſe that the force of gravity was not a continued one, but that it acted at certain intervals, ſo that ſome little time ſhould paſs between one of its impulſes and the next. Thus for inſtance, ſuppoſe that, when the body was thrown in the line AV, the force of gravity did not act upon it, till it was come to B; and then let this force, which acts always towards the earths center, or in the direction BE, be to the projectile force in the line AV as BE to BC. The body acted upon by theſe two forces would, by propoſition 16, deſcribe the diagonal BD of a parallelogram whoſe two ſides are BE and BC. Suppoſe

PLATE IV.

Fig. 1.

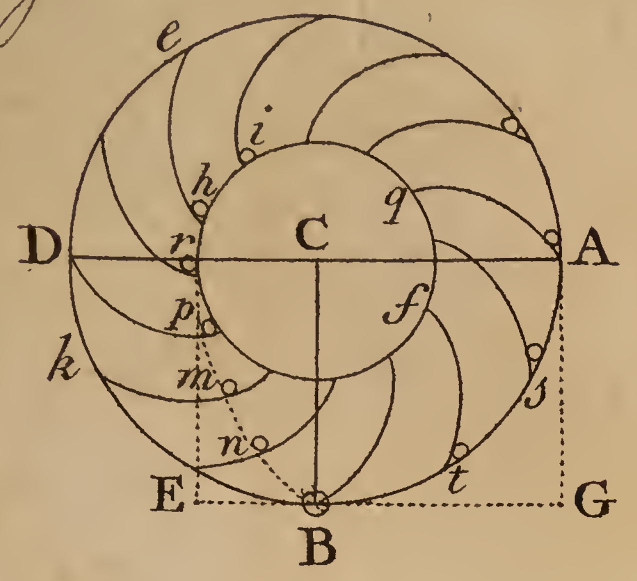


Fig. 2.

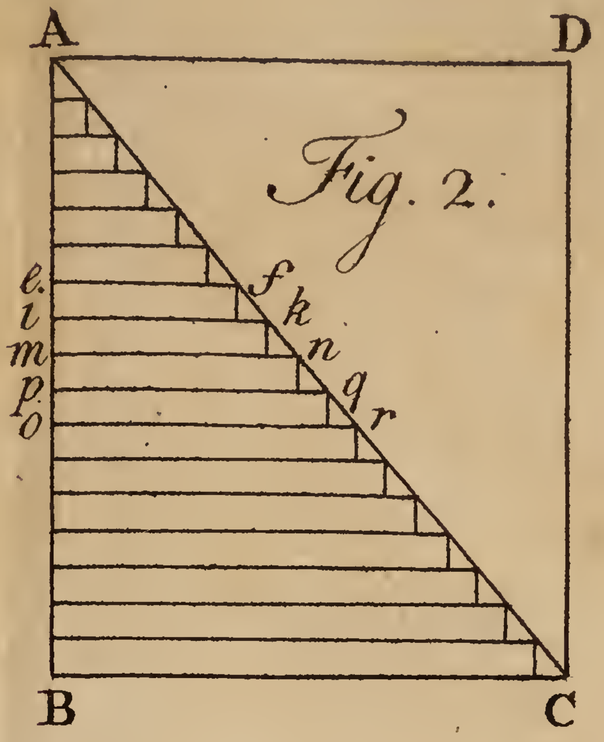


Fig. 3.

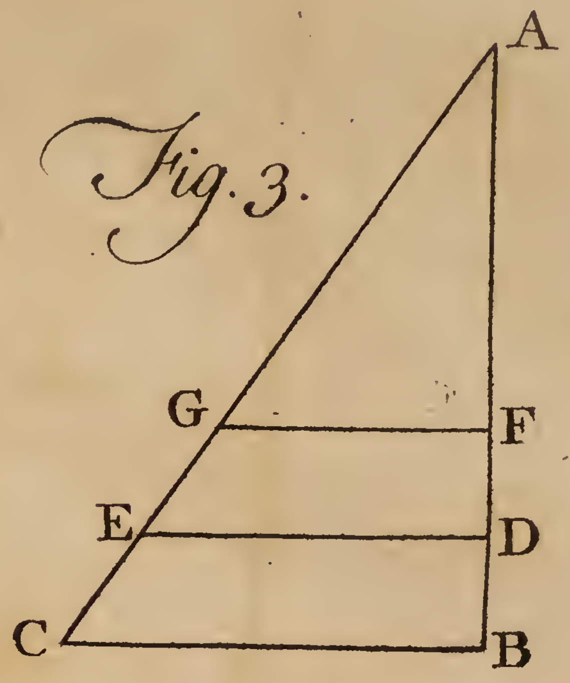


Fig. 4.

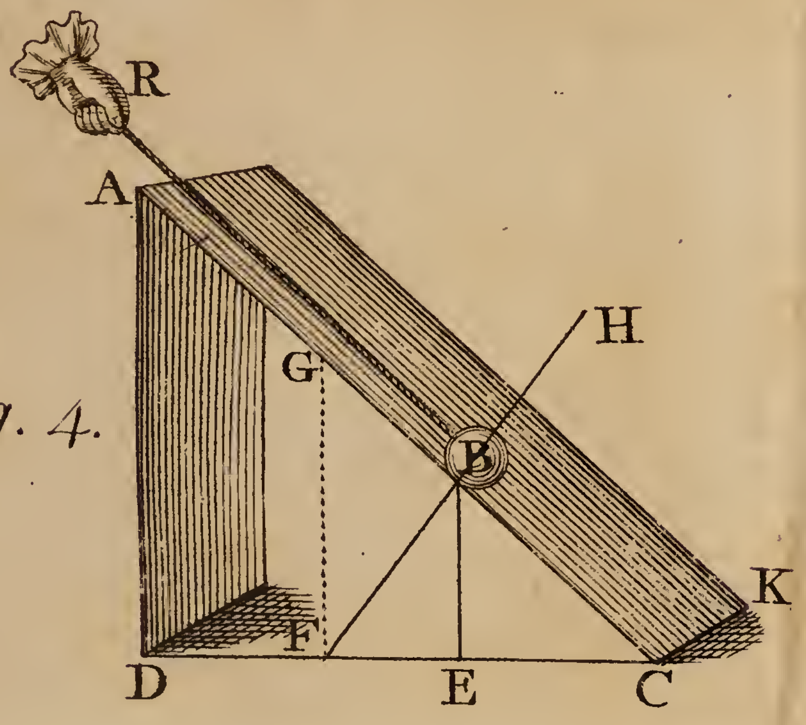


Fig. 5.

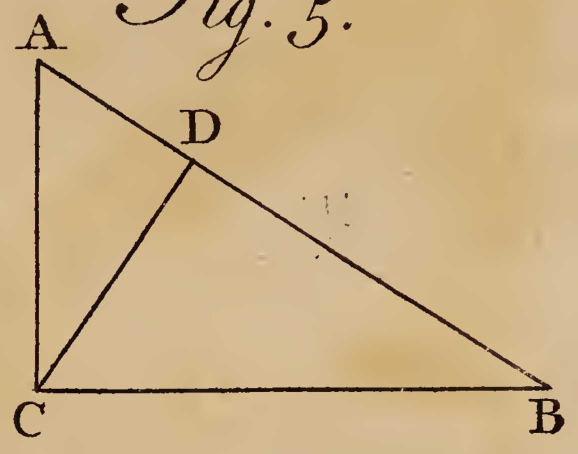


Fig. 6.

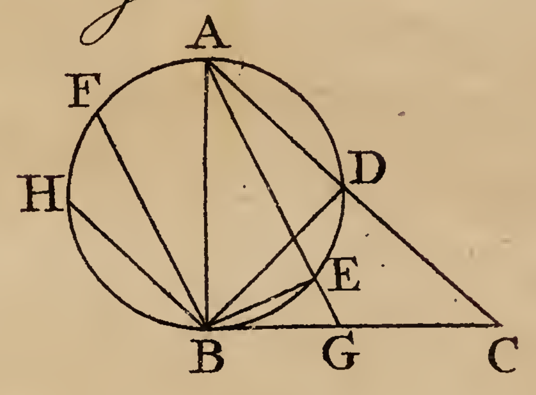


Fig. 7.

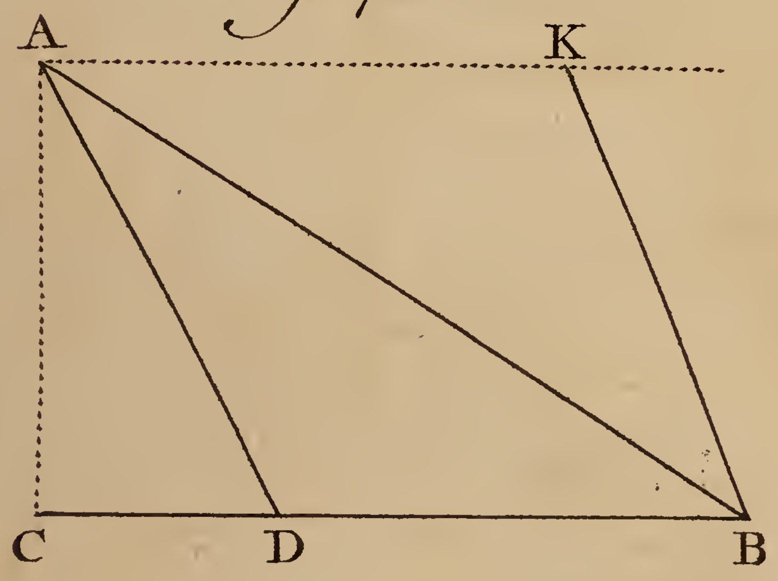


Fig. 8.

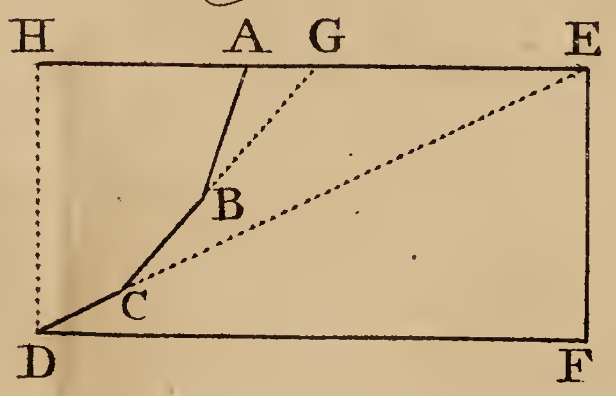


Fig. 9.

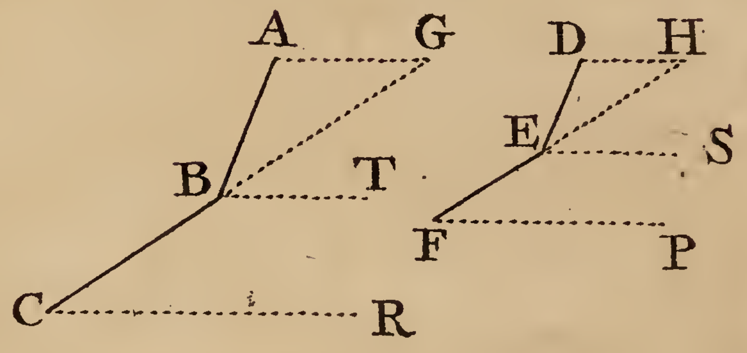


Fig. 10.

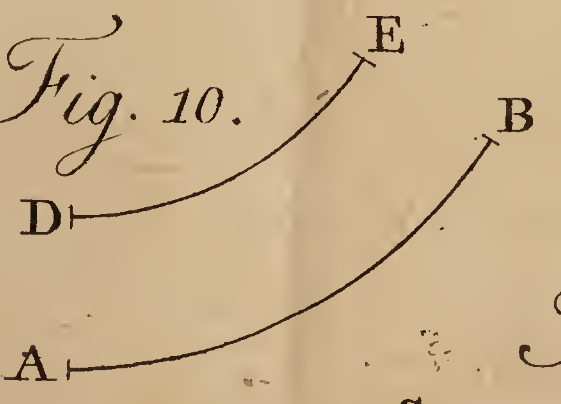


Fig. 11.

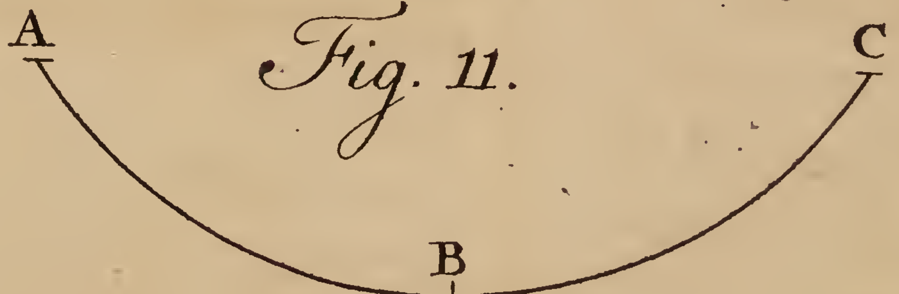


Fig. 12.

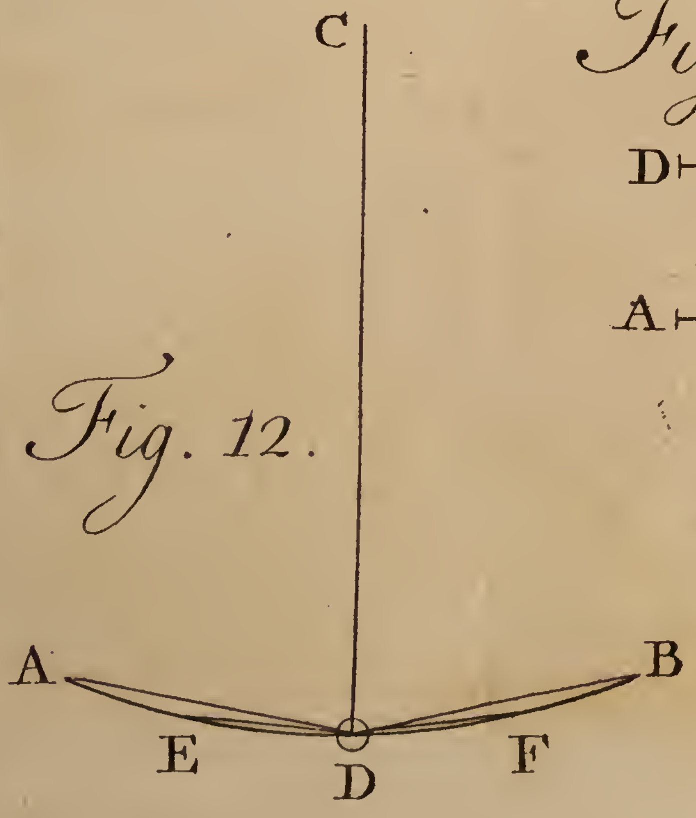
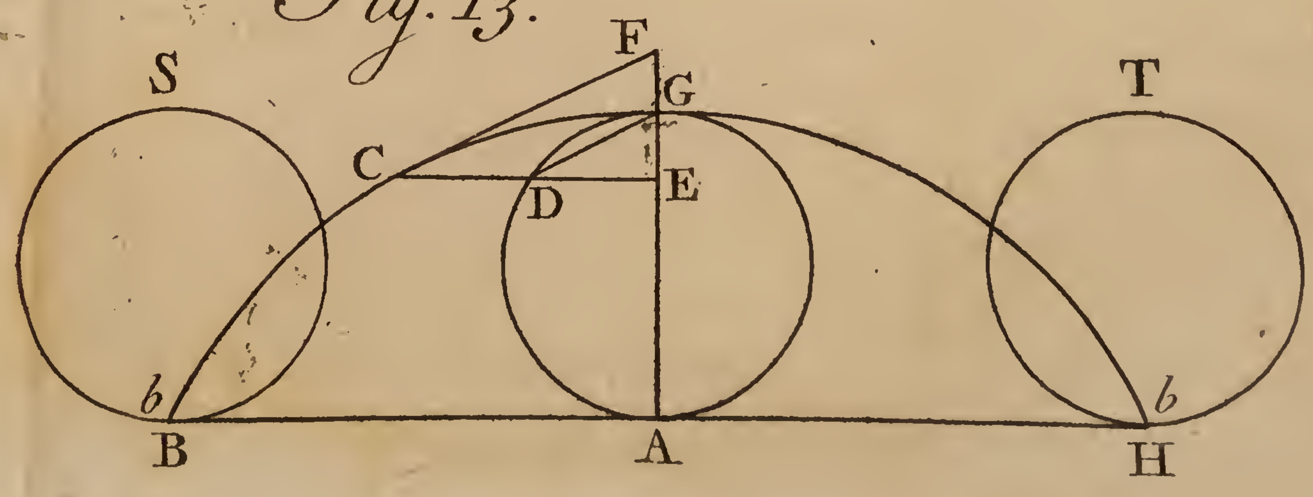


Fig. 13.



poſe the interval of time between this firſt impulſe of gravity and the next to be long enough for the body to paſs from B to D. Now if no other impulſe beſides the firſt was to act upon it at all, then, by the firſt law of motion or propoſition 11, it would continue to move uniformly in the line BDF. But when the body comes to D there is a ſecond impulſe of gravity ſuppoſed to act upon it in a direction parallel to FG, and let the force of this impulſe be to the projectile force as FG to DF, then making theſe two lines the two ſides of a triangle and drawing DG the third ſide, it is evident that this triangle is half a parallelogram of which DF the projectile force and FG equal to the force of gravity are two ſides and DG is the diagonal. Therefore the body acted upon by two forces in the directions DF and FG the two ſides of the triangle will deſcribe the third ſide DG, by propoſition 16. The body, when once moving in this direction, would, if left to itſelf, go on uniformly in the right line DGH. But a third impulſe of gravity towards the earths center, or in a direction parallel to HI, being ſuppoſed to act upon it, when it comes to G, if this impulſe is to the projectile force as HI to GH, then, as before, making GH and HI two ſides of a triangle, the body acted upon by two forces as GH and HI will deſcribe the third ſide GI. And if, after another interval of time ſufficient for the body to move as far as I in the new-acquired direction GL, a fourth impulſe of gravity was to act upon it, the projectile force IL and an impulſe of gravity acting at I in a direction parallel to LK would make it deſcribe IK. Thus by the projectile force, or force with which it is thrown, and the force of gravity together, ſuppoſing an interval of time to paſs between the impulſes of gravity, the body is made to deſcribe the polygon ABDGIK. Now the length of the ſides of this polygon as AB, BD, DG, GI, and IK, depends upon the interval of time between any one impulſe of gravity and the next. For if the action of gravity is a continued one, that is, if the intervals between any one impulſe and the next are infinitely ſhort, the body in ſuch an interval or inſtant of time could only move to an infinitely ſmall diſtance. And ſince the length of theſe ſides is the ſpace deſcribed by the body between any two impulſes, it follows that if theſe impulſes were continued ones, or the intervals between them were infinitely ſhort, theſe ſides muſt be infinitely ſhort too. Since likewiſe in the time, which the body takes up to move from A to K, there muſt be innumerable ſuch inſtants, therefore when the ſides of the polygon are infinitely diminiſhed in length, they will be infinitely increaſed in number. But the impulſes of gravity are continued, and do not act at any interval of time from each other: therefore the body will deſcribe a polygon of innumerable and infinitely ſhort ſides; and ſuch a polygon is a curve; ſuch an one as is repreſented Plat. V. fig. 7. Therefore a heavy body, which is thrown obliquely, is by the projectile force and force of gravity together made to deſcribe a curve.

We need not be particular in proving this propoſition when the body is thrown horizontally. For it is only ſuppoſing the body to be thrown originally in the direction DGH Plat. V. fig. 4. and then the proof will be the ſame as that already made uſe of in tracing out the line DGIK, which a heavy body deſcribes in the latter part of its motion, when it is thrown obliquely.

144. *A heavy body, which is thrown either horizontally or obliquely, deſcribes a parabola.*

Suppoſe the body to be thrown in the direction AF Plat. V. fig. 5. The projectile velocity is uniform, by the fiſt law of motion, for all bodies acted upon by only one force move uniformly, and conſequently this velocity would carry the body in the line AF through equal ſpaces in equal times: ſo that if it deſcribed AB in one ſecond of time, it would deſcribe BC in the next, CD in the third, DE in the fourth, and EF in the fifth. Upon ſuppoſition that all theſe lines are reſpectively equal to each other. Or if the body moved from A to B, or through a ſpace, which we will call 1, in 1 ſecond of time, and at the end of that ſecond was at B, then in 2 ſeconds it would move twice as far from A or through the ſpace AC = 2 and at the end of this next ſecond would be at C. In the ſame manner in 3 ſeconds it would have moved from A through the ſpace AD = 3. In 4 ſeconds through the ſpace AE = 4. And in 5 ſeconds through the ſpace AF = 5. Thus the ſpaces deſcribed by the force of projection in the times 1, 2, 3, 4, 5, are to each other as 1, 2, 3, 4, 5, or as the times, in which they are deſcribed. But the body is not carryed on by the projectile force alone, but will, becauſe it is heavy, fall towards the earths center. And the motion produced by the force of gravity is uniformly accelerated, by propoſition 102. So that if we compute the ſpaces through which the body falls from the line AF in each of the 5 ſeconds of time, theſe ſpaces are to each other as the uneven numbers 1, 3, 5, 7, 9, by propoſition 108. Thus if the body falls from A to G or through 1 pole in the firſt ſecond, it will fall through GH or 3 poles in the next, through HI or 5 in the third, through IK or 7 in the fourth, and through KL or 9 in the fifth, by propoſition 108. Since therefore if we reckon from the beginning of the fall the body falls from A to G or 1 pole in 1 ſecond, from A to H or 4 poles in 2 ſeconds, from A to I or 9 poles in 3 ſeconds, from A to K or 16 poles in 4 ſeconds, and from A to L or 25 poles in 5 ſeconds; it follows that in the unequal times 1, 2, 3, 4, 5, the ſpaces deſcribed are as the ſquares of the times, that is as 1, 4, 9, 16, 25. Let us next trace out the line, which the two forces of projection and gravity together will make the body deſcribe. By the force of projection it is carryed in the firſt ſecond from A to B and by the force of gravity it falls through AG or 1 pole, and by both forces together it deſcribes A*b* the diagonal of a parallelogram, of which theſe two forces are the ſides. It cannot in the next ſecond be carryed in the line of projection from B to C, becauſe at the beginning of the ſecond it is at *b* and not at B: but however the force of projection acts upon it, when it is at *b*, juſt as it would have done if it had been at B, and therefore will make it deſcribe in that ſecond *bk* parallel and equal to BC. But the force of gravity in this ſecond will make it fall through 3 poles or through $bh = \mathrm{GH}$: and conſequently both forces together will make it deſcribe *bc*. In the ſame manner the projectile force will carry it in the third ſecond through $c\delta = \mathrm{CD}$ and the force of gravity will make it fall through $ci = \mathrm{HI}$ or 5 poles; and both forces together will make it deſcribe *cd*. In the fourth ſecond it will deſcribe $d\varepsilon = \mathrm{DE}$ by one force, and $d\kappa = \mathrm{IK}$ or 7 poles by the other, that is *de* by both together. And in the fifth ſecond it will deſcribe $e\varphi = \mathrm{EF}$ by one, and *el* by the other, that is *ef* by both together. Thus by the projectile force and force of gravity together the body will deſcribe the curve A*bcdef*. Now the reader will remember that the unequal ſpaces AB, AC, AD, AE and AF, deſcribed by the projectile force are as the times in which the body deſcribes them: and that the ſpaces B*b*, C*c*, D*d*, E*e*, and E*f*, the ſpaces through which the body is made to fall by the force of gravity, are as the ſquares of the times in which the body deſcribes them. But ſince the ſpaces deſcribed in the line AF are as the times, and the ſpaces through which the body falls from AF to the curve A*bcdef* are as the ſquares of the times, it follows that theſe latter ſpaces are to each other as the ſquares of the former, or that B*b*, C*c*, D*d*, E*e*, and F*f*, are to each other as AB^q, AC^q, AD^q, AE^q, and AF^q, or taking AB, AC, AD, AE and AF, in the proportion of 1, 2, 3, 4, 5, B*b*, C*c*, D*d*, E*e*, and F*f* will be in the proportion of 1, 4, 9, 16, 25. If the reader underſtands this he will not find any great difficulty in the following demonſtrations.

Suppoſe a heavy body to be thrown horizontally, or in the direction AB, Plat. V. fig. 6. then, by what has been proved already, taking AC, AD, AE, AF, AB, in the proportion of 1, 2, 3, 4, 5, theſe will be the ſpaces deſcribed in the line AB by the projectile force alone, in the times 1, 2, 3, 4, 5. And taking CH, DI, EK, FL, BP, in the proportion of 1, 4, 9, 16, 25, theſe will be the ſpaces through which the body would fall by the force of gravity alone in the times 1, 2, 3, 4, 5. But when both forces act together the body deſcribes a curve, by propoſition 143.

Let AB be a tangent to A the higheſt point or vertex of the curve: from A draw AN perpendicular to this tangent; and draw HG, IM, KN, parallel to the tangent. AN is called the axis of the curve, which the projectile deſcribes. HG, IM, and KN are called ordinates to the axis. And AG, AM, AN, the parts intercepted between the vertex and each of theſe ordinates, are called abſciſſes in the axis. Now, ſince AC is equal to GH, and MI is equal to AD, and NK is equal to AE; becauſe they are oppoſite ſides of a parallelogram, Euc. b. I. prop. 34. ſince likewiſe, for the ſame reaſon, AG is equal to CH, AM equal to DI, and AN equal to EK, it follows, that if HG, IM, and NK, &c. are taken in the proportion of 1, 2, 3, &c. then AG, AM, AN, &c. will be in the proportion of 1, 4, 9, &c. that is, the abſciſſes in the axis AG, AM, AN, &c. are to each other as the ſquares of the ordinates GH, MI, NK, &c. But the mathematicians prove this to be a property of no curve but the parabola. Therefore the curve which a heavy body deſcribes, when it is thrown horizontally, is a parabola.

If the body is thrown obliquely, as in the line AB Plat. V. fig. 7. then let AB be a tangent to the curve at the point A from whence the body is thrown, draw AN parallel to the axis and HG, IM, KN, parallel to the tangent AB. AN is called a diameter of the curve, HG, IM, and KN are ordinates to the diameter, and AG, AM, AN are abſciſſes in the diameter. Now if AC, AD, AE, AF, and AB, are to each other as 1, 2, 3, 4, 5, or if the ſpaces deſcribed in unequal times by the projectile force are as the times, then from what has been ſaid already it appears, that CH, DI, EK, FL, and BP, or the ſpaces through which the body falls by the force of gravity are as the ſquares of the times, or as 1, 4, 9, 16, 25. And ſince, as before, AC = GH, AD = MI, and AE = NK; ſince likewiſe AG = CH, AM = DI, and AN = EK, it follows, that if GH, MI, and NK are to each other in the proportion of 1, 2, 3, then AG, AM, and AN, will be as the ſquares of GH, MI, and NK, or as the ſquares of 1, 2, 3, that is as 1, 4, 9. Therefore in the curve, which a heavy body deſcribes, when it is thrown obliquely, the abſciſſes in the diameter are as the ſquares of the ordinates. But this property belongs only to the parabola. And conſequently a parabola is the curve, which a body projected obliquely will deſcribe.

145. *Though a heavy body, when it is thrown perpendicularly upwards, appears to aſcend and deſcend in a right line; yet, ſuppoſing the earth to turn round its axis, the body does really deſcribe a parabola.*

If a ſtone upon the earths ſurface would move in a certain time, as in 5 ſeconds, from f, Plat.V. fig. 5. to L, this motion, which the ſtone has in common with the earth, is not deſtroyed by throwing it in the direction fF perpendicular to the earths ſurface. But after the projection the ſtone will be moved by two forces, one in the direction fL, the other in the direction fF; and by both of them together will begin to riſe in the diagonal of a parallelogram whoſe ſides are fF and fL. So that though the projectile force acts perpendicularly to the earths ſurface, yet this compounded with the force that carryes the ſtone along with the earth produces a motion in it, as it riſes, in the direction fe, which is oblique to the earths center. And thus the caſe of a ſtone, though thrown perpendicularly, is reduced to the caſe of an oblique projection as explaned in the laſt propoſition, and conſequently the ſtone muſt deſcribe a parabola; which it will do by riſing in the half parabola fA and by falling again in the other half of the ſame parabola.

To be more particular; ſuppoſe that the earths ſurface and ſtone along with it, if the ſtone was not thrown upwards but reſted on the ſurface, would move from f to P in one ſecond, from P to O in another ſecond, from O to N in a third, from N to M in a fourth, and from M to L in a fifth. But ſuppoſe, when the ſtone was at f, it was to be thrown upwards with ſuch a force as would carry it in 5 ſeconds to its greateſt height fF; then the ſpaces through which it would aſcend in each of the five ſeconds taken ſeperately, would be to each other as the five firſt uneven numbers taken backwards, or as 9, 7, 5, 3, 1. by propoſition 115. That is, it would aſcend through 9 poles in the firſt ſecond, through 7 in the next, through 5 in the third, through 3 in the fourth, and through 1 in the fifth. Now to trace out the line which the ſtone will deſcribe, we muſt compound theſe two motions together. For the motion, which the ſtone had in common with the earth, is not deſtroyed by throwing it perpendicularly upwards, but continues the ſame, whilſt the ſtone is in the air, that it was, whilſt it laid upon the earths ſurface. By one of theſe motions it is to be carryed in a ſecond from f to P and by the other it is to aſcend through $f\varphi$ ſuppoſing $f\varphi$ equal to 9 poles, and therefore by both together it will deſcribe fe the diagonal of a parallelogram, whoſe ſides are fP and $f\varphi$. In the next ſecond the motion, which the ſtone has in common with the earth, will carry it through a ſpace equal to PO, but the ſtone at the beginning of this ſecond is not at P but at e, therefore it will be carryed through $e\varkappa = PO$: and ſince it will in the mean time by the projectile force aſcend through $e\varepsilon$ equal to 7 poles, by both forces together it will deſcribe ed, by propoſition 16. In the third ſecond it will be carryed in the ſame direction as the earth from d to i or through a ſpace equal to ON, and will riſe from d to δ or 5 poles; therefore by both motions compounded together it will deſcribe *dc*. In the fourth ſecond, the motion common to the ſtone and earth will carry it through *cb* = NM, and the projectile force will make it aſcend through *ck* or 3 poles, therefore by the compound motion it will deſcribe *cb*. In the fifth ſecond it moves as if it was ſtill on the earths ſurface through *b*G = ML and aſcends 1 pole, by both which motions together it will be carryed in the line *b*A, and will then be arrived at its greateſt height. After this it will deſcend again but will in the mean time ſtill be carryed on by the motion common to itſelf and the earth: and therefore for the ſame reaſon that it aſcended in the ſemi-parabola *fedcb*A it will deſcend in the other half of the ſame parabola. But to any ſpectator on the earths ſurface, as particularly to the perſon who throws the ſtone, it will appear to aſcend and deſcend in a right line. For the ſpectator is carryed along with the earth juſt as faſt as the ſtone is and in the ſame direction: ſo that when the ſtone was at *e* the ſpectator would be at P, when the ſtone was at *d* he would be at O; and in every point either of the ſtones aſcent or its deſcent, it would be directly over his head; and it will fall at laſt on the point *f* from whence it was thrown, becauſe this point has moved as faſt as the ſtone has and in the ſame direction. The ſpectator in the mean time conſidering his own place as at reſt, does not attend to the motion, which is common to himſelf and the ſtone, but ſeeing the ſtone always over his head and attending only to the aſcent and deſcent of it, it appears to him to have riſen and fallen in a right line.

If *f*F is the maſt of a ſhip, and an arrow is ſhot directly up the ſide of it; then ſuppoſing the ſhip to move uniformly in a right line *f*L and to deſcribe this line, whilſt the arrow is aſcending to the top of the maſt; if we divide the line *f*L into 5 equal parts, when a fifth part of the whole time is over, the maſt will be at PE, and the arrow, as appears from what has been ſaid already, will, from its motion upwards and the motion which it has in common with the ſhip, be at *e*; and by the ſame ſort of reaſoning as was uſed above, when the maſt is come to OD the arrow will be at *d*, and ſo on till when the maſt is at LA the arrow will be at A; and thus in the courſe of its motion it will have deſcribed the line *fedcb*A. But it has all along aſcended up the ſide of the maſt: and, will for the ſame reaſon, as the ſhip goes on, deſcend again down the ſide of it, and will alight at the foot of it: and conſequently to thoſe, who are in the ſhip and do not attend to the motion of the maſt, the arrow will appear to have deſcribed a line as ſtrait as the ſide of the maſt is.

146.

146. *When a heavy body is thrown obliquely with a given velocity; if the ſpace, through which it muſt have fallen perpendicularly to acquire that velocity, is made the diameter of a circle, the height to which the body will riſe is equal to the verſed ſine of double the angle of elevation.*

Suppoſe a heavy body to be thrown, or a bullet to be ſhot from a gun, in the direction BE with a certain velocity; and ſuppoſe this velocity to be the ſame that this or any other heavy body would acquire by falling perpendicularly down AB, Plat.V. fig. 8. Now as the bullet moves in the direction BE it partly aſcends and is partly carryed forwards in an horizontal direction: and if AB is made the diameter of a circle the greateſt height to which the bullet will aſcend in this oblique projection will be BD.

This I ſhall prove immediately, after having ſhewn that BD is the verſed ſine of double the angle of elevation. IBL is in the plane of the horizon, and the angle EBI, which the projectiles direction makes with the horizon, is called the angle of elevation. But ſince the line IL is a tangent to the circle, and BE, which is drawn from the point of contact, cuts the circle, therefore the angle EAB in the alternate ſegment is equal to the angle EBI, which EB makes with the tangent, that is, EAB is equal to the angle of elevation, Euc. b. III. prop. 32. And ECB an angle at the center is double EAB an angle at the circumference, which ſtands on the ſame arc EB, Euc. b. III. prop. 20. Therefore ECB is double the angle of elevation EBI. But BD is the verſed ſine of ECB, that is, it is the verſed ſine of double the angle of elevation.

I am now to prove that if a body is thrown in the direction BE with ſuch a velocity as might be acquired by falling down AB, the greateſt height, to which it will aſcend in this oblique projection, is equal to BD. Let the line BE repreſent, not only the direction of the body, but likewiſe the velocity with which it is thrown. Then, ſince this velocity is ſuch as might be acquired by falling down AB, it follows by propoſition 115, that if the body was thrown perpendicularly upwards with the velocity BE, ſo that the whole of the velocity might be employed in aſcending, it would riſe to the height BA. But the body is thrown obliquely; and therefore only part of the velocity will be employed in aſcending, and the other part in carrying the body forwards horizontally. The proportion of theſe parts to each other and to the whole velocity may be thus found. Reſolve the oblique velocity BE, by propoſition 20, into two others, one in the direction BD perpendicular to the horizon, and the other in the direction DE parallel to it. Then the aſcending velocity is to the horizontal velocity as BD to DE, and to the whole velocity as BD to BE. We know already how high the whole velocity would carry the body, it would carry it to the height BA. We only want to find, when the body is thrown obliquely, how high it will be carryed by BD, or by only that part of the whole velocity which is then employed in its aſcent. Now the heights to which different velocities will carry a body are to each other as the ſquares of thoſe velocities, by propoſition 112. Therefore as the ſquare of BE to the ſquare of BD, ſo would be the height of a body aſcending with the whole velocity BE, to the height when it aſcends with the part BD. But AEB is a right angle becauſe it is in a ſemicircle, Euc. b. III. prop 31. and ED is perpendicular to AB, becauſe it is parallel to the horizon. Therefore the triangle EDB is ſimilar to the triangle AEB, Euc. b. VI. prop. 8. and BD is to EB as EB is to BA. Therefore ſince BD, BE, and BA are in continued geometrical progreſſion, the firſt or BD is to the third or BA as the ſquare of the firſt or BD^q is to the ſquare of the ſecond or BE^q. But the height to which the velocities BE and BD will make the body aſcend are to each other as BE^q to BD^q. that is, as BA to BD. And it has been proved already that the firſt of theſe velocities or BE would make the body aſcend to a height equal to BA, therefore the other or BD, which is that part of the whole velocity that the body aſcends with when it is thrown in the direction BE, will make it aſcend to a height equal to BD, which is the verſed ſine of double the angle of elevation.

In the ſame manner we might ſhew, if the direction had been BG, and the angle of elevation GBI, that GAB is equal to GBI, and GCB is double the angle of elevation, and that BH the verſed ſine of GCB, or of double the angle of elevation GBI, is the greateſt perpendicular height to which a body can riſe, when it is thrown in the direction BG with ſuch a velocity as it might have acquired by falling down AB. Or if FBI, which is half a right angle, had been the angle of elevation, then BC, which is the verſed ſine of FCB, or of a whole right angle, or of double the angle of elevation, would have been the greateſt height to which the ſame velocity of projection would make it aſcend.

147. *When a heavy body is thrown obliquely with a given velocity, if the ſpace, through which it muſt have fallen perpendicularly to acquire that velocity, is made the diameter of a circle, the random will be equal to four times the ſine of double the angle of elevation.*

By what has been ſhewn in demonſtrating the laſt propoſition, if EBI is the angle of elevation, ECB is double that angle, and conſequently DE, which is the ſine of ECB, is the ſine of double the angle of elevation. I affirm therefore, that if a body is thrown from the point B and in the direction BE, with ſuch a velocity as might be acquired by falling perpendicularly down AB, then the random or horizontal diſtance from B, at which the body will fall, is equal to four times DE. For it has been obſerved already under propoſition 146, that the whole velocity, if BE is put to repreſent it, may be reſolved into two others, BD, which makes the body aſcend, and DE, which carryes it forwards in the direction DE or horizontally. Since therefore the aſcending velocity is to the horizontal velocity as BD to DE, it follows by propoſition 10, that, if theſe two velocities were to continue uniform, the ſpaces deſcribed in equal times would be as the velocities themſelves, and conſequently, whilſt the body was riſing to its greateſt perpendicular height BD, it would be carryed forwards through a ſpace equal to DE. Now of theſe two velocities only the horizontal one or DE is uniform, or continues always the ſame; for the force of gravity can neither accelerate nor retard a motion in this direction, which neither carryes the body farther from the earths center nor brings it nearer to it. But the aſcending velocity or BD does not continue the ſame that it was when the body firſt begins to riſe, but is uniformly retarded, according to propoſition 109; therefore, by propoſition 110, it will be twice as long in aſcending to its greateſt height BD as it would have been, if the aſcending velocity had been uniform or had continued the ſame that it was at firſt. But if the firſt aſcending velocity had been uniform, then in the time of aſcent through BD the body would have been carryed forwards through once DE, as has been proved already, therefore in double the time, that is, in the time of aſcent through BD with an uniformly retarded velocity, it will be carryed forwards through twice DE. And ſince, by propoſition 115, the time of deſcent is equal to the time of aſcent, it will likewiſe in the time of deſcent be carried forwards through twice DE. And therefore in the times of aſcent and deſcent together it will move forwards through four times DE. Therefore a body thrown in the direction BE with ſuch a velocity as might be acquired by falling down AB the diameter of a circle, if it is thrown from the point B, will fall at the diſtance of four times the ſine of double the angle of elevation.

In the ſame manner we might ſhew, if the angle of elevation had been GBI, equal to GAB, that the random would have been equal to four times HG, which is the ſine of GCB or of double GAB, that is of double the angle of elevation. Or if FBI half a right angle had been the angle of elevation then the random would have been four times CF, which is the radius, whole ſine, or ſine of a right angle, that is of double the angle of elevation.

148. *When the velocity is given, the random of a projectile will be the greateſt poſſible with that velocity, if the angle of elevation is half a right one or an angle of 45 degrees.*

For when the velocity is given, the height from whence a heavy body muſt have fallen to acquire that velocity, or the diameter of the circle AB is a given quantity. And in a given circle the longeſt ſine that can be drawn is the ſemidiameter, whole ſine, radius, or ſine of a right angle. Thus FC, Plat. V. fig. 8. is longer than GH or than ED, or than any other ſine which can be drawn in the circle whoſe diameter is AB. Therefore four times the radius is longer than four times any other ſine: and conſequently the random will be the greateſt, when it is equal to four times the radius or four times the ſine of a right angle; and this will be the caſe, by propoſition 147, when the double angle of elevation is a right one, or when the angle of elevation itſelf is half a right one.

149. *When the velocity is given, the random of a projectile will be the ſame at two different elevations, provided thoſe elevations differ equally from half a right angle.*

Before we prove this it muſt be obſerved, that two angles, which differ equally from a right one, have the ſame or equal ſines. Thus GCB and GCA, Plat. V. fig. 8. are two angles which differ equally from a right one. For GCB is greater than FCB a right one by the difference GCF: and GCA is leſs than FCA a right one by the ſame difference GCF. But GH is the ſine both of the angle GCA and of the angle GCB. Therefore two angles, which differ equally from a right one, have the ſame or equal ſines. Now, if the angles of elevation differ equally from half a right one, the double angles of elevation will differ equally from a whole right one. Thus, if EBI is an angle of 30 degrees, or 15 degrees leſs than 45, or than half a right one, then twice EBI is 60 degrees or 30 degrees leſs than 90, or than a whole right one. And if GBI is an angle of 60 degrees, or 15 degrees greater than 45 or than half a right one; then twice GBI is 120 degrees, or 30 degrees greater than 90, or than a whole right one. Therefore when the two angles of elevation differ each of them 15 degrees from half a right one, the double angles of elevation will each of them differ 30 degrees from a whole right one. And ſo it will be in all other caſes. From hence it follows, that if two projectiles are thrown at different elevations, which differ equally from half a right angle, the double angles of elevation will have equal ſines, becauſe theſe angles differ equally from a whole right one: and conſequently four times their ſines, that is, by propoſition 147, the randoms of the two projectiles, will be equal. Thus when the elevation is EBI the random is four times DE, and when the elevation is GBI the random is four times HG. But ſuppoſing EBI and GBI to differ equally from half a right angle, then double theſe angles or GCB and ECB will differ equally from a whole right angle; therefore the ſines of theſe double angles, or GH and ED, will be equal, and conſequently four times GH and four times ED, or the randoms of the projectiles thrown at theſe two different elevations, will be the ſame.

150. *When the velocity of a projectile is given, its greateſt random is double the height, to which it would riſe, if it was to be thrown perpendicularly upwards with the ſame velocity.*

The greateſt random of a projectile is, when it is thrown at the elevation of 45 degrees, or when its direction makes half a right angle with the horizon. This random, when the velocity is given, is, by propoſition 148, equal to four ſemidiameters of a circle whoſe diameter is the height from whence a heavy body muſt have fallen to acquire the velocity of projection. But if the body was to be thrown perpendicularly upwards with this velocity, the height, to which it would aſcend being, by propoſition 115, the ſame that it muſt have fallen from to acquire this velocity, would only be equal to the diameter of ſuch a circle. Therefore the greateſt random, or four ſemidiameters, is double this perpendicular height.

If a bullet was to be ſhot in the direction BF, Plat. VI. fig. 8. making half a right angle with the horizontal plane IBL, or from a gun elevated to an angle of 45 degrees; and, if the bullets velocity was equal to what a heavy body would acquire in falling down AB, then, by propoſition 148, the random will be the greateſt, and will be equal to four times CF, or to twice BA. But if the bullet had been ſhot perpendicularly upwards with the ſame velocity, then the height to which it would have riſen would, by propoſition 115, be equal to BA. Therefore, when the velocity is given, the greateſt random is double the height to which the ſame velocity would carry a body, if it was to be thrown perpendicularly. The reader will not want to be put in mind, that when, by propoſition 117, we have determined to what height a gun will carry, it will be eaſy to determine the guns greateſt random from this propoſition, when the charge of powder and the weight of ball is the ſame.

151. *The randoms of projectiles, whoſe elevation is given, are as the ſquares of their velocities.*

By propoſition 147, if a body is thrown in any given direction, as ſuppoſe in the direction BE, its random will be equal to four times DE, or to

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four times the ſine of double the angle of elevation. But then we muſt remember that DE is always to be the ſine of double the angle of elevation in a circle whoſe diameter AB is the height from whence a heavy body muſt fall to acquire the velocity of projection. And ſince, where we are comparing the randoms of projectiles which are thrown with different velocities, we conſider the velocity as a variable quantity, greater in ſome bodies than in others; conſequently the height AB from whence a heavy body muſt have fallen to acquire the velocity of projection muſt likewiſe be conſidered as a variable quantity. For a body muſt fall from a greater height to acquire a great velocity, than it need fall from to acquire a ſmall one. But if AB was variable, if it was either longer than in the figure, or ſhorter, then DE would vary in the ſame proportion, or DE is directly as AB, longer as AB is longer, and ſhorter as AB is ſhorter. For in the triangle ECD the angle at D is invariable, whether AB increaſes or decreaſes: becauſe, as the angle of elevation or EBI is ſuppoſed to be given, ECD which is double the angle of elevation muſt be given too. And DE by the conſtruction is perpendicular to AB, becauſe it is the ſine of ECD, for which reaſon the angle EDC in the ſame triangle is likewiſe given, for it is always a right one. And conſequently, ſince the three angles of every triangle are only equal to two right ones, the third angle or CED is given. Euc. b. I. prop. 32. corol. 2. From hence it follows, that in all ſtates of AB, whether it is longer or ſhorter, the triangle ECD is always equiangular and ſimilar to itſelf: therefore its ſides are always in the ſame proportion to each other, or ED varyes in the ſame proportion with E C, that is, ED is directly as EC. Euc. b.VI. prop. 4. But EC is a ſemidiameter of the circle, whoſe diameter AB is the height from whence a heavy body muſt fall to acquire the velocity of projection; and the ſemidiameter varyes in the ſame proportion as the diameter, or EC is directly as AB. Now ſince ED is directly as EC, and EC is directly as AB, ED or the ſine of a given angle muſt be directly as AB the diameter. But four times the ſine is directly as the ſine itſelf; that is, the random at the given elevation being equal to four times ED is greater as ED is greater or leſs as ED is leſs. And ED is directly as AB, therefore the random at the given elevation will be directly as AB, or directly as the height from whence a heavy body muſt fall to acquire the velocity of projection. But, by propoſition 106, the height from whence a heavy body muſt fall to acquire the velocity of projection, whatever it is, will be as the ſquare of that velocity. Therefore the random of a given elevation will likewiſe be as the ſquare of that velocity. Thus if two balls were ſhot at the ſame elevation but with different velocities, and theſe velocities were to each other as 2 to 3, the randoms would be in the proportion of the ſquares of 2 and 3, or of 4 and 9: ſo that if one of the balls was to alight at the diſtance of 400 feet the other would fly 500 feet farther; that is, it would not alight till it came to the diſtance of 900 feet from the place where it was diſcharged.

CHAP. XII.

Of central forces.

152. *A body, which is conſtantly drawn or impelled towards any point, may be made to deſcribe round that point as a center a curve returning into itſelf; if the body is thrown in a line that does not paſs through the center.*

IF T, Plat. V. fig. 9. is the earths center and ADEI its ſurface; the force of gravity will conſtantly draw or impell any heavy body towards the point T. And if ſuch a body is thrown in any direction as GH, which does not paſs through the point T, it may deſcribe round T as a center the curve GKMF which returns into itſelf; that is, after its firſt projection from G in the line GH it may continue to revolve round T in the orbit GKMF. If a ball was let fall from G it would drop downright to A: but if it was ſhot from a cannon in the direction GH, the projectile force and the force of gravity together would make it deſcribe a curve GB, by prop. 143. 144. If the projectile force was to be encreaſed it would make the ball deſcribe the curve GC, and if it was ſtill farther encreaſed the ball would deſcribe the curve GD. For the greater the force is with which the ball is ſhot, the greater muſt neceſſarily be the diſtance to which it will fly, before it falls to the earths ſurface. It is therefore poſſible to imagine a heavy body to be thrown with a force great enough to carry it half round the earth, as far as M, without ſuffering it to fall to the ſurface. But if the body has not fallen to the ſurface in going half round, it can never fall thither at all. For ſuppoſe it to have been conſtantly approaching the earth, ſo as to be nearer to the center, when it is at M, than when it was at G: then the body in its deſcent, for ſo its motion from G to M may be called, muſt have acquired velocity, and conſequently will be moving faſter when it is at M, than it moved at firſt, when it was thrown from G. But the velocity which it had at G was ſo great as not to ſuffer it to fall to the ſurface in going half round the earth. Therefore certainly the velocity, which it has at M, will be ſufficient to carry it the other half round, and will

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make it complete its orbit. As it fell or approached the center T in the curve GKM, ſo it will riſe or depart from the center in the ſimilar curve MFG. And ſince the velocity, which the body muſt loſe in riſing from M to G, muſt neceſſarily be the ſame that it acquired in falling from G to M; its velocity upon the whole, when it returns to G, will be the ſame that it had, when it firſt ſet out from thence. And conſequently this ſame velocity will produce the ſame effect as before, that is, it will make the body deſcribe the orbit GKMF a ſecond time. And as the velocity, when it returns to G, will always be the ſame, it will continue to revolve always in the ſame orbit GKMF round the center T.

To make the caſe we have been deſcribing a little more intelligible, and to ſhew the reaſon why the body, when it has kept approaching to the center, whilſt it moved from G to M, ſhould afterwards depart from it and riſe from M to G; it will be neceſſary to conſider another caſe of a body projected or thrown in like manner from G, but with a greater velocity than was ſuppoſed in the former caſe.

When the body is thrown in the line GH, the projectile force, which acts in that direction, carryes the body farther from the center T; and the force of gravity impells it towards the center. Now if the force, with which the body is carryed from the center in the line GH, is exactly equal to that, with which it is impelled towards it; the body muſt always keep at the ſame diſtance from the center, and the curve that it deſcribes muſt be a circle. And ſince, when the body has performed half a revolution and is arrived at N in the circle GN, it is neither nearer to the center T nor farther from it than it was, when it firſt ſet out from G, it cannot in its motion from G to N have either acquired any velocity or loſt any; and conſequently it will go on from N in the ſame manner as at firſt from G, and will deſcribe the other ſemicircle, ſtill keeping as before at the ſame diſtance from the center T. But if, when the body arrived at N, the velocity of it had been greater than when it firſt ſet out, that is, if the projectile force in the line NR after half a revolution was greater than in the line GH at firſt; then it is evident that as the body moved from N it would have been conſtantly carryed off farther from the center, ſo that the diſtance NT would have kept encreaſing as the body moved on.

Now this is the caſe, when the body performs the firſt half of its revolution in the curve GKM. For the velocity at M, after half a revolution, is greater than it was at firſt, when the body began to move from G. Therefore as the body moves on from M, the diſtance TM will keep encreaſing. And ſince the body in its aſcent from the center loſes velocity in the ſame manner and by the ſame degrees that it acquired velocity in its deſcent; the curve MFG, in which it riſes, will be ſimilar and equal to the curve GKM, in which it fell. This may perhaps be more evident, if we conſider how the body would move from M, ſuppoſing that when it comes thither, any force ſhould, without altering its velocity, change the direction of its motion, and make it move back again. Upon this ſuppoſition there can be no doubt, but it would return to the point G in the ſame curve MKG, in which it had before deſcended to M. Now in order to turn the body back again, no alteration is made in the force which impells it to the center T: for this force acts in the ſame manner and in a line drawn from the body to T which way ſoever the body is moving. The only alteration, that is made, is in the direction of the projectile force. Thus, when the body is at M, it is impelled towards the center T, and is carryed on at the ſame time by the projectile force in the direction ML. Juſt as, when it was G it was impelled towards the ſame center, and was carryed on by the projectile force in the line GH. But if, when the body is at M, the projectile force acts in the direction ML, and to turn it back again nothing need be altered but the direction of the projectile force; then it is evident that, when the body is arrived at M, and is moving by the projectile force in the line ML, if by any means it is made to move in the contrary direction MP with the ſame velocity, this alteration together with the conſtant action of the force, that impells it towards T, will turn it back again in the curve MKG which it had deſcribed before. Now if a projectile force in the direction MP can make the body deſcribe the curve MKG, then an equal force in the direction ML will make it deſcribe a ſimilar and equal curve MFG. But when the body, after deſcending in the curve GKM, arrives at the point M; its projectile velocity in the direction ML is the ſame as the velocity, with which we ſuppoſed it to be thrown back in the direction MP. Therefore when the body has deſcended in the curve GKM, it will go on to move in the curve MFG, and will return to the point from whence it ſet out.

We may now conſider the third caſe of a heavy body thrown from G and impelled conſtantly towards the center T. We have ſeen already that a certain projectile velocity would make the body revolve in the curve GKMF; in which caſe it conſtantly approaches the center, till it arrives at M, and conſtantly departs from it, after it leaves M, till it arrives at G. Therefore in the orbit GKMF, the point M is neareſt of all to the center T, and the point G, from whence we ſuppoſed the body to ſet out, is fartheſt of all from that center. We have ſeen likewiſe that a greater projectile velocity would make the body revolve in the circle GN; all the parts of which are at the ſame diſtance from the center T. And from hence it is obvious, in the third place, that a projectile velocity ſtill greater would make the body begin to depart from the center T at its firſt ſetting out; and might make it deſcribe the curve GO. For as the body departs from the center its velocity will conſtantly decreaſe; becauſe the force, which impells it to the center, acts in a direction contrary to the bodys motion and therefore muſt retard it. When the body has aſcended in the curve GO; and the force, which impells it towards T, acts again at right angles to the projectile force in the line OQ, as it did in the line GH, when the body was firſt thrown from G; then the body will begin to deſcend towards the center T in a curve OG ſimilar and equal to GO the curve in which it aſcended from thence; ſo that this third caſe is like the firſt, with this difference only, that in the firſt caſe the point G, from whence the body ſets out, is fartheſt of all from the center T, whereas in this third caſe the point G is neareſt of all to to that center.

The curve GKMF, as alſo the other curve GO, is an ellipſis. Such a curve is drawn upon two centers as a circle is upon one. The manner of drawing it is this. Plat.V. fig. 10. A ſtring F*df*, which is longer than F*f*, being faſtened to two pins at F and *f*, is kept ſtretched by a pen *d*: and as this pen is moved round the two points F*f*, in ſuch a manner as to return to the place, from whence it ſets out, it will deſcribe the elipſis *mbnb*. The two points F and *f* are called the focuſes. A line *mn* drawn through the focuſes F and *f* is called the longer axis of the ellipſis. The point *c* which divides the longer axis into two equal parts, is called the center of the ellipſis. And a line *bcb*, drawn through this center perpendicular to the principal or longer axis, is called the ſhorter axis.

Plat.VI. fig. 1. Let S be the earth, and let any heavy body, the gravity of which impells it conſtantly towards the earth, be thrown at A in the direction A*t*. Then if the force, with which the body is thrown, is to the force, which impells it towards the earth, as A*l* is to AM; that is, if in the ſame time that one force would carry the body along A*l*, the other would make it deſcend through AM; the body by both forces together will deſcribe the diagonal AB of a parallelogram whoſe ſides are A*l* and AM, by propoſition 16. Now this diagonal is a part of the ellipſis ADPG, one of whoſe focuſes is S, where the earth is ſuppoſed to be, and the other is *s*. Therefore if the moon gravitates towards the earth and was thrown in the direction A*l*, or was moving, according to the firſt law of motion, in the right line A *l*, it would by this force and the force of gravity together be made to ſet out in an ellipſis with the earth in one of its focuſes: and at the point A from whence it is thrown it would be at its greateſt diſtance from the earth. This is the firſt of the three caſes, before explaned.

If the heavy body was to be thrown with a greater force, ſuch an one as ſhould be to the force, that impells it to the center S, as A*r* to AM; then A*r* and AM will be the ſides of the parallelogram, and A*m* will be the diagonal in which the body will begin to move. Now A*m* is a part of a circle, which has S for its center. Therefore, if a certain degree of velocity would make the heavy body ſet out from A to move in ſuch an ellipſis that A ſhould be the greateſt diſtance from the point S towards which it is impelled; a greater degree of velocity might make it ſet out to move in a circle having S for its center, and to revolve round S keeping always at the ſame diſtance from it. This is the ſecond of the three caſes before explaned.

A ſtill greater force, one for inſtance, that is to the force which impells the body to the point S, as A*t* to AM, would make it deſcribe the diagonal A*n*, and this A*n* is part of ſuch an ellipſis as has S for one of its focuſes, and AS for the leaſt diſtance of the revolving body from the point S, towards which it is impelled. This is the laſt of the three caſes before explaned.

The firſt of theſe caſes may eaſily be reduced to the laſt by only ſuppoſing the body to be thrown originally from P inſtead of A, in the direction P*p*, and with the ſame velocity that it will acquire in moving from A to P, that is, with a velocity too great to ſuffer it to keep always at the ſame diſtance from S, or too great to make it deſcribe a circle round S having PS for its ſemidiameter.

We may now anſwer ſome queſtions, which are uſually aſked in relation to a body that is made to deſcribe an ellipſis by the action of two forces, one of which is a projectile force, that acts in a tangent to the orbit, as in the line A *t*, when the body is at A, in the line D *d*, when it is at D, in the line P *p* when at P, and in the line F *f*, when at F; and the other force impells the revolving body, not towards the center of the ellipſis, but towards S one of its focuſes. When the body is at A or at the greateſt diſtance from S the two forces act at right angles to each other, one in the line A *t*, the other in the line AS. They act likewiſe at right angles again, when the body is at P, or at the leaſt diſtance from S. Now, if a body revolves in a circle, the projectile force, which acts in the tangent, is always in every part of the circle at right angles to the other force, which impells the revolving body towards the center of the circle, as is obvious from a bare inſpection of Plat. VI. fig. 2. where BG the tangent or direction of the projectile force when the body is at B is perpendicular to BA, or to the direction in which it is impelled towards the center A. So likewiſe CF is perpendicular to CA, DH to DA, and LK to LA: and univerſally the tangent is always perpendicular to a line drawn from the point of contact to the center. Euc. b. III. prop. 18. Therefore, ſince this is the direction of the two forces in reſpect of each other, when the revolving body deſcribes a circle; it may be asked, why the body ſetting out from A, Plat. VI. fig. 1. in the line A*t* perpendicular to AS does not move in a circle having S in its center, and AS for its ſemidiameter, rather than in an ellipſis having S in its focus. Or why, when it is come to P, where the two forces are perpendicular again, does not it go on to move in a circle having PS for its ſemidiameter. The reader is, I imagine, by this time able to tell any one who ſhould make ſuch an enquiry, that the revolving bodys velocity at A is too ſmall to make it deſcribe a circle having AS for its ſemidiameter, and at P is too great to make it deſcribe a circle having PS for its ſemidiameter. The force, with which the body, when it is at A, moves in the tangent A*t*, is ſo ſmall, that the other force, which impells it towards S, will make it approach nearer to S than it is at A, or the diſtance of it from S will keep decreaſing as the body moves on. Till at laſt it arrives at P where the two forces are perpendicular again. But then, as in caſe the third, the velocity is ſo great in the tangent P*p*, that the force, which impells the body towards S, will not be ſufficient to prevent its running out to a greater diſtance from S than PS, ſo that its diſtance from S, will, as it goes on from P, keep conſtantly encreaſing in the ſame manner that it decreaſed before, and the body will return from P to A in a curve PGA equal and ſimilar to the curve ADP, in which it deſcended from A to P.

But ſince the velocity of the revolving body is too ſmall, when at A, and too great, when at P, to make it deſcribe a circle round S; it may be farther aſked, whether, in ſome intermediate part of the orbit between A and P, the velocity is not ſuch as would make it deſcribe a circle; and if there is ſuch a velocity, why does not the body, when it has that velocity go on to deſcribe a circle for the future? There may, and indeed will be, ſuch a velocity of the revolving body, in ſome point of its orbit, whilſt it is moving from A to P. And ſuppoſe the body to be at the point D, when it has this velocity. Then the queſtion is, why it does not go on from the point D to deſcribe a circle round S at the diſtance DS?

Now it muſt be obſerved, in anſwer to this, that when the revolving body is at D, the projectile force acts in the line D*d*, which is a tangent to the orbit at the point D: for if the force, which impells the body towards the center, was to ceaſe acting, when it is at that point, it would then be acted upon by the projectile force only, and would, by the firſt law of motion, run out in a right line D*d*. But ſince that force does not ceaſe to act, the body, when it is at the point D, will be acted upon by both the forces, by one of them in the direction D*d*, and by the other in the direction DS. Theſe two lines D*d* and DS make an acute angle with each other, and conſequently the direction of the two forces is not ſuch as it ought to be in order to make the body deſcribe a circle: for when a body revolves in a circle, the projectile force and the force, which impells it towards the center, act at right angles to each other; as was juſt now obſerved.]

When the body by the projectile force is moving in the line *Dd*, and the force, which impells it to the center, acts at an acute angle in the line DS, this latter force does in ſome ſort conſpire with the bodys motion, and conſequently will accelerate it. For this reaſon, though the velocity of the body, when at D, was ſuch as would make it deſcribe a circle round S at the diſtance DS, if the two forces had acted at right angles to each other; yet as they act obliquely to each other, this velocity will immediately be changed, and the body will go on to deſcribe an ellipſis. In like manner, when the body is aſcending from P towards A, ſuppoſe the velocity of it at G ſhould be ſuch as would make it deſcribe a circle round S at the diſtance GS, if the two forces acted at right angles to each other; yet as the projectile force acts in the line G*g*, and the force, which impells it towards the center, acts in the line GS at an obtuſe angle to G*g*, this latter force does in ſome meaſure oppoſe the bodys motion and conſequently retards it: for which reaſon the velocity, that would have made it move in a circle, if the two forces had acted at right angles, is by their acting at an obtuſe angle immediately altered, and the body goes on to deſcribe an ellipſis.

A ſtone whirled round in a ſling deſcribes a circle; and ſo do moſt bodies that are made by artificial motions to deſcribe curves returning into themſelves. And as we are moſt uſed to this ſort of orbit, and ſcarce ever ſaw a body deſcribe an ellipſis; we are apt to think a motion in an ellipſis leſs natural than the other, and to make the enquiryes that have juſt now been explaned. For, if it was not on account of this prejudice, we might as well aſk why a body, when it is moving in a circle, does not change the form of its orbit, and move in an ellipſis, as why a body, when it is moving in an ellipſis, does not change the form of its orbit, and move in a circle.

We may explane the motion of a body in an elliptical orbit, when it is impelled towards one of the focuſes, in another manner, ſo as to give the reader a farther notion of the reaſon why ſuch a body, after it has approached the focus towards which it is impelled, whilſt it is moving from A, Plat. VI. fig. 1. to P in the line ADP, ſhould again depart from that focus, notwithſtanding the ſame force continues to impell it thither, and ſhould go round from P to A in the line PGA.

If the body, inſtead of being thrown from A at firſt in the direction A*l*, had received, when it was at A, no impulſe at all from any projectile force, but had only been impelled towards the center S, or which amounts to the ſame thing had been let fall from A, it would have deſcended directly to S, in the line AS. But if it is thrown in the line A*lrt* which makes a right angle with AS, or is impelled in that direction by any projectile force, at the ſame time that the other force impells it towards S, theſe two forces by their joynt action will make the body deſcribe a curve. One impulſe in the line A *lrt* will be ſufficient: for, by the firſt law of motion, a body once put in motion will always continue to move on uniformly in the ſame direction, if no other cauſe prevents it. Now the force, which conſtantly impells the body towards S, is ſuch a cauſe as muſt change its direction, and may prevent it from moving uniformly. That this force will neceſſarily change the bodys direction, is evident after what has been ſhewn in propoſition 16. For when the body is thus acted upon by two forces, inſtead of moving in the direction of either of them, it will deſcribe the diagonal of a parallelogram, whoſe two ſides will lie in the directions of theſe two forces. And conſequently the diagonal, which the body deſcribes cannot lie in the ſame direction in which the projectile force acted at firſt. Thus if AB is the diagonal, and A *l* and AM the two ſides, the body acted upon by two forces in the directions A *l* and AM, will move in neither of theſe two directions, but in the direction of the diagonal AB. If the force which urges the body towards S was to ceaſe acting, when the body arrives at B, there is no occaſion for any new impulſe of the projectile force to carry it on ſtrait forwards from B in the line B *b*, for a body once in motion will, by the firſt law of motion, continue to move on in the ſame right line, unleſs ſome external cauſe prevents it. But, inſtead of running out in the tangent B*b*, it is made by this projectile force and the other force together to deſcribe the diagonal of a parallelogram, one of whoſe ſides is in the direction B*b* in which the projectile force acts when the body is at B, and the other is in the direction BS, in which the body is at the ſame time impelled towards S. By the joynt action of theſe two forces the body is made to revolve round S in the curve ADPG.

Here we are to obſerve that the angle *l* AS which is contained between the line A *l* in which the projectile force acts, and the line AS, in which it is urged towards S, is a right angle, when the body is at A. And ſince A *l* is a tangent, and ſince, if the body moves in a circle, and is impelled towards the center of its circular orbit, AS would be a ſemidiameter of that circle, it follows that theſe two forces will always act at right angles to each other in every part of ſuch an orbit. Becauſe a tangent to any point in a circle is always perpendicular to a ſemidiameter drawn from the center to that point. Euc. b. III. prop. 18. We have already ſeen that if the body, when it is at A, was to be thrown with a force which is to the centripetal force as A *l* to AM, the diagonal of the parallelogram, which the joynt action of theſe two forces would make the body deſcribe, is AB, or a part of the ellipſis ADPG. But if the projectile force is to the centripetal one as A *r* to AM, then the diagonal will be A *m*, which is part of a circle having AS for its ſemidiameter. From hence then we may conclude that when the body ſets out from A the projectile force, which could make it deſcribe a circle, muſt be greater than that which makes it deſcribe the ellipſis ADPG. For the former is to the latter as A*r* to A*l*. Now if the body had not been thrown at all from A, but, inſtead of being put in motion in the direction A*l*, had been ſuffered to fall towards S, by the action of the centripetal force only it would, as we have obſerved before, deſcend in the line AS. And if this line AS would be the direction of the bodys motion, ſuppoſing it not to have been acted upon by any projectile force; it follows that the leſs the projectile force is, the nearer will the direction, in which the body moves, approach to AS. But ſuch a projectile force as would be ſufficient to make the body deſcribe a circle, would keep the angle at A, or the angle made by the projectile and centripetal forces, in this or in any other part of the orbit, always a right one. Therefore, ſince with a leſs projectile force, or with ſuch an one as would make the body begin to revolve in the ellipſis ADPG, the direction of the bodys motion will approach nearer to AS, it follows that as the body goes on from A the projectile and centripetal forces will approach nearer to each other, that is, the angle made by theſe two forces will become leſs than a right one, as it is at B, where B *b* is the direction of the body as it moves by the projectile force only, and BS is the direction of the centripetal force, and the angle *b* BS is an acute one.

As long as the projectile velocity, or the velocity with which the body would run off in a tangent to its orbit, is leſs than the velocity that is requiſite to make the body revolve in a circle round S; ſo long the direction of its motion, for the reaſons juſt now explaned, will keep approaching to the line AS, if the body was at A, or to BS, if it was at B, or to CS, if it was at C. That is, as long as the projectile force is too ſmall to make the body move in a circle, the direction of the projectile force will keep approaching to the direction of the centripetal force, or the angle made by theſe two forces will keep diminiſhing. Thus the angle *b*BS is leſs than *l*AS. And *c*CS is leſs than *b*BS. And *d*DS is leſs than *c*CS.

But ſuppoſe the body, by deſcending from A to D, to acquire ſuch a velocity as would be ſufficient to make it revolve round S in a circle whoſe ſemidiameter ſhould be DS. Then, after the body is paſſed the point D, the angle contained between the projectile and centripetal forces will begin to encreaſe. So that, when the body is arrived at E, the angle *e*ES will be greater than *d*DS. This is evident from what has been ſaid before. For if when the velocity is leſs than what is required to make the body revolve in a circle, the direction of the projectile force will approach to that of the centripetal force, or the angle made by theſe two forces will conſtantly diminiſh, it follows, that when the velocity is greater than what is required to make it revolve in a circle, the direction of the bodys motion will conſtantly depart from the line DS, and the angle made by the projectile and centripetal forces will conſtantly encreaſe, as the body moves on from the point D.

When the body arrives at P, and this angle has encreaſed to SP*p* and ſo is become a right one; the body is then at its leaſt diſtance from S. And as it was eaſy to underſtand, how the body, when thrown from A in the direction A *l* at right angles to AS, might have a leſs projectile force in the line A *l* than is requiſite to keep it always at the ſame diſtance AS from the point S, and might upon this account conſtantly approach to the point S in the curve ADP. So it ſeems to be as eaſy to underſtand, that, when the body is at P, and the projectile force, which acts in the line P*p*, is again at right angles to the centripetal force, which acts in the line PS, the projectile force in the line P*p*, or the velocity, with which the body is then moving in that direction, may be greater than it ought to be for the body to continue to revolve round S in a circle having PS for its ſemidiameter. But if the velocity of the body in the line P*p* is ſo great that the centripetal force, which impells it towards S, is not ſufficient to keep the body always at the ſame diſtance from it; becauſe the body by its projectile force is more carryed off from the center in the line Pp than it is impelled towards it in the line PS; then one difficulty about a bodys moving in an ellipſis, whilſt it is impelled towards one of the focuſes, is quite removed. For though the body has kept approaching towards S, whilſt it deſcended from A to P, yet it cannot poſſibly approach towards it any longer: ſince that force, which is not ſufficient to keep the body always at the ſame diſtance PS from the point S and to make it deſcribe a circle round that point, muſt certainly be inſufficient to bring the body nearer to S, or to make it approach towards S, after it is paſſed P.

This may be a little farther explaned, if we go on with conſidering the angle made by the projectile and centripetal forces. This angle is the leaſt of all when the body is at D, and keeps encreaſing conſtantly, till it becomes a right one, when the body arrives at P. The reaſon of its encreaſing after the body has paſſed by D is, becauſe the projectile velocity, or velocity with which the body endeavours to move in the tangent, is greater than what would be requiſite to make the body revolve round S in a circle. But the ſame reaſon holds when the body is at P; its velocity is ſtill too great to ſuffer it to move in a circle having PS for its radius or ſemidiameter. Therefore the angle SPp, as the body moves on, will keep ſtill encreaſing, and conſequently muſt become greater than a right one. From hence it is evident that after the body is arrived at P, that is, after the angle made by the projectile and centripetal forces is become a right one, it can approach towards S no longer. For if the body was to approach nearer to S, the angle made by the lines Pp and PS muſt decreaſe: juſt as the angle made by the lines Al and AS decreaſed in the approach of the body towards S, when it ſet out from A. But the angle at P will encreaſe, as has been proved, whilſt the body moves forward from P. Therefore it will not approach nearer to S than when it is at P: but the point P, where this angle is a right one, will be its leaſt diſtance from S.

When the body goes on from P, the velocity being too great for it either to approach nearer to S or to keep at the ſame diſtance from S, it will depart from S, and the angle made by the projectile and centripetal forces will encreaſe; ſo that when the body arrives at F it will be SFf greater than a right one. And becauſe, as the body riſes or departs from S, the centripetal force, though it is not ſufficient to draw it back again, will retard its motion, the projectile velocity will keep decreaſing. Till at laſt, in the return of the body towards A, this velocity will be ſuch an one as would juſt be ſufficient to make the body revolve in a circle round the point S. Now, whilſt the bodys velocity was greater than this, the direction, in which it endeavours in every point of its orbit to fly off in a tangent to that point, that is, the direction of the projectile force departed farther from the direction of that force by which it is impelled towards the center, or the angle made by theſe two forces encreaſed. Thus when the body was at P, the angle was SP*p*; when it was at F, the angle was at SF*f*; and when it arrives at G, the angle becomes SG*g*. But ſince, as the body goes on from G, it will be ſtill retarded, the tangent, or direction of the projectile force, will approach towards the line, in which the centripetal force acts, that is, this angle will conſtantly decreaſe; at H it will be SH*h*, leſs than SG*g*; at I, it will be SI*i* leſs than SH*h*; and at laſt at A, it will be ſo far diminiſhed as to be a right one again.

Three queſtions may here be aſked in relation to the bodys motion, when it is at or about the point D. The firſt is this. Since the velocity at D is, by the ſuppoſition, juſt ſuch an one as is requiſite to make it deſcribe a circle round S with the radius or ſemidiameter DS, or to keep it always at the ſame diſtance DS from the point S; whence does it happen that in going on from D it does not keep at the ſame diſtance from S, but continues to approach to S? This queſtion has been anſwered already; but we may now put that anſwer into another form. The body, when it has paſſed by the point D, continues to approach to S, not becauſe its projectile velocity is then too ſmall to keep it always at the ſame diſtance from S, but becauſe this velocity lies in ſuch a direction as would make it approach to S even though no force impelled it thither. For if the force, which impells the body thither when it is at D, was to ceaſe acting, and the body was to run out from D in a tangent, this tangent would be D*d*, and the body, when it arrives at *d*, would be nearer to S than when it ſet out from D. Therefore certainly the force, which impells the body towards S, and the projectile force together will make it approach S, after it has paſſed by D: and conſequently the body cannot move in a circle, notwithſtanding, when it is at D, it has ſuch a velocity in the tangent D*d*, as would be proper for that purpoſe provided this tangent was perpendicular to SD.

The ſecond queſtion which may be aſked in relation to the body, when it paſſes by the point D is, why the velocity, which it then has, ſhould not continue, till the body arrives at P, and the two forces act at right angles to each other? The moſt obvious anſwer to this queſtion is, that the body, after it has paſſed by the point D, continues ſtill to approach or fall towards S; and conſequently in falling will acquire velocity. Or otherwiſe; when the body is at D, the force, which impells it towards S, and the projectile force act at an acute angle, the former in the line DS and the latter in the line D*d*. Suppoſe therefore the former force to be to the latter as D*δ* to D*d*. Make theſe the two ſides of a parallelogram, and the diagonal DE will be the line, that the body will deſcribe by the two forces together in the ſame time that it would have deſcribed D*d* by the projectile force alone, agreeably to what has been proved in propoſition 16, and the velocity produced by both forces together will be to the projectile velocity as DE to D*d*. Now the diagonal DE, being drawn from an acute angle, is longer than either ſide of the parallelogram, and conſequently longer than D*d*. Therefore the bodys velocity produced by both forces, or the velocity with which it moves in its orbit, will be greater than its projectile velocity. That is the body will be accelerated, as long as the two forces continue to act at an acute angle. For thus again at E, where the angle *e*E*ε* is an acute one, if the force which impells the body towards S, is to the projectile force as E*ε* to E*e*; the body by the joynt action of theſe two forces will deſcribe EP, and the velocity with which it moves in the diagonal EP, will be to the projectile velocity as EP to E*e*. And EP the diagonal drawn from an acute angle is longer than the ſide D*e*. Therefore the body is ſtill accelerated.

The third queſtion that may here be aſked is this. Since, if a body revolves in a circle the angle made by the two forces is a right one, and ſince as ſoon as the body has paſſed by the point D, its projectile velocity becomes too great for it to move in a circle having S for its center, why does not the angle made by theſe forces immediately become greater than a right one, and the body immediately begin to depart from S? For as this is the reaſon given why the body, when it is at P, approaches to S no longer, but begins to depart from S, becauſe the velocity is too great for it to continue to move in a circle; ought not the ſame reaſon to take place as ſoon as ever the body has paſſed by D; and ought not the body then to begin to depart from S, ſince the velocity is then too great for it to continue to move in a circle? To this it may be anſwered; that if the angle made by the projectile and centripetal forces was a right one, as ſoon as the body is paſſed by D, then upon the velocitys becoming too great for it to move in a circle, this angle would immediately become greater than a right one, and the body would immediately begin to depart from S. But the angle both at D and after the body is paſſed by D is an acute one, and the body by the projectile force alone will continue to approach S, inſtead of departing from it. Therefore, ſince it would not depart from S even by the projectile force alone, it is certain that the action of the other force which impells it towards S, cannot make it depart from thence. But both theſe forces acting together will cauſe the body ſtill to approach to S. The immediate effect produced by the velocitys being too great for the body to move in a circle round S. is not to make the body depart farther from S, but to make the angle contained between the tangent and a line drawn to S, or the angle contained between the two forces, begin to encreaſe. And as the body moves on from D to P this angle keeps encreaſing, till at P it becomes a right one, and then, and not till then, the body will begin to depart from S.

It is eaſy to apply theſe queſtions and the anſwer to them to the motion of the body as it aſcends from P towards A. For firſt, though the body at G be ſuppoſed to have ſuch a velocity as in proper circumſtances would make it revolve round S in a circle at the diſtance GS; yet, when the body paſſes on from G, it ought not to keep at the ſame diſtance GS from the point S, becauſe the tangent G*g* or direction, in which the body would move by the projectile velocity only, is not perpendicular to SG the line in which the body is impelled towards S, but makes a greater angle with it than a right one: conſequently the projectile force in this line G*g* makes the body depart farther from S in any given time than ſuch a projectile force would do as acted in a line perpendicular to GS. Secondly, ſince the body after it leaves the point G keeps departing from S, the velocity, which it had at G, will not continue till it arrives at A, for the body in its aſcent will be retarded. Or otherwiſe. When the angle made by the two forces is ſo obtuſe as it is at G, the diagonal drawn from the obtuſe angle SG*g* will be ſhorter than either ſide of a parallelogram having one ſide in the direction G*g* and the other ſide in the direction GS. For which reaſon the velocity in the diagonal, or the velocity produced by both the forces, will be leſs than the velocity in the ſide G*g*, or than the velocity ariſing from the projectile force alone. Thirdly. Though the velocity of the body as ſoon as it is paſſed G becomes too little to make it move in a circle; yet it will not immediately begin to approach to S, as it does after it has arrived at A. But as the body goes on from G towards A the angle made by the projectile and centripetal forces will conſtantly diminiſh, till the body comes to A, at which time this angle will be a right one; and after that, as the angle diminiſhes ſtill farther, the body having paſſed A begins to approach towards S as before.

If the revolving body deſcribes the ellipſis *mbnb*, Plat.V. fig. 10. and F is the point, towards which it is conſtantly impelled; then *m* is the greateſt diſtance of the revolving body, *n* is its leaſt diſtance, and *b* is its middle diſtance from the point F.

That *m* is the greateſt diſtance, and that *n* is the leaſt diſtance is apparent; and if the reader thinks it worth his while to attend to what follows, it will be no difficult matter to prove that when the body is at *b* it is at the middle diſtance from F or at a diſtance which differs equally from the greateſt and the leaſt. That is, *b*F is juſt as much leſs than *m*F as it is greater than *n*F.

When the pen *d* in drawing the ellipſis is at *m*; that is, when *d* and *m* coincide, *df* is equal to *mf* and *d*F is equal to *m*F. But *df* added to *m*F is always equal to the whole length of the ſtring. Therefore *mf* added to *m*F is equal to the length of the ſtring. In the ſame manner it may be proved that *n*F added to *nf* is equal to the length of the ſtring. Again: when the pen is at *m* the ſtring is ſtretched out that way, and when the pen is at *n* the ſame ſtring is ſtretched out the other way. Now we may take it for granted in this caſe, that the ſame ſtring ſtretched out upon the ſame two points F and *f* will reach to an equal diſtance from thoſe points either way; and conſequently *mf* and *n*F are equal to each other. Therefore *mf* added to *nf*, that is the longer axis, is equal to *mf* added to *m*F. But *mf* added to *m*F has juſt been proved equal to the length of the ſtring. Therefore *mf* added to *nf*, that is the longer axis *mn*, is equal to the length of the ſtring. From hence it follows, ſince *c* divides the longer axis into two equal parts, that *mc* or *nc* are equal to half the length of the ſtring.

Now as *mc* and *nc* are equal, and likewiſe *mf* and *n*F equal, *mc*—*mf* or *fc* is equal to *nc*—*n*F or F*c*: that is, the center *c* divides the diſtance between the focuſes F and *f* into two equal parts. When the pen in deſcribing the ellipſis is at *b*, then *bf* and *b*F is the length of the ſtring, and *bf* is equal to *b*F. For in the two triangles *bcf* and *bc*F the angles at *c* are both of them right ones and conſequently equal to each other; the ſide *bc* is common; and the ſide *fc* in one is equal to the ſide F*c* in the other. From whence it follows, Euc. b. I. prop. 4. that *bf* the third ſide in one is equal to *b*F the third ſide in the other: and ſince *bf* and *b*F together make up the whole ſtring, *b*F muſt be half the ſtring. But *mc* or *nc* are equal to half the ſtring. Therefore *b*F is equal to either *mc* or *nc*. That is, *b*F, which I am to prove to be the middle diſtance, is equal to half the longer axis.

*m*F is the greateſt diſtance, which is greater than *mc*, or than half the longer axis, by the quantity *c*F; and *n*F is the leaſt diſtance, which is likewiſe leſs than *nc*, or than half the longer axis, by the ſame quantity *c*F. Therefore half the longer axis, or *b*F which is equal to half the longer axis, differs equally from the greateſt and leaſt diſtances; and conſequently is the middle diſtance.

The reader is deſired to obſerve farther, that, if F and *f* the two focuſes of the ellipſis, or the two pins, to which the ends of the ſtring are faſtened, were to be brought cloſe together at *c*, then the pen, which deſcribes the ellipſis, would in going round be always at the ſame diſtance from *c*, and this diſtance would be equal to half the length of the ſtring, to *bf* or *b*F. Conſequently the curve deſcribed in this caſe, would be a circle, whoſe ſemidiameter would be equal to *b*F, or, which amounts to the ſame to *mc*, or to *nc*. From hence it appears, that a circle may be conſidered as an ellipſis, the focuſes of which coincide, or an ellipſis may be conſidered as a circle drawn upon two centers.

153. *A body that revolves in an orbit, endeavours in every point of the orbit to fly off in a right line, which is a tangent at that point, and to depart from the center, towards which it is impelled.*

Whilſt a ſtone is whirled round in a ſling, it deſcribes a circle; becauſe it is conſtantly impelled towards the hand, which is the center, and is kept from flying off by the force of the ſtring. But if the ſtring breaks, or if one end of it is ſlipped, then the ſtone, being no longer impelled towards the hand or kept from flying off, will continue its circular motion no longer, but ſetting out from the place where it was, when the ſtring broke, and when the force, which before impelled it to the center, ceaſed to act upon it any longer, it will, by the firſt law of motion propoſition 11, go on to deſcribe a right line. If BCDL Plat. VI. fig. 2. is the circle, which the ſtone deſcribes, and one end of the ſling was to be ſlipped, or the ſtring was to break, when the ſtone is at B; the ſtone, by the firſt law of motion, would run off from the point B in the right line BG, which is a tangent to the circle at B. In the ſame manner, if the ſtring was to break, when the ſtone is at C, it would fly off in the line CF; if at D, in the line DH; if at L, in the line LK; or if at any other point of the circle, it would fly off in a tangent at that point.

This is equally true of bodies that revolve in any other orbit, as of thoſe that revolve in a circle. For if the force, which impells a body to the point S, Plat. VI. fig. 1. and makes it deſcribe the ellipſis ADPG, was to ceaſe acting, when the body is at P, or at H, or at C, or at any other place in its orbit, it would, by propoſition 11, run out in a right line which is a tangent to the ellipſis at that point.

Now the ſame force with which the body would fly off, if it was not impelled towards the center, conſtantly acts upon it, whilſt it is deſcribing the curve and is impelled thither. Therefore ſince this force would make the body fly off in a tangent, if no other cauſe prevented it, the ſame force conſtantly acting, whilſt the body revolves in an orbit, makes it endeavour to fly off in the ſame manner, and is prevented from producing its effect by the other force, which conſtantly impells the body towards the center.

Now as the body moves on in the tangent, it will be every where farther from the center, towards which it is impelled, than if it was to move in the curve. Thus, Plat. VI. fig. 3. if the body deſcribes the tangent B*a* inſtead of the arc B*n* it departs from the center C, or is farther from that center, when it is at *m*, or at any other point in the tangent, than it would have been if it had deſcribed B*n*. In like manner, Plat. VI. fig. 1. if a body is conſtantly impelled towards S and revolves in the curve ADPG the force, with which it endeavours to fly off in a tangent, makes it likewiſe endeavour to depart from the point S. If the body is at D, then in deſcribing the tangent D*d*, though it approaches the center, and is nearer to it when it comes to *d*, then when it was at D; yet by deſcribing the tangent it does not approach S ſo much as it would do by deſcribing the curve, or in the tangent it is every where farther from S than it would have been, if it had gone on in the arc DE. For though the body by deſcribing the tangent D*d* is nearer to the center S at *d* than at D, yet it is farther off from the center, when it is at *d*, than it would have been, if it had deſcribed the arc DE, and ſo had been at E. Therefore ſince the body will be every where farther from the center by deſcribing the tangent than it will by deſcribing the curve, it follows that the ſame force, which makes it endeavour to fly off in a tangent, makes it likewiſe endeavour to depart from the center, towards which it is impelled.

From hence we may ſee the reaſon, why we may turn a pail full of water over our heads, and none of the water be ſpilled, when the pail has its bottom upwards. For the pail and the water in it being turned round in a circle, the water will every where endeavour to depart from the center of that circle. Therefore when the pail in this circular motion is over our heads and its bottom is upwards, though the weight of the water would make it fall out of the pail, yet as it cannot fall without approaching to the center of the circle which it deſcribes, the force with which it avoids this center will, if the pail is whirled round faſt enough, be ſufficient to keep it in the pail.

154. *The force, which impells a body towards the center, when it revolves in an orbit, is called the centripetal force.*

155. *The force, by which a body revolving in an orbit endeavours to depart from the center, is called the centrifugal force.*

156. *When a body deſcribes a curve returning into itſelf, the centripetal force is equal to the centrifugal force; and they are called by one common name the central forces.*

If a body revolves in the circle BD, Plat.VI. fig. 3. then in the time that it deſcribes the arc B*n*, it will have been impelled towards the center through the ſpace *an*; for if it had been left to itſelf and had not been impelled thither at all, the projectile force alone would have carryed it in that time along the tangent from B to *a*. Therefore ſince the line *an* is the ſpace deſcribed by means of the centripetal force, this force is proportional to *an*. But if, when the body was at B no centripetal force had acted upon it, then inſtead of deſcribing the arc B*n* it would in the ſame time have moved along the tangent B*a*, and the line *na* would have been the ſpace through which it would have departed from the center. Therefore the centrifugal force is proportional to *na*. And conſequently, as both theſe forces are repreſented by one and the ſame line, they muſt be equal to each other.

Though three forces are here ſpoken of, the centripetal, the centrifugal, and the projectile force; yet two are all that are required to make a body revolve in an orbit. For the centrifugal and projectile forces differ from each other only as a part differs from the whole. The projectile force is that, with which the body would run out in a tangent to its orbit, if there was no centripetal force to prevent it; and the centrifugal force is that part of the projectile force, which carryes the body off from the center, whilſt it is deſcribing the tangent. Thus if the body revolved in the orbit BD, Plat.VI. fig. 3. the projectile force is what would make it deſcribe the tangent B*a*, if the centripetal force was to ceaſe acting. But in the mean time the whole force B*a* does not carry it off from the center C; when it is arrived at *a*, it is farther from the center than when it was at B only by the quantity *an*; and as much of the projectile force as acts in this line *an* is what carryes the body off from the center, and is called the centrifugal force.

157. *The periodical time of a body revolving in an orbit is the time that it takes up to perform one complete revolution.*

158. *The plane of an orbit, in which a body revolves, paſſes through the line of projection, and through the center to which the centripetal force is directed.*

Suppoſe ADPG, Plat.VI. fig. 1. to be the orbit, which the body deſcribes, and S to be the point or center, to which the centripetal force is directed; for though S is not the center of the ellipſis but one of its focuſes,

cuſes, yet as it is the point towards which the body is impelled, or the point in which the centripetal force acts, we therefore call it the center. What is affirmed in the propoſition amounts to this. The plane of the orbit paſſes through A *t* the line of projection and S the center in which the centripetal force acts. Or, ſince the projectile force continues to act upon the body in every point of its whole revolution, and in a line which is a tangent to that point, by propoſition 153, the orbit lies in the ſame plane with the line C*c* in which the projectile force acts, when the body is at C; or in the ſame plane with P*p*, or with G*g*, or I*i*, lines in which the projectile force acts in different parts of the orbit; and likewiſe in the ſame plane with S the center towards which the centripetal force tends.

If S is the earth and ADPG the moons orbit; upon ſuppoſition that the moon is impelled towards the earth conſtantly by the force of gravity, and that this is the centripetal force, which retains it in its orbit, and prevents it from flying off in a tangent; the plane of the moons orbit muſt paſs through any line as A*t*, D*d*, F*f*, or H*h*, in which the projectile force acts, and through S the earth.

So likewiſe if S is the ſun, and ADPG is the orbit of any planet, as of Jupiter for inſtance; upon ſuppoſition that the planet gravitates towards the ſun, and that gravity is the centripetal force, which retains it in its orbit; the plane of the orbit deſcribed by it paſſes through any line as A*t*, D*d*, P*p*, or G*g* in which the projectile force acts, and through S the ſun, or point towards which the centripetal force is directed.

Since the whole motion of the body ariſes from two forces, of which one acts in the line A*l*, and the other in the line AS; it ſeems to follow that the firſt of theſe forces or the projectile force can never make the body leave the plane, in which the line A*l* is, and that the other can never make it leave the plane, in which AS or S is. Therefore it muſt always be found in the ſame plane with A*l* the projectile force and with S the point towards which it is impelled by the centripetal force.

This may be proved more exactly in the following manner. Let ABCD, Plat. VI. fig. 4. repreſent a part of the orbit, which is deſcribed by a body that is conſtantly impelled towards S. BC, CD, or DE, or any other part of this orbit lies in the ſame plane with AZ the line of projection and with S the point, in which the centripetal force acts and towards which the revolving body is impelled. If the body begins to move by the projectile force from A in the direction ABZ; then, according to the firſt law of motion propoſition 11, if no other force was to act upon it, and to alter the courſe of its motion, it would continue to move on uniformly in a right line in the ſame direction AZ. But the centripetal force bends the courſe of its motion towards S, and makes it conſtantly change the direction of its motion ſo as to deſcribe an orbit round S, of which ABCDE is a part. Let us ſuppoſe that the action of this centripetal force is not continued, but that it exerts itſelf by ſeparate ſtrokes after certain intervals of time. Thus for inſtance; let us ſuppoſe, that the body, after it has ſet out from A, may move on uniformly by the projectile force for as long a time as is requiſite to deſcribe the line AB; but when it is arrived at B, and would, if left to itſelf, move on in the ſame uniform manner, and in an equal time deſcribe B*c*, which is equal to AB; then ſuppoſe one ſtroke of the centripetal force to act upon it, and to bend its courſe or alter its direction from B*c* to BC, ſo that both forces together ſhall make it deſcribe BC in the ſame time that the projectile force alone would have made it deſcribe B*c*. Again, when the body is arrived at C, ſuppoſe another ſtroke of the centripetal force to act upon it, and to bend its courſe towards S, ſo as to make it deſcribe CD, inſtead of moving ſtrait forwards from C in the direction C*d*, as it would do according to the firſt law of motion, if none but the projectile force was to act upon it. And ſuppoſe in like manner, after an equal interval of time was paſſed and the body was arrived at D, that another ſtroke of the centripetal force was to act upon it, and to bend its courſe from the direction CD into the direction DE. In this caſe either BC, or CD, or DE, are in the ſame plane with AZ and with the point S. When the body is at B, there are two forces act upon it; the projectile force in the direction BZ and the centripetal force in the direction BS. Now as BZ and BS repreſent the directions of the two forces, ſo let B*c* taken in the direction of the projectile force and BG taken in the direction of the centripetal force repreſent the proportion of theſe two forces to each other. That is; let the projectile force be to the centripetal force as B*c* to BG. From what was proved in propoſition 16, it follows that the body thus acted upon by two forces B*c* and BG, will deſcribe BC the diagonal of a parallelogram, of which B*c* and BG are the ſides. But BC the diagonal of a parallelogram is in the ſame plane with its two ſides B*c* and BG. Euc. b. XI. prop. 2. And B*c* which is part of the right line AZ is in the ſame plane with the whole line, Euc. b. XI. prop. 1. as alſo BG, which is part of the right line BS, is in the ſame plane with the whole line. Therefore BC is in the ſame plane with AZ the line of projection and with BS, and conſequently with S the point, to which the centripetal force is directed. In like manner it may be proved that the next part of the orbit CD is in the ſame plane with C*d* and CH the two ſides of a parallelogram, of which CD is the diagonal. For here, as before, the projectile force alone would make the body move ſtrait forwards from C in the direction C*d*. But at C the centripetal force impell sit towards S in the direction CS. And, ſuppoſing theſe two forces to be to each other as C*d* to CH, their joynt action will make the body move in the diagonal CD, according to propoſition 16. Therefore CD is in the ſame plane with C*d* and with CH, that is, with CS, and conſequently with S. But C*d*, being the right line BC continued, is in the ſame plane with BC. Euc. b. II. prop. 1. and BC has been proved to be in the ſame plane with AZ. Therefore CD is likewiſe in the ſame plane with AZ the line of projection, and with S the point to which the centripetal force is directed. The ſame thing may be proved in the ſame manner of DE and of every other part of the orbit. And conſequently the whole orbit is in the ſame plane with the line AZ and with the point S.

ABCDE is not a curve, but a polygon, which is owing to the ſuppoſition we ſet out with, that the centripetal force does not act continually, but by ſeparate ſtrokes repeated at certain intervals of time For upon this ſuppoſition, the length of AB, BC, CD, or DE, will always be equal to the ſpace, which the body can deſcribe in the interval of time between any one ſtroke of the centripetal force and the next that follows it. Becauſe between any one of theſe ſtrokes and the next, the body is acted upon only by the projectile force, and therefore will, by the firſt law of motion, move on in a right line during that interval. But, if we correct this ſuppoſition and make the caſe of the revolving body what in fact it is; that is, if we conſider the centripetal force as a continued action; then the interval between any two ſtrokes of it will be infinitely ſhort; and conſequently the ſpace deſcribed by the revolving body during ſuch an interval will be infinitely ſhort too. That is, the ſides AB, BC, CD, DE, &c. of the polygon will be infinitely ſhort. And, as in the time of a whole revolution there muſt be innumerable ſuch ſtrokes of a continued centripetal force, theſe ſides will be innumerable, as well as infinitely ſhort. But a polygon conſiſting of innumerable infinitely ſhort ſides is a curve. And if the propoſition is true in one ſort of polygon, it will neceſſarily be true in every other ſort. Therefore, when a body revolves in a curve, its orbit is always in the ſame plane with the line of projection and with the point to which the centripetal force tends.

159. *If a body revolves in a curve returning into itſelf, it deſcribes, by a radius drawn to the point where the centripetal force acts, equal areas in equal times, and in unequal times areas proportional to the times.*

If a body revolves in the curve *mbnb*, Plat. V. fig. 10. and F is the point where the centripetal force acts, or the center towards which the body is conſtantly impelled; and if a radius or line as F*m* is drawn from the center F to the body at *m*, and this radius or line is ſuppoſed to turn round upon the point F and to attend the body in its motion, ſo that when the body is at *l*, the radius is F*l*, when the body is at *r*, the radius is F*r*, when as *s*, F*s*, and when at *n*, F*n*. Then it is obvious, that whilſt the body moves from *m* to *l* the radius F*m* by moving at the ſame time from F*m* to F*l* will deſcribe or ſweep the area of the mixt triangle *m*F*l*: in like manner, whilſt the body moves from *l* to *r*, a radius drawn from the body to F deſcribes or ſweeps the area or ſpace *l*F*r*: whilſt the body moves from *r* to *s*, the radius deſcribes or paſſes over the area *r*F*s*: and whilſt the body moves from *s* to *n*, the radius deſcribes the area *s*F*n*. The propoſition affirms, that the areas, which the radius thus deſcribes in equal times, are equal to each other. For inſtance, if the body takes up juſt as much time to move from *m* to *l* as it does to move from *r* to *s*, then *ml* and *rs* are the arcs deſcribed in equal times by the body, and *m*F*l* and *r*F*s* are the areas deſcribed in equal times by a radius drawn to the center; now we are not to prove that the arcs *ml* and *rs* are equal, though they are deſcribed in equal times, for theſe arcs are lines and not areas; and in the figure before us they are plainly unequal. But we are to prove that the area or ſpace contained within the triangle *m*F*l*, which is ſwept by the radius in a certain time, is equal to the area or ſpace contained within the triangle *r*F*s*, which is ſwept by the radius in an equal time. The propoſition farther affirms that the areas deſcribed in unequal times by the radius round the point F are proportional to the times, in which they are deſcribed. By this we mean, that if a radius or line drawn from the revolving body to the point F, where the centripetal force acts, deſcribes the area *m*F*l* in a certain time; then in twice that time, it will deſcribe, an area double to *m*F*l*; in thrice that time it will deſcribe a triple area; and in all caſes the area deſcribed by ſuch a radius will be greater or leſs in proportion as the time is greater or leſs, which is taken up to deſcribe it.

Having thus explaned the meaning of the propoſition, we may apply it to an inſtance or two, that the reader may underſtand the uſe of it, before we go on to prove it. If *mbnb* is the moons orbit, and F the place of the earth round which the moon revolves. Then ſuppoſe the moon to be heavy as other bodies are and to gravitate towards the earth; and ſuppoſe likewiſe that its weight or gravity, is the centripetal force, by which it is conſtantly impelled towards the earth, and made to deſcribe an orbit inſtead of moving on in a right line, as it would do, if no centripetal force acted upon it. A right line or radius drawn from the moon to the earth muſt, upon theſe ſuppoſitions, deſcribe round the earth equal areas in equal times. In like manner, if F is the ſun and *mbnb* is the orbit of any planet revolving round the ſun; upon ſuppoſition, that the planet is continually impelled towards the ſun by its own weight or by the force of gravity, it will follow, if this propoſition is true, that ſuch a planet by a radius drawn to the ſun deſcribes equal ſpaces round it in equal times. A farther uſe of this propoſition will appear in propoſition 161.

We have ſeen already that as the revolving body approaches the point F it moves faſter than it did, when it was at *m*. Therefore if two arcs as *ml* and *rs* are deſcribed in equal times, it is evident that *ml* the baſe of the triangle *m*F*l*, where the body moves ſlowly, muſt be ſhorter than *rs* the baſe of the triangle *r*F*s* where the body moves faſter. For this reaſon though the triangle *m*F*l* is higher than the triangle or area *r*F*s*, yet the former will be narrower, or will have a ſhorter baſe in proportion; and conſequently one of theſe areas or triangular ſpaces will be equal to the other.

To prove this more exactly let us conſider ABCDE, Plat. VI. fig. 4. as part of an orbit deſcribed by a body, that revolves round the point S by means of a centripetal force, which impells the body towards that point. And let us ſuppoſe here, as we did in the laſt propoſition, that the action of this centripetal force is not continued, but that its impulſes are repeated after certain intervals of time. So that the body may move from A to B in a ſtrait line, and not be impelled towards S by the centripetal force that acts there, till it is arrived at B. The body left to itſelf would move on from B, by the firſt law of motion, in the ſtrait line B*c*. But if it is there impelled towards S by the centripetal force, it will deſcribe neither B*c* which is in the direction of the projectile force, nor BG, which is in the direction of the centripetal force; but BC which is the diagonal of a parallelogram, whoſe two ſides are B*c* and BG the two forces that joyntly act upon the body. When the body has been juſt as long moving in the direction BC as it was before in the direction AB, that is, after an equal interval of time, ſuppoſe another impulſe of the centripetal force to be exerted upon it and to draw it, when it is arrived at C, out of the direction C*d* into the direction CD. And when, after an equal interval of time is paſſed, the body arrives at D,

let

let it be impelled towards S by another ſtroke of the centripetal force, and be made to deſcribe DE. Since AB, BC, CD, DE, &c, are the lines deſcribed in equal times by the revolving body, the areas ſwept round S in equal times by a radius drawn from the body to S are ASB, BSC, CSD, DSE, &c,. Theſe areas we muſt prove to be equal to each other upon ſuppoſition, that S is the point towards which the body is impelled. We will therefore ſhew that upon this ſuppoſition, the triangle ASB is equal to the triangle BSC. Now AB is the ſpace, which the body deſcribes in a certain time by the projectile force only: and if only this force acted upon the body, it would go ſtrait forwards in the ſame right line and would in an equal time deſcribe B*c*, which is equal to AB. For, by the firſt law of motion, a body acted upon by only one force continues to move on uniformly in the ſame right line. And if it moves on the ſame pace after it is paſſed B, that it did in coming to it, the ſpace AB deſcribed in a certain time muſt neceſſarily be equal to the ſpace B*c*, which the body left to itſelf, or not impelled by the centripetal force, would deſcribe in an equal time. From hence it follows that the triangle ASB is equal to the triangle BS*c*: for their baſes AB and B*c* are equal, and as they are both terminated at the ſame point S their perpendicular heights are the ſame. Eucl. b. I. prop. 38. Now the centripetal force, which impelled the body, when it was at B, is ſuppoſed to act in the direction BS: and let the centripetal force in this direction be to the projectile force in the direction AZ as BG to B*c*. The body impelled by both forces together is made to deſcribe BC the diagonal of a parallelogram of which B*c* is one ſide, and the other ſide or BG is in the direction BS. But when one ſide is BG and the diagonal is BC, the oppoſite ſide or *c*C is parallel to BG. Eucl. b. I. prop. 34. Therefore *c*C and BS, of which BG is a part are likewiſe parallel to each other. But BS is the baſe both of the triangle BSC and of the triangle BS*c*; and theſe two triangles are both of them between the lines BS and C*c*, which we have juſt proved to be parallel to each other. Therefore, ſince all triangles upon the ſame baſe and between the ſame parallels are equal, Eucl. b. I. prop. 37. the triangle BSC is equal to the triangle BS*c*. And ſince ASB has been likewiſe proved to be equal to BS*c*, it follows that ASB and BSC, the areas deſcribed in equal times round S, are equal to each other; becauſe each of them is equal to BS*c*.

When the body is arrived at C, the projectile force, if it was left to itſelf, would carry it ſtrait on from thence in the ſame direction, in which it was moving before: and, becauſe all motion ariſing from the ſingle action of one force is uniform, by propoſition 11, the body, after it is paſſed C, would take up juſt as much time to deſcribe C*d* equal to BC, as it took up before in deſcribing BC. That is to ſay, BC and C*d* the ſpaces deſcribed in equal times by the projectile motion are equal; becauſe the body, when it is once put into motion by the projectile force, will continue always to move on the ſame pace, and conſequently will deſcribe equal ſpaces in equal times. But ſince BC and C*d* the baſes of the two triangles BSC and CS*d* are equal, and the height of theſe two triangles is the ſame, as they are both terminated at the ſame point S, it follows that the triangles are equal. Eucl. b. I. prop. 38. Now the time of the bodys motion from B to C being the interval between two ſtrokes of the centripetal force; upon its arrival at C it will be impelled towards S; as S by the ſuppoſition is the point to which this force tends. Let the centripetal force in the direction CS be to the projectile force in the direction C*d* as CH to C*d*: and the body acted upon by both forces together will deſcribe the diagonal CD of the parallelogram whoſe two ſides are CH and C*d*, by propoſition 16, and the area paſſed over by a radius drawn from the body to the center u will be CSD. Now the triangle CSD is equal to the triangle CS*d*. For, ſince HC*d*D is a parallelogram, the two oppoſite ſides are parallel; that is, D*d* is parallel to HC; and conſequently D*d* is parallel to SC, of which HC is a part. Therefore the triangles CSD and CS*d*, being between the two lines D*d* and CS, or between the ſame parallels, and being likewiſe upon the ſame baſe SC, are equal. But BSC has likewiſe been proved to be equal to CS*d*. Therefore BSC the area ſwept by the radius, whilſt the body is paſſing from B to C, is equal to CSD the area ſwept by the ſame radius whilſt the body is paſſing from C to D. And conſequently, ſince the body is juſt as long in paſſing from B to C as from C to D, the areas ſwept in equal times round S, the point where the centripetal force is placed, are equal to each other.

What has here been proved to be true of a body moving in a polygon, if the centripetal force acts at certain intervals, is equally true, when a body moves in a curve; ſince a curve may be conſidered as a polygon conſiſting of innumerable infinitely ſmall ſides. And if the intervals between the ſeveral impulſes of the centripetal force are infinitely ſhort, that is, if the centripetal force is continued, or acts conſtantly upon the revolving body in every inſtant of its motion, then the ſides AB, BC, CD, DE, &c, muſt be infinitely ſhort. Becauſe theſe are the lines deſcribed by the revolving body between any two impulſes of the centripetal force: which lines muſt neceſſarily be infinitely ſhort, when theſe impulſes are continued, or ſucceed ſo cloſely to each other that no time paſſes between any one of them and the next that follows it. And when the ſides are infinitely ſhort, there will of courſe be innumerable ſuch ſides in the whole orbit deſcribed by the body.

But after we have ſhewn that the areas deſcribed by the radius in equal times round the point S, where the centripetal force acts, are equal; it follows that in double times a double area will be deſcribed, and in triple times a triple area, that is, that the areas deſcribed in unequal times are greater in proportion as the times of deſcribing them are greater, or leſs as the times are leſs.

160. *When a body by a radius drawn to any immoveable point within its orbit deſcribes equal areas in equal times round that point, or in unequal times areas proportional to the times, it is impelled towards that point by the centripetal force, which retains it in its orbit.*

This propoſition is the converſe of the foregoing one. In the laſt propoſition we ſuppoſed that S, Plat. VI. fig. 4. was the point towards which the body is made to tend by the centripetal force: and from thence we proved that the body by a radius drawn to S muſt neceſſarily deſcribe equal areas in equal times round S. But in this propoſition we ſuppoſe that equal areas are deſcribed in equal times round S; and from thence we are to prove that S is the point towards which the body is impelled by the centripetal force.

The uſe, which is made of this propoſition, will help to explane the meaning of it. If from obſervation it appears that the moon by a radius drawn to the earth deſcribes equal areas in equal times; and if the earth is an immoveable point within the moons orbit; then, from this propoſition, it will follow that the moon is retained in its orbit and kept from running off in a tangent by a centripetal force, which acts in the earth and conſtantly impells it thither.

In like manner, if it is found by obſervation, that all the planets by a radius drawn to the ſun deſcribe equal areas in equal times; and if the ſun is an immoveable point within the reſpective orbits of the planets; then, from this propoſition, we may ſhew that the planets are conſtantly urged towards the ſun by a centripetal force; or that the force, which retains them in their orbits, acts in the ſun and by conſtantly impelling the planets makes them tend thither.

If a body revolves in the orbit *mbnb*, Plat. V. fig. 10. and deſcribes by a radius drawn to F equal areas in equal times: for inſtance, if it is juſt as long in moving from *m* to *l* as from *r* to *s*; and *m*F*l*, which is the area paſſed over, whilſt the body moves from *m* to *l*, is equal to *r*F*s* the area paſſed over, whilſt it moves for an equal time from *r* to *s*; then ſince we have ſeen by the laſt propoſition that a centripetal force acting in F or conſtantly impelling the body thither will make the body move in ſuch a manner, it is probable, that where there is ſuch a motion as this it is owing to a centripetal force acting in the point F.

This we may demonſtrate more exactly in the following manner. Let AB, BC, CD, DE, &c, Plat. VI. fig. 4. be the lines which a revolving body deſcribes in equal times, that is, ſuppoſe the body to be juſt as long in moving from A to B as from B to C, or from C to D, or from D to E. Suppoſe farther that the centripetal force acts at certain intervals of time, and that one ſtroke of it impells the body when at B, another when at C, a third when at D. And let S be the point round which equal areas are deſcribed in equal times. Then from the ſuppoſition it follows, that, ſince the body takes up an equal time to move from A to B, and from B to C; the area ſwept by the radius whilſt the body is moving from A to B, muſt be equal to the area ſwept by it, whilſt the body is moving from B to C; that is, the triangular ſpace ASB by the ſuppoſition is equal to the triangular ſpace BSC. Now as only the projectile force acts upon the body, whilſt it moves from A to B, the line AB will be deſcribed with an uniform velocity, by the firſt law of motion, propoſition 11. And if no centripetal force was to act upon the body, when it arrives at C, it would continue to move on uniformly as before in the ſame right line, and in an equal time would deſcribe the line B*c* equal to the line AB. From the equality of the two lines AB and B*c*, which are the baſes of the triangles ASB and BS*c*, it follows, that theſe two triangles, being both of the ſame height, are equal to each other. Eucl. b. I. prop. 38. But ASB and BSC are equal by the ſuppoſition. Therefore BS*c* and BSC are equal. Now BSC and BS*c* becauſe they are upon the ſame baſe SB, and are equal to each other, muſt be between the ſame parallels. Eucl. b. I. prop. 39. But they are between the lines C*c* and SB; and conſequently C*c* and SB are parallel to each other. When the body arrived at B, if it had been left to itſelf, the motion communicated by the projectile force would have carryed it on uniformly from B to *c*. But the centripetal force impelled it at B and made it move in the line BC inſtead of B*c*. Since therefore the body when at B, was acted upon by two forces, the projectile force in the line B*c* and the centripetal force, and theſe two forces together made it deſcribe BC, it follows that BC is the diagonal of a parallelogram, of which B*c* is one ſide, and the centripetal force acts at B in the direction of the other ſide. The queſtion then is, in what direction this other ſide of the parallelogram lies at B; for when this is determined the direction of the centripetal force will be known. Now in a parallelogram, whoſe diagonal is BC and one of whoſe ſides is B*c*, C*c* muſt be another: and as the oppoſite ſides of a parallelogram are always parallel to each other, the direction of the centripetal force, when the body was at B, muſt be parallel to *c*C. But, from the deſcription of equal areas round the point S, we have juſt now proved that *c*C and BS are parallel to each other. Therefore from the deſcription of equal areas round S it follows that the centripetal force, when the body is at B acts in the direction BS or makes the body tend towards S.

When the body is moving in the direction BC and arrives at C; it would, if no new force was to act upon it, go on uniformly from C to *d* in the ſame right line, in which it was moving before. But at C another impulſe of the centripetal force changes its direction and makes it move in the line CD. Therefore CD is the diagonal of a parallelogram, and ſince C*d* is one of its ſides, *d*D muſt be another. And, when the body was at C the centripetal force impelled it in a direction parallel to this other ſide or to *d*D. But, becauſe equal areas are deſcribed in equal times round S, CS is parallel to *d*D; therefore the line CS is parallel to *d*D, and conſequently, the centripetal force impelled the body, when at C in the direction CS or made it tend towards S. The ſame may be ſhewn in every other point of its orbit. Therefore the centripetal force tends to that point round which the body by a radius drawn thither deſcribes equal areas in equal times. In what manner it follows from the deſcription of equal areas in equal times round S, that *d*D is parallel to CS has been ſhewn already.

But to make it the more intelligible and to fix it the better in the readers memory; we will go over the proof of it again in a different manner. BC and CD are deſcribed by the body in equal times, and therefore BSC and CSD the areas deſcribed in the mean time by a radius drawn to S are equal by the ſuppoſition. But ſince CD is the diagonal, and C*d* is one ſide of the parallelogram C*d*DH, it follows, by propoſition 16, that the body deſcribes this diagonal CD by the centripetal and projectile forces together, in the ſame time that it would deſcribe C*d* by the projectile force alone. But ſince BC and CD are deſcribed in equal times, and likewiſe CD and C*d*, it follows that BC and C*d* would be deſcribed in equal times, if the body was left to be carryed on by the projectile force only: and as the velocity ariſing from the action of this one force is uniform, the ſpaces deſcribed by ſuch a velocity in equal times muſt be equal. Therefore BC is equal to C*d*. From hence it follows that, ſince BC and C*d* the two baſes of the triangles BSC and CS*d* are equal and ſince theſe triangles have the ſame height, their areas are equal, or BSC is equal to CS*d*. But BSC is likewiſe equal to CSD by the ſuppoſition. Therefore CSD is equal to CS*d*. Now two triangles ſuch as CSD and CS*d* upon the ſame baſe SC and equal to each other are between the ſame parallels, Euc. b. I. prop. 39. But CSD and CS*d* are between the lines *d*D and CS. Therefore *d*D and CS are parallel. And when the body was at C the centripetal force impelled it in a line parallel to *d*D, that is in the line CS towards the point S, round which it deſcribes equal areas in equal lines.

What has thus been proved in this part of the orbit ABCD, might be ſhewn to be equally true in every other part of the orbit. And though ABCD is a polygon and not a curve; yet in a curve likewiſe, if equal areas are deſcribed in equal times by a radius drawn to S, S is the point in which the centripetal force acts. Becauſe a curve is only a polygon of innumerable infinitely ſhort ſides. The line here deſcribed, as has been obſerved already under the two foregoing propoſitions, is a polygon and not a curve for no other reaſon, but becauſe we ſuppoſed the ſeveral ſtrokes of the centripetal force to be repeated ſeparately from each other and after certain intervals of time: ſo that, if the body begins to move from A, there ſhould be time enough allowed for it to arrive at B before any impulſe of that force urged it towards S, and after this ſtroke there ſhould be an interval of time long enough for the body to move from B to C before it is impelled by a ſecond: and the third ſtroke ſhould not act upon it till it had deſcribed CD. Now if theſe intervals are infinitely ſhort, that is, if the action of the centripetal force is continued, which is the caſe when the body revolves in a curve; theſe impulſes will be infinitely near to each other: and conſequently the ſpaces AB, BC, CD, &c. which are deſcribed between any one impulſe and the next will be infinitely ſhort. And ſince in the time of a whole revolution there muſt be innumerable ſtrokes or impulſes of the centripetal force, if they are thus repeated without any interval between them; it follows that when theſe ſides AB, BC, CD, &c. are infinitely ſhort, there will be innumerable of them, and the polygon will be changed into a curve.

161. *The velocity of a body, which is impelled towards an immoveable center, and revolves in ſpaces where it meets with no reſiſtance, is in different parts of the orbit inverſely as a perpendicular let fall from that center upon a tangent to that part.*

Let *mbnb*, Plat. V. fig. 10. repreſent the orbit, which any planet deſcribes round the ſun at reſt in the point F. Then if the planet is conſtantly attracted by the ſun or impelled towards it by the force of gravity, and moves in a vacuum ſo as to meet with no reſiſtance; the velocity of the planet in any one part of its orbit as *m* may be thus compared with its velocity in any other part as *r*. To the point *m* draw the tangent *tmt*, and to the point *r* draw the tangent *trt*, and from F draw F*m* perpendicular to *tmt*, and F*e* perpendicular to *trt*. The velocities at theſe two points *m* and *r* will be to each other as theſe perpendiculars F*m* and F*e* inverſely: by which we mean that the velocity of the planet at *m* is to its velocity at *r* as F*m* to F*e* inverted, or as F*e* to F*m*. That is to ſay the velocity of it at *m* is as much leſs than at *r* as F*m* is greater than F*e*, or as F*e* is leſs than F*m*.

This follows from propoſition 159. For ſince a radius drawn from the body to the point F deſcribes equal areas in equal times round F, *m*F*l* and *r*F*s* being ſuppoſed to be areas deſcribed in equal times are two equal triangles. Therefore as much higher as *m*F*l* is than *r*F*s*, ſo much broader *r*F*s* is than *m*F*l*. For if *r*F*s* had not in breadth what it wants in height it could not be equal to *m*F*l*. Now the height of *m*F*l* means its perpendicular height or F*m*, and its breadth is its baſe *ml*. So likewiſe the height of *r*F*s* is its perpendicular height F*e*, and its breadth is its baſe *rs*. Therefore, becauſe the triangles are equal, *ml* is as much leſs than *rs* as F*m* is greater than F*e*. Or the baſes *ml* and *rs* are to each other as F*m* to F*e* inverted, that is as F*e* to F*m*. But the velocity of the body either at *m* or at *r* is proportional to the ſpace deſcribed by it in equal times. And ſince *ml* and *rs* are the arcs deſcribed by the body, whilſt the radius drawn to the center F deſcribes equal areas; theſe arcs or the baſes *ml* and *rs* are the ſpaces deſcribed by the body in equal times; and conſequently the velocities at *m* and *r* muſt be to each other in the ſame proportion as theſe baſes *ml* and *rs*. But the baſes are to each other inverſely as the perpendiculars let fall from F to the tangents at the points *m* and *r*. Therefore the velocities of the body at *m* and *r* are likewiſe inverſely as thoſe perpendiculars.

Here it will not be improper to obſerve that, if a body revolves in an elliptical orbit, ſuch as *mbnb*, and the centripetal force impells it towards one of the focuſes F, the tangent is perpendicular to the line in which this force acts, when the body is either at the greateſt diſtance *m*, or at the leaſt diſtance *n*. Therefore at theſe two diſtances, the perpendiculars are the diſtances themſelves: and ſince the velocity in every point of the orbit is inverſely as the perpendicular to a tangent at that point, the reſpective velocities of the body at the greateſt and leaſt diſtances will be as Fm to Fn inverſely, or inverſely as the diſtances of the revolving body from F.

162. *In circles, and in all curves which have the ſame curvature with circles, the naſcent or evaneſcent ſubtenſe of the angle of contact is as the ſquare of the conterminous arc.*

To the point A Plat. VI. fig. 5. in the ſemicircle ADC draw the tangent AB. Then the angle BAD, which this tangent makes with the curve at the point A, is called the angle of contact. Draw BD and likewiſe *bd* parallel to AC the diameter. Either of theſe lines BD or *bd* is called the ſubtenſe of the angle of contact; for both of them, though at different diſtances from A, ſubtend the angle at A. AD an arc, which begins at the point of contact A and is terminated at BD, is called the conterminous arc to the ſubtenſe BD. And A*d* is the conterminous arc to the ſubtenſe *bd*. The right line AD, which is the chord of the arc AD, is called the conterminous chord to the ſubtenſe BD. And the right line A*d* is called the conterminous chord to the ſubtenſe *bd*. From D draw the line DC; and from *d* draw the line *d*C. If theſe two lines DC and *d*C are conceived to turn round upon the point C as upon a center, ſo that the two points D and *d*, and with them the two ſubtenſes BD and *bd*, may approach towards A; then it is evident that as theſe ſubtenſes BD and *bd* come nearer to A, they will be ſhorter, and will, as they keep moving on, continue to decreaſe till at laſt they become points, coincide with A, and ſo vaniſh. The ultimate ratio of theſe ſubtenſes is their laſt proportion, or that proportion, which they bear to each other when they are evaneſcent, that is, when they are the neareſt poſſible to A, or at the inſtant when they vaniſh.

The propoſition affirms that, whatever may be the proportion of BD and *bd* to each other at any other time, yet ultimately, at the inſtant they vaniſh, or when they are the neareſt poſſible to A, BD is to *bd* as the ſquare of the arc AD is to the ſquare of the arc A*d*. For, from what has been ſaid, it appears that this is what we mean by affirming that the evaneſcent ſubtenſes of the angle of contact are ultimately as the ſquares of the conterminous arcs.

In order to prove this we muſt firſt ſhew that the ſubtenſes BD and *bd*, not only when they vaniſh, but at all times, and in all ſtates or conditions, are to each other as the ſquare of the chord or right line AD to the ſquare of the chord or right line A*d*. Let ED be drawn parallel to AB, and *ed* parallel to A*b*. Then, becauſe AB is a tangent at the point A and conſequently is perpendicular to the diameter AC, Eucl. b. III.

prop.

prop. 18. ED which is parallel to AB muſt likewiſe be perpendicular to AC. And the ſame may be ſhewn to be true of *ed* for the ſame reaſon. Now the angle ADC, and likewiſe the angle A*d*C, are right ones: becauſe they are both in a ſemicircle. Eucl. b. III. prop. 31. Therefore in the right-angled triangle ADC the line DE is drawn from the right angle at D perpendicular to AC the baſe, and conſequently AED one of the ſmaller triangles into which this line divides ADC is ſimilar to ADC. Eucl. b.VI. prop. 8. So that AE the ſhorteſt ſide of the triangle AED bears the ſame proportion to AD its longeſt ſide, that the ſame AD, which is the ſhorteſt ſide of the triangle ADE, bears to AC its longeſt ſide. Eucl. b.VI. prop. 4. That is to ſay, AE is to AD, as AD to AC. Or, AE, AD, and AC, are in continued geometrical progreſſion. Therefore AE multiplyed into AC is always equal to the ſquare of AD. Eucl. b.VI. prop. 17. But AE and BD, being parallel to each other, by the conſtruction of the figure, are the oppoſite ſides of a parallelogram and conſequently are equal to each other. Eucl. b. I. prop. 34. Therefore BD multiplyed into AC is equal to AE multiplyed into AC, or, by what has juſt been proved, to the ſquare of AD. That is, the ſubtenſe of the angle of contact multiplyed into the diameter is always equal to the ſquare of the conterminous chord. Now, if two quantities are always equal to each other, when one of the quantities decreaſes, or is by any means made leſs, the other muſt neceſſarily decreaſe in the ſame proportion. For if one of them was to decreaſe in a greater proportion than the other does, then that quantity which decreaſes the faſteſt would become leſs than the other; whereas they are ſuppoſed to be always equal. But BD multiplyed into AC is equal to the ſquare of AD: and for the ſame reaſon *bd* multiplyed into AC is equal to the ſquare of A*d*. That is, when the ſubtenſe is BD, the product of it multiplyed into the diameter AC is equal to the ſquare of the conterminous chord AD. And when the ſubtenſe is nearer to A, and conſequently is leſs than BD, as ſuppoſe it equal to *bd*, ſtill *bd* the ſubtenſe multiplyed into AC the diameter is equal to the ſquare of the conterminous chord A*d*. Therefore the product ariſing from BD multiplyed into AC always decreaſes in the ſame proportion with the ſquare of AD. But in all multiplication, where the multiplicand is given, the product will be proportional to the multiplicator. And in the product ariſing either from BD multiplyed into AC, or from *bd* multiplyed into AC, or from any ſubtenſe whatever multiplyed into the diameter, this diameter, which is the multiplicand, is a given quantity, and conſequently the product will be proportional to the ſubtenſe. Thus BD multiplyed into AC, is to *bd* mul-

PLATE V.

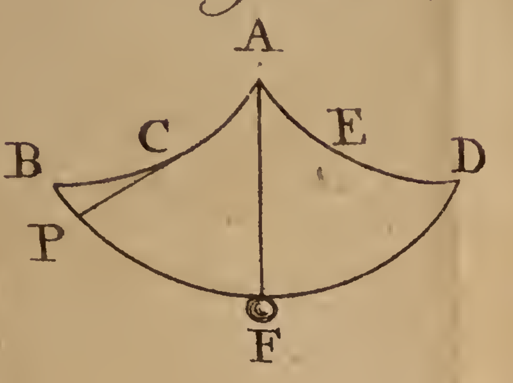


Fig. 1.

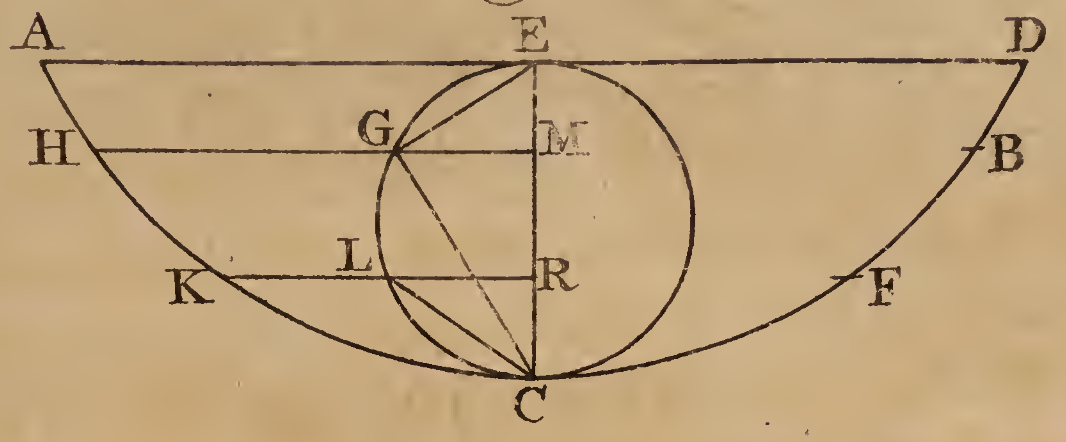


Fig. 2.

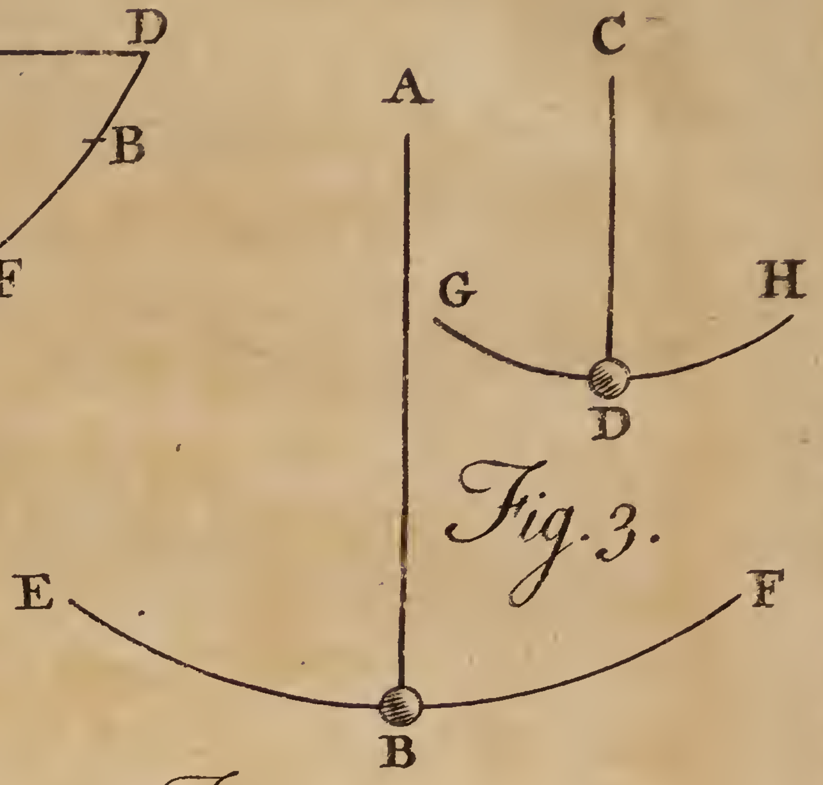


Fig. 3.

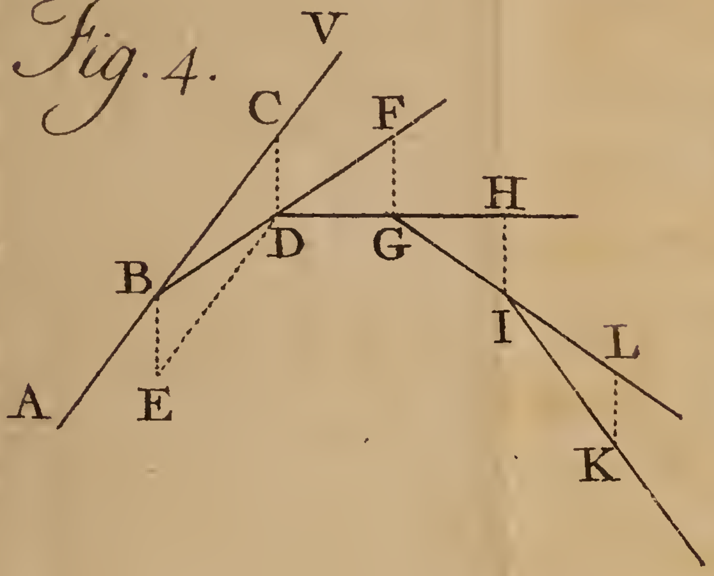


Fig. 4.

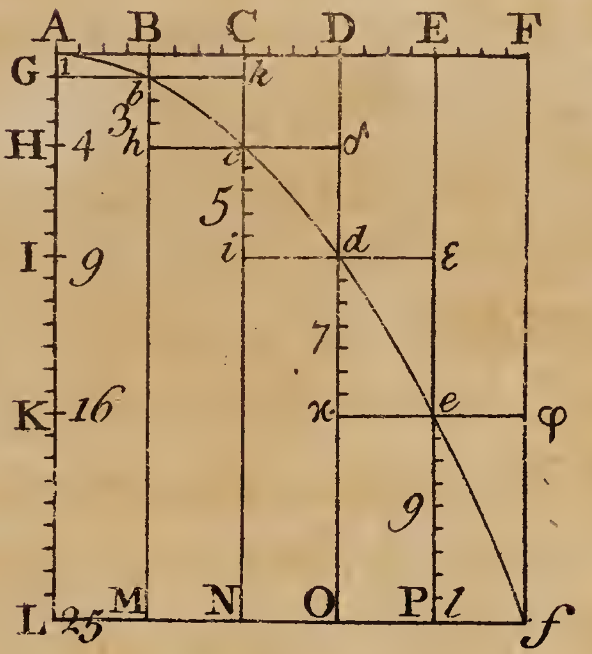


Fig. 5

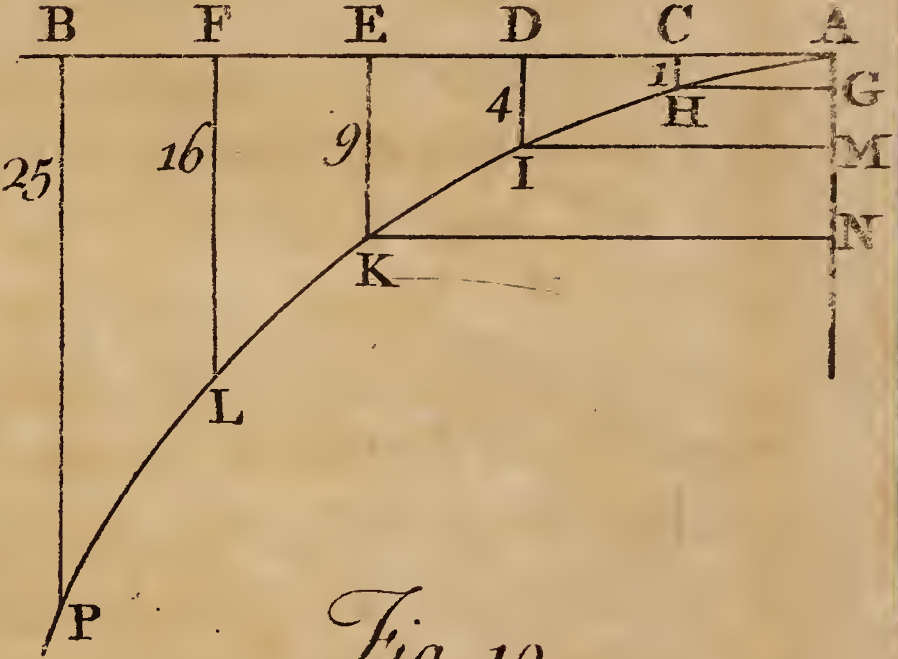


Fig. 6.

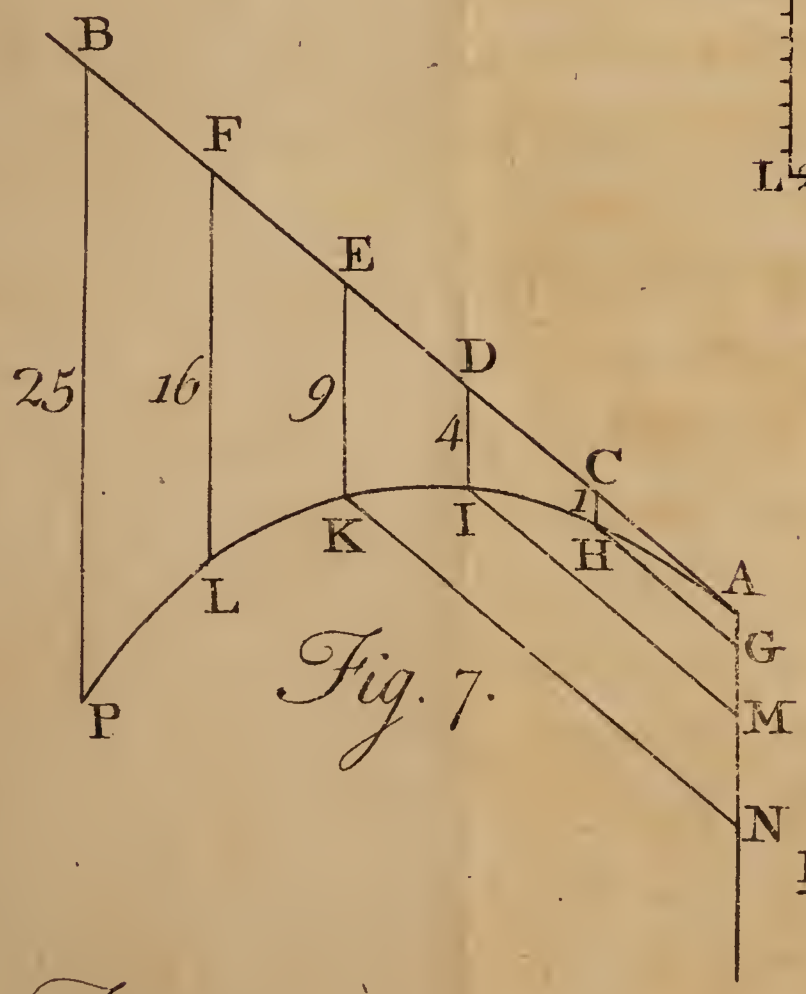


Fig. 7.

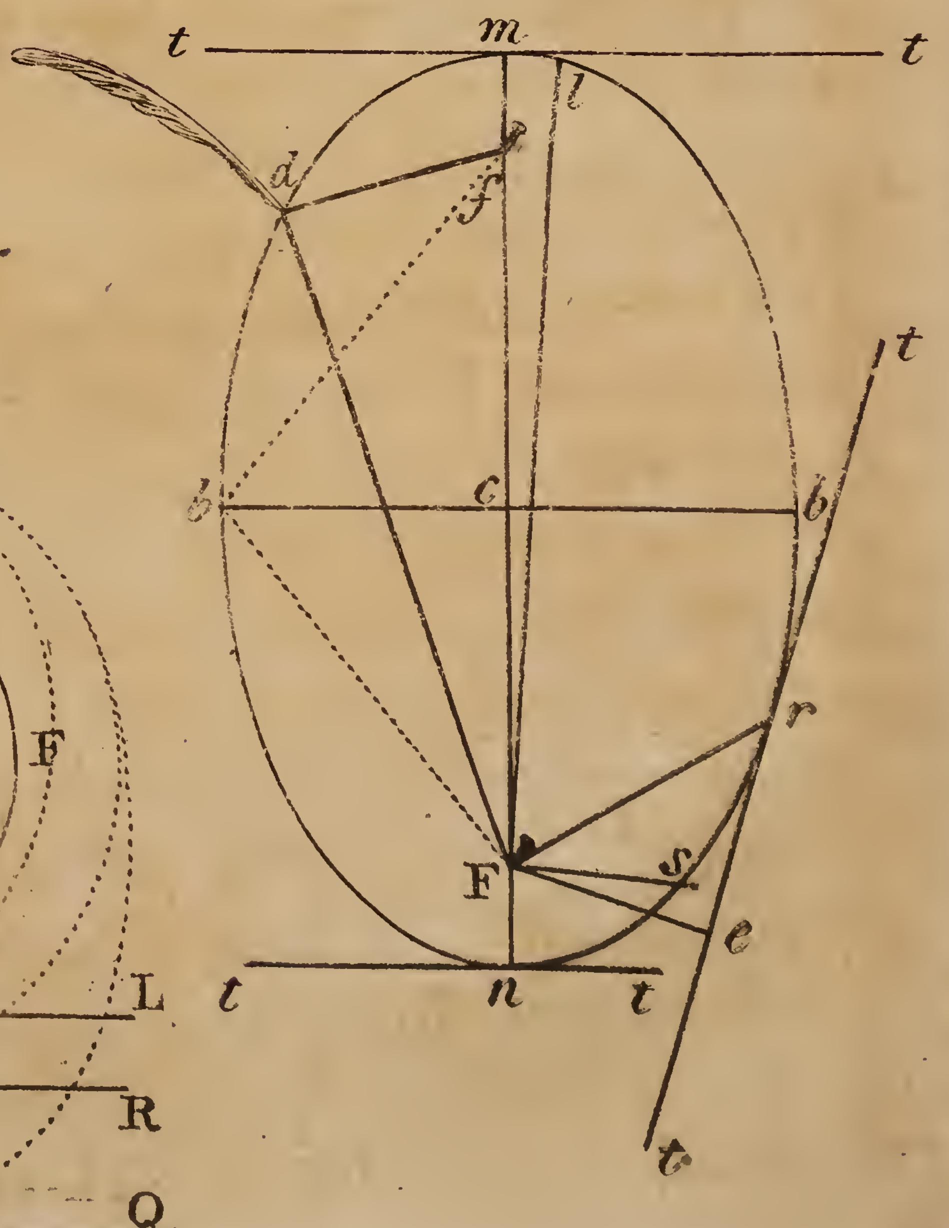


Fig. 10.

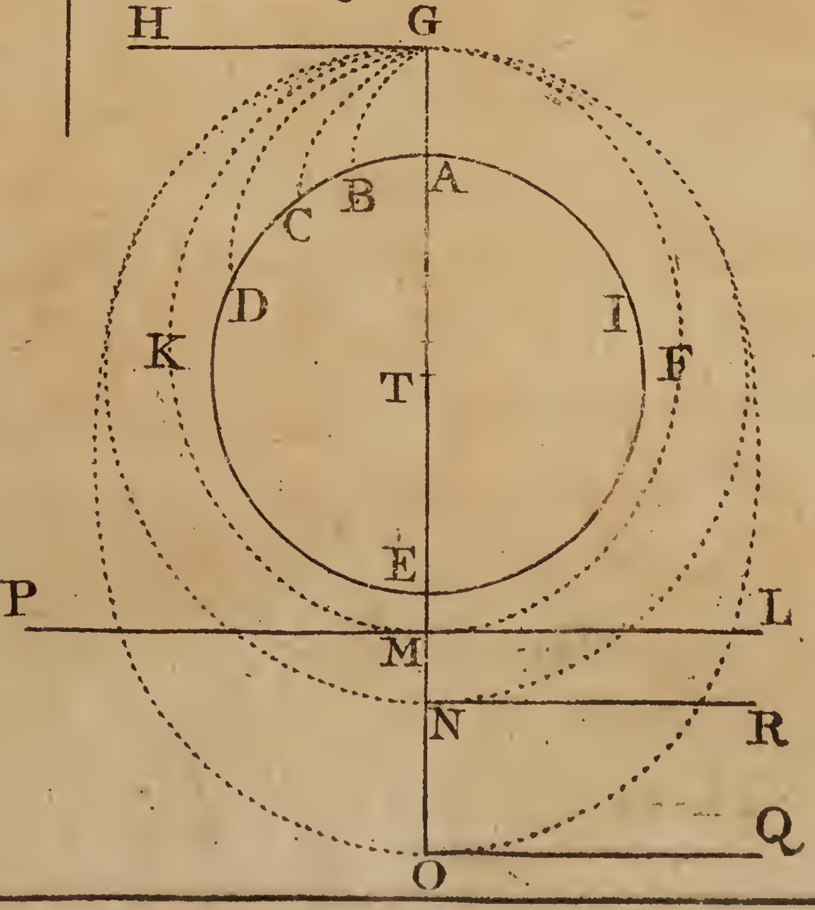


Fig. 9.

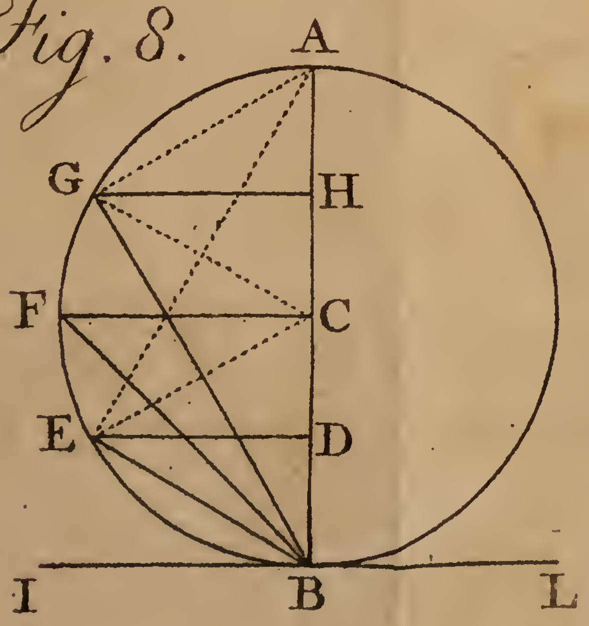


Fig. 8.

tiplyed into AC, as BD to *bd*. But theſe products of the diameter multiplyed into the ſubtenſes are proportional to the ſquares of the conterminous chords. BD multiplyed into AC bears the ſame proportion to *bd* multiplyed into AC, that the ſquare of AD bears to the ſquare of A*d*. Therefore, becauſe AC is a given quantity, BD is to *bd* as the ſquare of AD to the ſquare of A*d*. That is, the ſubtenſes are proportional to the ſquares of the conterminous chords.

But when BD and *bd* are evaneſcent, that is, when they are the neareſt poſſible to A, or at the inſtant when they vaniſh, the arcs AD and A*d* are ſo ſmall as not to differ from right lines or from their reſpective chords. Therefore whatever is the proportion of the chords the ſame is ultimately the proportion of the conterminous arcs. And ſince the ſubtenſes of the angle of contact are univerſally as the ſquares of the conterminous chords; it follows, that, when they are evaneſcent they are, ultimately, as the ſquares of the conterminous arcs. In the conſtruction of this figure, the ſubtenſes BD and *bd* were ſuppoſed to be drawn parallel to the diameter AC, and the foregoing demonſtration depends upon this ſuppoſition. But yet, though the ſubtenſes ſhould not be parallel to the diameter, we may ſtill demonſtrate that they are ultimately as the ſquares of the conterminous arcs, provided they are only parallel to each other.

By what has been already proved *mn* and *fg*, Plat.VI. fig. 3. being ſubtenſes of the angle of contact at B drawn parallel to the diameter BD are ultimately as the ſquares of the centerminous arcs B*n* and B*g*. Draw *og* and *an* two ſubtenſes, which are parallel to each other, but not to the diameter; theſe ſubtenſes are likewiſe ultimately as the ſquares of their conterminous arcs. For ſince B*a* is a tangent at the point B, and is therefore perpendicular to BD: Eucl. b. III. prop. 18. and ſince *mn* and *fg* are parallel to BD, it follows that *mn* and *fg* are likewiſe perpendicular to BD, and conſequently the angles *ofg* and *amn* are both of them right ones and equal to each other. But becauſe *og* and *an* are parallel to each other, by the conſtruction, the angle *fog* is equal to the angle *man*, for one is the external angle and the other is the internal oppoſite angle on the ſame ſide made by the line B*a* croſſing the two parallel lines *og* and *an*. Eucl. b. I. prop. 29. And ſince in the triangle *fgo* there are two angles reſpectively equal to two angles in the triangle *mna*, and all the three angles of each are only equal to two right ones, it follows, that the third angle at *g* in one of them is equal to the third angle at *n* in the other. Eucl. b. I. prop. 32. coroll. 2. But ſimilar or equiangular triangles have their ſides proportional. Eucl. b. VI. prop. 4.

Therefore *an* is to *og* as *mn* is to *fg*. And we have already proved that *mn* is ultimately to *fg* as the ſquares of the conterminous arcs. From whence it follows that *an* is likewiſe ultimately to *og* as the ſquares of the conterminous arcs B*n* and B*g*.

Or laſtly, ſuppoſe *an* and *og* were both of them directed to C the center of the circle. In this caſe they would each of them be a ſemidiameter continued from *g* and *n* reſpectively to the tangent B*a* and would neither be parallel to each other nor to BD in any but their evaneſcent ſtate. But when they are the neareſt poſſible to B, though they are directed to C, they will be parallel to each other: for when *an* and *og* vaniſh at B, they muſt be parallel to each other; becauſe then they both coincide with the ſame point B in the ſame right line BC, and therefore, by the foregoing demonſtration, they will be ultimately as the ſquares of the conterminous arcs.

When D and *d* coincide with A, Plat. VI. fig. 5. ſuppoſe them to ſet out from A and to move back again, the lines DC and *d*C turning ſtill upon the point C as they did before, only the contrary way. In this caſe, as D and *d* move away from A, the ſubtenſes, which were nothing when they coincided with A, will begin to exiſt, and the firſt proportion which they bear to each other, when they thus begin, is their prime ratio in their naſcent ſtate. Now the reader will eaſily conceive that they begin, when D and *d* are thus moving from A, in the ſame proportion that they before ended, when D and *d* were conſidered as moving towards A. Therefore the naſcent as well as the evaneſcent ſubtenſe of the angle of contact is as the ſquare of the conterminous arc.

This propoſition is true of other curves as well as circles, provided they are ſuch that a circle may be drawn of the ſame curvature. Let B*g* and B*n*, Plat. VI. fig. 6. be naſcent arcs in an ellipſis. The points *g* and *n* in the naſcent arcs B*g* and B*n* are the neareſt poſſible to B; and therefore they are not only arcs in the ellipſis, but are likewiſe arcs in a circle which has the ſame curvature that the ellipſis has at B. But, ſince *sn* and *rg*, or the naſcent ſubtenſes in any circle, have been proved to be as the ſquares of the naſcent arcs B*n* and B*g*, it follows that the ſame ſubtenſes *sn* and *rg*, when thoſe arcs are in an ellipſis, are likewiſe in the ſame proportion.

163. *The naſcent or evaneſcent ſubtenſe of the angle of contact in circles is equal to the ſquare of the conterminous arc divided by the diameter.*

BD, Plat. VI. fig. 5. is equal to the ſquare of AD divided by AC; if BD and AD are in their naſcent ſtate, that is, if the point D is ſuppoſed to be the neareſt poſſible to A. We have already ſeen in the foregoing propoſition, that univerſally BD the ſubtenſe, AD the conterminous chord, and AC the diameter, are in continued geometrical progreſſion; and that, when D is taken the neareſt poſſible to A, the chord and arc AD do not differ from one another. Therefore the naſcent ſubtenſe BD, the naſcent arc AD, and the diameter AC are in continued geometrical progreſſion. And conſequently BD multiplyed into AC is equal to the ſquare of AD, Eucl. b.VI. prop. 17. that is, the product BD × AC, where BD is the multiplicator and AC the multiplicand, $= AD^q$. But whenever two quantities are equal, if they are divided by equal diviſors the quotients will be equal. Therefore if $BD \times AC$ and AD^q are each of them divided by AC the quotients will be equal. But if $BD \times AC$ is divided by AC the quotient is BD: becauſe when any product is divided by the multiplicand the quotient is the multiplicator. Therefore BD is equal to the ſquare of AD divided by AC, that is $BD = \frac{AD^q}{AC}$.

164. *When bodies deſcribe circles, and the centripetal forces tend to the centers of thoſe circles; theſe forces in different circles, and likewiſe the centrifugal forces, are to one another as the ſquares of the arcs that the bodies deſcribe in the ſame time, divided by the ſemidiameters of the circles.*

Let there be two bodies revolving round the center C; one in the circle B*n*D, Plat. VI. fig. 3. and the other in the circle R*l*E: and ſuppoſe that the centripetal force, which retains them in their reſpective orbits, impells them towards C. One of theſe bodies ſetting out from B would by the firſt law of motion go on in the right line B*a*, if there was no centripetal force which acted upon it, and which by impelling it conſtantly towards C made it deſcribe the curve B*n*. When the body has moved in this curve through the arc B*g*, or from B to *g*, the ſpace through which it has been drawn from the tangent B*a* towards the center C is *fg*: and conſequently, ſince it is the centripetal force, which has thus drawn it, this force is proportional to the ſpace which it has made the body deſcribe, or proportional to *fg*. In like manner the body which revolves in the circular orbit R*l*E deſcribes the arc R*l* and does not run out from R in the tangent R*s*, as it would have done by the action of the projectile force alone. And the centripetal force which has drawn it from this tangent muſt be proportional to the effect produced by it. But the effect produced by it, whilſt the body is moving from R to *l*, is to impell or draw the body from the tangent through the ſpace *hl*. Therefore the centripetal force is proportional to *hl*. Now if B*g* and R*l* are the arcs which theſe two bodies deſcribe in equal times in their reſpective orbits, fg and hl, or the centripetal forces, are to each other as the ſquare of Bg divided by BC is to the ſquare of Rl divided by RC.

For ſuppoſe Bg and Rl to be the leaſt arcs poſſible, or naſcent arcs, then fg is equal to the ſquare of Bg divided by BD; and hl is equal to the ſquare of Rl divided by RE. That is, $fg = \frac{Bg^q}{BD}$ and $hl = \frac{Rl^q}{RE}$, by propoſition 163. Therefore fg is to hl as $\frac{Bg^q}{BD}$ is to $\frac{Rl^q}{RE}$. For ſince the firſt term or fg has been proved to be equal to the third or to $\frac{Bg^q}{BD}$ and the ſecond term or hl has likewiſe been proved to be equal to the fourth or to $\frac{Rl^q}{RE}$, it follows that the firſt muſt bear the ſame proportion to the ſecond that the third bears to the fourth. Now the proportion, which $\frac{Bg^q}{BD}$ and $\frac{Rl^q}{RE}$ bear to each other, will not be altered by changing the two diviſors BD and RE for two other diviſors that are in the ſame proportion. But BC, which is half BD, and RC, which is half RE, are in the ſame proportion to each other with BD and RE. Therefore $\frac{Bg^q}{BD}$ is to $\frac{Rl^q}{RE}$ as $\frac{Bg^q}{BC}$ is to $\frac{Rl^q}{RC}$. And conſequently fg and hl are to each other as $\frac{Bg^q}{BC}$ to $\frac{Rl^q}{RC}$. That is the naſcent ſubtenſes fg and hl, and conſequently the centripetal forces, by which the revolving bodies are impelled towards C, are as the ſquares of the arcs Bg and Rl divided reſpectively by the ſemidiameters BC and RC.

This will be equally true though Bg and Rl are not naſcent arcs or the ſmalleſt arcs poſſible, provided they are arcs, which the bodies deſcribe in the ſame time, that is, provided one of the bodies deſcribes the arc Bg, whilſt the other is deſcribing the arc Bl. For whatever are the velocities with which one body ſets out from B and the other from R, the arcs which they deſcribe will begin in the ſame proportion with thoſe velocities; the naſcent arc at B will be juſt as much greater or leſs than that at R, as the velocity of the body which ſets out from B is greater or leſs than the velocity of the other body which ſets out from R. But the arc, which one body deſcribes in any given time, will likewiſe be to the arc, which the other deſcribes in an equal time as the velocity of the former body is to the velocity of the latter. Therefore the arcs deſcribed in a given time are to one another as the naſcent arcs;

and conſequently the ſquares of the arcs deſcribed in a given time are as the ſquares of the naſcent arcs. But the centripetal forces have been proved to be as the ſquares of the naſcent arcs divided by the ſemidiameters. Therefore the centripetal forces are likewiſe as the ſquares of the arcs deſcribed in a given time divided by the ſemidiameters.

We may apply this reaſoning to another caſe by obſerving farther that the two curves B*n*D and R*l*E are ſimilar and conſequently BD is to RE, or BC to RC, as the whole curve B*n*D is to the whole curve R*l*E. Upon which account, we may change the two diviſors BC and RC in the fractional quantities $\frac{Bg^q}{BC}$ and $\frac{Rl^q}{RC}$ for two others B*n*D and R*l*E without changing the proportion of the fractions. That is, $\frac{Bg^q}{BnD}$ bears the ſame proportion to $\frac{Rl^q}{RlE}$ that $\frac{Bg^q}{BC}$ bears to $\frac{Rl^q}{RC}$. But the centripetal forces, by which the two bodies are retained in their reſpective orbits, are to each other as $\frac{Bg^q}{BC}$ to $\frac{Rl^q}{RC}$ and conſequently are likewiſe to each other as $\frac{Bg^q}{BnD}$ to $\frac{Rl^q}{RlE}$. That is the centripetal forces are as the ſquares of the arcs deſcribed in the ſame time divided reſpectively by the whole curves.

What has here been ſaid concerning bodies that revolve in circles, may be applyed to bodies that revolve in ſimilar elliptical orbits. For if they deſcribe equal areas in equal times round one of the focuſes, that is, by propoſition 160, if the centripetal force tends to one of the focuſes, then in parts of their orbits, which are ſimilar and likewiſe ſimilarly placed in reſpect of that focus, the centripetal force will be as the ſquare of the arc deſcribed in a given time divided by the diſtance from the focus. Suppoſe that one body revolves in the ellipſis BT, Plat. VI. fig. 6. and another in the ellipſis RP, and that each of them deſcribes equal areas in equal times round F. Then, by propoſition 160, the centripetal force tends to the point F, that is, the force, which retains each body in its orbit, impells it conſtantly towards this focus. Now if B*g* and R*l* are the ſmalleſt arcs poſſible, and are deſcribed in the ſame time, the ſpaces, through which the bodies are reſpectively impelled towards F, whilſt one of them is deſcribing the arc B*g* and the other the arc R*l*, are *rg* and *hl*. From whence it follows that theſe ſubtenſes *rg* and *hl* are to each other as the centripetal forces, which make the bodies tend to F. And if B and R, have the ſame curvature or are ſimilar places of theſe elliptical orbits, and are likewiſe ſituated in the ſame manner in reſpect of F, as they will be for inſtance when B is the greateſt diſtance from F in one orbit, and R is the greateſt diſtance from F in the other; then *rg* is to *hl*, as the ſquare of the arc Bg divided by BF is to the ſquare of the arc Rl divided by RF. That is, fg is to hl as $\frac{Bg^q}{BF}$ to $\frac{Rl^q}{RF}$. For from what has juſt been proved concerning bodies revolving in circular orbits, it follows that as B and R are ſimilar parts of ſimilar orbits and are likewiſe ſimilarly ſituated in reſpect of F, rg is to hl as the ſquare of Bg divided by the whole curve BgT to the ſquare of Rl divided by the whole curve RlP. That is rg is to hl as $\frac{Bg^q}{BgT}$ is to $\frac{Rl^q}{RlP}$. But BF is to RF as BgT to RlP; becauſe the curves are ſimilar. Therefore in the fractional quantities $\frac{Bg^q}{BgT}$ and $\frac{Rl^q}{RlP}$ the two diviſors BgT and RlP may be changed into BF and RF without affecting the proportion of the fractions. And conſequently rg is to hl as $\frac{Bg^q}{BF}$ is to $\frac{Rl^q}{RF}$. That is, the centripetal forces are as the ſquares of the arcs deſcribed in the ſame time divided by the diſtances from F.

What has here been proved of the centripetal force need not be particularly ſhewn to be true likewiſe of the centrifugal force; ſince it cannot poſſibly be otherwiſe: becauſe, by propoſition 156, theſe two forces are always equal to one another.

165. *When bodies revolve in orbits of equal ſemidiameters, the central forces are proportional to the ſquares of the velocities with which the bodies move, or are as thoſe ſquares directly.*

If two bodies revolve round different centers in equal orbits, or round the ſame center C, Plat.VI. fig. 3. in one and the ſame orbit, where BC is the common ſemidiameter of the orbit they deſcribe; then, ſuppoſing their velocities to be unequal, the forces that impell them towards this center muſt be unequal too. A greater force will be requiſite to retain that body in its orbit, which moves the faſter of the two, than what will be ſufficient to retain the other body, which moves ſlower, in the ſame or in an equal orbit. And ſuppoſing the velocity of one to be to the velocity of the other as 5 to 2, the centripetal forces muſt be as the ſquares of theſe velocities, that is, as the ſquare of 5 to the ſquare of 2, or as 25 to 4. And whatever may be the proportion of the velocities, with which the bodies revolve, the centripetal forces, when the ſemidiameters of the orbits are equal, muſt always be as the ſquares of the velocities.

The naſcent arcs are proportional to the velocities, with which the bodies begin to move. Thus the naſcent arc Bn, is to the naſcent arc Bg, as the velocity of one body is to the velocity of the other. The centripetal forces, which have acted in the mean time upon theſe two bodies to retain them in their orbit, are to each other as *mn* to *fg*, that is as the naſcent ſubtenſes of the angle of contact. But theſe ſubtenſes in the ſame circle or in equal ones, by propoſition 162, are to each other as the ſquares of the conterminous arcs B*n* and B*g*. And ſince theſe arcs are proportional to the velocities with which the bodies revolve; the ſquares of theſe arcs muſt be proportional to the ſquares of the velocities. Therefore the centripetal forces being as the naſcent ſubtenſes are likewiſe as the ſquares of the conterminous arcs, or as the ſquares of the velocities with which the bodies move.

We may likewiſe deduce this propoſition from the foregoing one. For ſince the arcs deſcribed in equal times are in all caſes as the velocity with which the body revolves, and ſince, by propoſition 164, the centripetal force is as the ſquare of the arc deſcribed in a given time divided by the ſemidiameter; it follows that the centripetal force is as the ſquare of the velocity divided by the ſemidiameter. Thus if one body deſcribes the arc B*n* in the ſame time that another body deſcribes the arc B*g*, the centripetal forces, by which theſe two bodies are impelled to the center C, will be to each other, as the quotient ariſing from the ſquare of B*n* divided by BC is to the quotient ariſing from the ſquare of B*g* divided by BC, that is as $\frac{Bn^q}{BC}$ to $\frac{Bg^q}{BC}$. But in all diviſion, where the diviſor is given the quotients are as the dividends. Now BC, or the diviſor is here a given quantity, becauſe the bodies are ſupppoſed to revolve either in the ſame circle or in circles of equal ſemidiameters. Therefore the quotients, which are here expreſſed by the two fractions $\frac{Bn^q}{BC}$ and $\frac{Bg^q}{BC}$, are to each other as the dividends, or as Bn^q to Bg^q, which are the numerators of thoſe fractions. And conſequently the centripetal forces are to each other as the ſquare of B*n* to the ſquare of B*g*, that is, as the ſquares of the arcs deſcribed in a given time, or as the ſquares of the velocities with which the bodies revolve.

The reader muſt remember that when we have proved a propoſition to be true of the centripetal force, it muſt neceſſarily be true of both the central forces, the centrifugal as well as the centripetal; becauſe one of theſe forces is equal to the other.

166. *When bodies revolve with equal velocities in unequal circles, the central forces are inverſely as the ſemidiameters of the circles.*

If one body revolves in the orbit B*n*D, Plat. VI. fig. 3. and another in the leſſer orbit R*l*E the centripetal force, which retains the former body in the orbit B*n*D is to the centripetal force, which retains the latter in the orbit R*l*E, as BC to RC inverſely, or as BC to RC inverted, that is as RC to BC. Or otherwiſe; when one body moves with the ſame velocity in the orbit B*n*D that another moves with in the orbit R*l*E; the force which impells the former body towards C is as much leſs than that which impells the latter body thither, as BC is greater than RC, or as RC is leſs than BC.

Let B*g* and R*l* be the arcs, which the two bodies deſcribe in equal times in their reſpective orbits. Then becauſe they move with equal velocities, by the ſuppoſition, the ſpaces deſcribed in equal times, or B*g* and R*l*, muſt be equal. And conſequently the ſquare of B*g* is equal to the ſquare of R*l*: that is, $Bg^q = Rl^q$. But, by propoſition 164, the centripetal forces, by which theſe bodies are impelled towards C, are to each other as the ſquares of the arcs deſcribed in the ſame time divided by the ſemidiameters; or as the quotient ariſing from the ſquare of B*g* divided by BC to the quotient ariſing from the ſquare of R*l* divided by RC: that is, as $\frac{Bg^q}{BC}$ to $\frac{Rl^q}{RC}$. But in all diviſion, if the dividends are equal, the quotients are inverſely as the diviſor, leſs as the diviſor is greater, and greater as the diviſor is leſs. Therefore ſince Bg^q and Rl^q are equal dividends, the quotients, which are expreſſed by the two fractions $\frac{Bg^q}{BC}$ and $\frac{Rl^q}{RC}$, are as BC to RC inverſely. And the centripetal forces, ſince they are as theſe quotients, are likewiſe as BC to RC inverſely, that is, as the ſemidiameters inverſely.

167. *When bodies revolve in circles and tend towards the center of theſe circles, the central forces are always as the ſquares of the velocities directly and as the ſemidiameters of the orbits inverſely.*

If the ſemidiameters are equal and the velocities different, then the central forces are as the ſquares of the velocities directly, by propoſition 165. And if the velocities are equal, but the ſemidiameters unequal, then the central forces are as the ſemidiameters inverſely, by propoſition 166. Therefore, where both the velocities and the ſemidiameters are different, the central forces will be as the ſquares of the velocities directly, and as the ſemidiameters inverſely. So that this propoſition is nothing more than the two foregoing ones joyned together.

Call the velocity with which one body revolves V, and the velocity with which any other body revolves v. Call the ſemidiameter of the orbit deſcribed by the former R, and the ſemidiameter of the orbit deſcribed by the latter r. By propoſition 164, the central forces are as the ſquares of the arcs deſcribed in the ſame time divided by the ſemidiameters, and becauſe the arcs deſcribed in the ſame time are as the velocities, the ſquares of thoſe arcs will be as the ſquares of the velocities. Therefore the central forces are as the ſquare of V divided by R to the ſquare of v divided by r, or as $\frac{V^q}{R}$ to $\frac{v^q}{r}$. And ſince in all diviſion the quotient is directly as the dividend and inverſely as the diviſor, theſe two quotients expreſſed by the two fractions $\frac{V^q}{R}$ and $\frac{v^q}{r}$ are as V^q to v^q directly and as R to r inverſely. But V^q and v^q are the ſquares of the velocities, and R and r are the ſemidiameters of the reſpective orbits deſcribed by the bodies. Therefore the central forces are as the ſquares of the velocities directly and as the ſemidiameters inverſely.

The 165, 166, and 167 propoſitions concerning the central forces of bodies revolving in circles have been all of them deduced from propoſition 164. Where we proved that the central forces are univerſally as the ſquare of the arc deſcribed in a given time divided by the ſemidiameter. Now this propoſition 164 was ſhewn to be true of bodies revolving in ſimilar curves, when they are in parts of theſe curves that are ſimilar and alſo ſimilarly placed in reſpect of the center towards which the bodies tend, provided we ſubſtitute the diſtances from that center for the ſemidiameter. In particular it was ſhewn that at B and R, Plat. VI. fig. 6. which are ſimilar points of two ſimilar elliptical orbits, and are ſimilarly placed in reſpect of F the focus, towards which the revolving bodies tend, the central forces are to each other as the ſquare of the arc B*g* divided by BF, to the ſquare of the arc R*l* divided by RF: that is, as the ſquares of the arcs deſcribed in the ſame time divided by the diſtances from the focus F, towards which they are impelled. From hence therefore it follows, that the 165, 166, and 167 propoſitions are applicable to bodies that revolve in elliptical orbits. And if two bodies were to revolve in the orbit B*g*T with unequal velocities the centripetal forces at B would be as the ſquares of thoſe velocities. If one body revolved in the orbit B*g*T, and the other in a ſimilar orbit R*l*P with the ſame velocity, then the centripetal forces at B and R would be as BF to RF inverſely. And univerſally in ſimilar parts that are ſimilarly placed in reſpect of the point F, towards which the revolving bodies are impelled, the centripetal force is as the ſquare of the velocity directly and as the diſtance from F inverſely.

168. *When bodies revolve in equal circles, the central forces are inverſely as the ſquares of their periodical times.*

If two bodies were to revolve either in the ſame circle B*n*D, Plat.VI. fig. 3. round the ſame center C, or in equal circles round different centers, and each body was to take up a different time to deſcribe its reſpective orbit, as ſuppoſe the periodical time of one of them to be 10 days and of the other 6 days: the ſquares of their periodical times are then to each other as 100 to 36. And theſe ſquares inverted are to each other as 36 to 100. The centripetal forces, which impell them towards the center of their orbits, will be in this proportion. And, whatever may be the proportion of the periodical times of the revolving bodies, the centripetal forces, and conſequently the centrifugal forces likewiſe, will always be as the ſquares of theſe times inverted.

The orbits, or the ſpaces which the bodies have to deſcribe in the time of a revolution, are equal by the ſuppoſition. Therefore the faſter any body moves the ſooner it will go round in its orbit, and the ſlower it moves the longer it will be in going round. That is, ſince the ſpaces to be deſcribed by each body are equal, the times taken up to deſcribe thoſe ſpaces, or the periodical times, will be inverſely as the bodys velocity, by propoſition 9. But, becauſe the periodical times are inverſely as the velocities, therefore the ſquares of the periodical times are inverſely as the ſquares of the velocities: or, which is the ſame thing, the ſquares of the velocities are inverſely as the ſquares of the periodical times. Now by propoſition 165, when the orbits are equal, the central forces are as the ſquares of the velocities. And by what has juſt been proved the ſquares of the velocities are inverſely as the ſquares of the periodical times. Therefore the central forces are inverſely as the ſquares of the periodical times.

169. *When bodies deſcribe unequal circles, if their periodical times are equal, the central forces are proportional to the ſemidiameters of their reſpective orbits, or are as thoſe ſemidiameters directly.*

If one body revolves in the orbit B*n*D and any other body revolves in an equal time in the orbit R*l*E; then their orbits are unequal and their periodical times are the ſame. In this caſe the centripetal force by which the former body is retained in the orbit B*n*D is as much greater than that which retains the latter in the orbit R*l*E, as BC the ſemidiameter of the former bodys orbit is greater than RC the ſemidiameter of the latters orbit.

Each body deſcribes its reſpective orbit in the ſame or an equal time, by the ſuppoſition. And, ſince each orbit contains 360 degrees, each body deſcribes 360 degrees in its reſpective orbit in the ſame time. But if, whilſt one is deſcribing 360 degrees, the other deſcribes juſt as many; then whilſt one is deſcribing 180 degrees or half its orbit, the other muſt likewiſe deſcribe 180 degrees or half its orbit; whilſt one is deſcribing 90 degrees or a quarter of its orbit, the other muſt likewiſe deſcribe 90 degrees or a quarter of its orbit; whilſt one is deſcribing 60 degrees the other muſt deſcribe 60; and univerſally, when the periodical times are equal, each body in any given part of its periodical time deſcribes an equal number of degrees in its reſpective orbit. But arcs, that contain an equal number of degrees, bear the ſame proportion to their reſpective orbits, or are ſimilar arcs: as for inſtance if B*n* and R*l* contain each of them 60 degrees, then the arcs B*n* and R*l* are ſimilar, that is, B*n* bears the ſame proportion to the whole orbit B*n*D, that R*l* bears to the whole orbit R*l*E. Therefore, when the whole periodical times of the revolving bodies are equal, the arcs deſcribed by them in any equal parts of thoſe times are ſimilar. But, by propoſition 164, the central forces are as the ſquares of the arcs deſcribed in the ſame time divided by the ſemidiameters; and conſequently; when the periodical times are equal, the central forces are as the ſquares of ſimilar arcs divided by the ſemidiameters. Thus if two bodies revolve round C one in the orbit B*n*D the other in the orbit R*l*E in equal times; any arcs as B*n* and R*l*, which they deſcribe in the ſame time in their reſpective orbits, will be ſimilar: and the centripetal force by which one of the bodies tends to C will be to the centripetal force by which the other tends thither, as the ſquare of B*n* divided by BC to the ſquare of R*l* divided by RC; that is, as $\frac{Bn^q}{BC}$ to $\frac{Rl^q}{RC}$. But ſimilar arcs are in the ſame proportion to each other that the whole orbits are, or that the ſemidiameters of the orbits are. Thus B*n* is to R*l* as BC to RC: and conſequently the ſquare of B*n* is to the ſquare of R*l* as the ſquare of BC to the ſquare of RC: that is Bn^q is to Rl^q as BC^q to RC^q. Therefore in the two fractions $\frac{Bn^q}{BC}$ and $\frac{Rl^q}{RC}$, BC^q may be ſubſtituted for Bn^q and RC^q for Rl^q without changing the proportion which thoſe fractions bear to each other, that is $\frac{Bn^q}{BC}$ is to $\frac{Rl^q}{RC}$ as $\frac{BC^q}{BC}$ is to $\frac{RC^q}{RC}$. But when the ſquare of any quantity is divided by the quantity itſelf, the quotient is likewiſe the ſame quantity. Therefore $\frac{BC^q}{BC}$ or the ſquare of BC divided by BC is equal to BC, and likewiſe $\frac{RC^q}{RC}$ is for the ſame reaſon equal to RC. From whence it follows that $\frac{Bn^q}{BC}$ is to $\frac{Rl^q}{RC}$ as BC to RC. But the centripetal forces are to each other as $\frac{Bn^q}{BC}$ to $\frac{Rl^q}{RC}$. Therefore they are likewiſe to each other as BC to RC, or as the ſemidiameters of the orbits in which the bodies revolve.

170. *When bodies revolve in circles and are impelled towards the centers of thoſe circles, the central forces are always as the ſemidiameters of their orbits directly and as the ſquares of their periodical times inverſely.*

If the periodical times are equal, and the ſemidiameters unequal, the central forces are as the ſemidiameters directly, by propoſition 169. And if the ſemidiameters are equal, and the periodical times unequal, the central forces are inverſely as the ſquares of the periodical times by propoſition 168. Therefore where both the ſemidiameters and the periodical times are unequal, the central forces will be in a ratio compounded of both, or as the ſemidiameters directly and as the ſquares of the periodical times inverſely.

Call the ſemidiameter of one bodys orbit R, and the ſemidiameter of any other bodys orbit r. Call the periodical time of the former T and the periodical time of the latter t. The ſquares of the periodical times will be T×T and t×t, that is T^q and t^q, or TT and tt. For the ſquares of T and t are expreſſed in all theſe three different ways. Now ſince in all diviſion the quotient is directly as the dividend and inverſely as the diviſor, and ſince the central forces are as the ſemidiameters directly and as the ſquares of the periodical times inverſely, they will be as R divided by TT to r divided by tt, or as $\frac{R}{TT}$ to $\frac{r}{tt}$.

The 168, 169, and 170 propoſitions are deduced as we have ſeen from propoſitions 164 and 165, which have already been ſhewn to be applicable not only to bodies revolving in circular orbits, but alſo to bodies revolving in ſimilar elliptical ones, when they are in parts of theſe orbits, that are ſimilar and alſo ſimilarly placed in reſpect of the center towards which the bodies are impelled; provided we ſubſtitute the diſtances from that center for the ſemidiameters. Therefore with theſe reſtrictions the 168, 169, and 170 propoſitions are true of bodies revolving in elliptical orbits. The central forces, when the periodical times are equal, will be as the diſtances from the focus towards which the bodies are impelled. When the diſtances from that focus are equal, they will be inverſely as the ſquares of the periodical times. And when neither the periodical times nor diſtances are equal, they will be in a ratio compounded of both, that is, as the diſtances directly and the ſquares of the periodical times inverſely; or calling the diſtances of two bodies, that revolve round the ſame point, D and d and their periodical times T and t, the central forces will be as D to d directly and as TT to tt inverſely, that is, by what has been ſhewn, as $\frac{D}{TT}$ to $\frac{d}{tt}$.

171. *When bodies revolve round the ſame center, if the ſquares of their periodical times are to each other as the cubes of their diſtances from that center, the central forces of thoſe bodies are inverſely as the ſquares of their diſtances.*

The planets revolve round the ſun and the ſquares of their periodical times are found by obſervation to be to each other, as the cubes of their diſtances. For inſtance, The ſquare of Jupiters periodical time bears the ſame proportion to the ſquare of Mercurys periodical time, that the cube of Jupiters diſtance from the ſun bears to the cube of Mercurys diſtance. And the ſame analogy, or proportion, holds good in all the other planets that revolve round the ſun and are impelled towards it. And from hence we are to prove that the force, with which the ſun acts upon them to retain them in their reſpective orbits, is at different diſtances from the ſun inverſely as the ſquares of thoſe diſtances.

Call the diſtance of Jupiter from the ſun D and the diſtance of Mercury d. Call the periodical time of Jupiter T, and the periodical time of Mercury t. Then, by what is ſuppoſed in the propoſition, TT or the ſquare of Jupiters periodical time, bears the ſame proportion to tt or to the ſquare of Mercurys periodical time, that $D \times D \times D$ or D^c or DDD or the cube of Jupiters diſtance bears to $d \times d \times d$ or d^c or to ddd, or to the cube of Mercurys diſtance. Now by what was ſhewn in the laſt propoſition the centripetal force that retains Jupiter in its orbit is to the centripetal force that retains Mercury in its orbit as $\frac{D}{TT}$ to $\frac{d}{tt}$. But DDD bears the ſame proportion to ddd that TT bears to tt, by the ſuppoſition. Therefore in the two fractions $\frac{D}{TT}$ and $\frac{d}{tt}$ inſtead of TT and tt, we may ſubſtitute DDD and ddd. And becauſe theſe quantities DDD and ddd bear the ſame proportion to each other with TT and tt, the proportion of the two fractions will not be altered by this change, but $\frac{D}{DDD}$ and $\frac{d}{ddd}$ will be to each other as $\frac{D}{TT}$ and $\frac{d}{tt}$. But the centripetal force of Jupiter is to that of Mercury as $\frac{D}{TT}$ to $\frac{d}{tt}$. Therefore, ſince the ſquares of their periodical times are as the cubes of their diſtances, the centripetal force of Jupiter is likewiſe to that of Mercury as $\frac{D}{DDD}$ to $\frac{d}{ddd}$. Now neither of theſe fractions are in their leaſt terms. The way of reducing them to their leaſt terms is this: let D the numerator of $\frac{D}{DDD}$ be divided by D, that is, by itſelf, and the quotient ariſing from thence will be 1; for the quotient ariſing from any quantity divided by itſelf is always 1: and let the denominator DDD, or the cube of D, be likewiſe divided by D, and the quotient will be DD; for when the cube of any quantity is divided by that quantity the quotient will be its ſquare. And taking 1 the firſt quotient for a new numerator and DD the other quotient for a new denominator we ſhall have a new fraction $\frac{1}{DD}$ equal to $\frac{D}{DDD}$. So likewiſe if d the numerator of the fraction $\frac{d}{ddd}$ is divided by itſelf we ſhall have 1 for a new numerator, and if ddd the denominator is divided by the ſame quantity d we ſhall have dd for a new denominator. And when the numerator and denominator of the fraction $\frac{d}{ddd}$ are thus reduced to their leaſt terms by dividing both of them by the ſame quantity d, the fraction is changed into $\frac{1}{dd}$, which is equal to $\frac{d}{ddd}$. But, ſince the centripetal forces are as $\frac{D}{DDD}$ to $\frac{d}{ddd}$, and ſince $\frac{D}{DDD}$ is equal to $\frac{1}{DD}$ and $\frac{d}{ddd}$ is equal to $\frac{1}{dd}$; it follows that the centripetal force of Jupiter is to that of Mercury as $\frac{1}{DD}$ to $\frac{1}{dd}$. But all fractions expreſs a quotient ariſing from a diviſion of their numerator by their denominator. And in all diviſion, where the dividend is given, the quotient is inverſely as the diviſor. Therefore all fractions, whoſe numerators are equal muſt be inverſely as their denominators. From hence it follows, that the two fractions $\frac{1}{DD}$ and $\frac{1}{dd}$, whoſe numerators are each of them 1 and conſequently equal, muſt be to each other inverſely as DD to dd. But DD and dd are the ſquares of the diſtances.

Therefore theſe fractions, and conſequently the centripetal forces, which are as theſe fractions, are to each other inverſely as the ſquares of the diſtances.

From this propoſition it appears that the force, which impells the planets towards the ſun, and by impelling them prevents them from running out in right lines and retains them in their reſpective orbits, may be the force of gravity. Since this force at different diſtances from the ſun is varyed in the ſame proportion that the force of gravity would be. It has been ſhewn, in propoſition 23, that the weight of bodies or the force of gravity that acts upon them at different diſtances from the earths center is inverſely as the ſquares of thoſe diſtances. From whence we may conclude that, if heavy bodies, ſuch as the planets are, tend towards the ſun, in the ſame manner that other heavy bodies tend towards the earth; the force of gravity, that acts upon the planets and impells them towards the ſun, would at different diſtances from the ſun be inverſely as the ſquares of thoſe diſtances. And ſince it is found by obſervation that the ſquares of the periodical times of the planets are as the cubes of their diſtances, it follows from this propoſition, that the force, which retains them in their reſpective orbits, is likewiſe at different diſtances from the ſun inverſely as the ſquares of thoſe diſtances: and conſequently this force may be the force of gravity; ſince at different diſtances from the central body both theſe forces are found to be varyed by the ſame law.

172. *If a body, that revolves in an ellipſis, deſcribes equal areas in equal times by a radius drawn to one of the focuſes; the centripetal force, which retains this body in its orbit, is, at different diſtances from the focus, inverſely as the ſquares of the diſtances.*

If the moon revolves in the elliptical orbit ADPG. Plat.VI. fig. 1. and a radius or line drawn from the moon to the earth at S in one of the focuſes deſcribes equal areas in equal times as the moon goes round in its orbit; then, by propoſition 160, the centripetal force conſtantly impells the moon towards the earth: and in this propoſition we are to ſhew that in this caſe the centripetal force, that retains the moon in its orbit, muſt at different diſtances from the earth be inverſely as the ſquares of the diſtances. In like manner we may apply this propoſition to any of the planets: for as they revolve in elliptical orbits having the ſun in one of their focuſes, and deſcribe by a radius drawn to the ſun equal areas in equal times, this propoſition affirms that the centripetal force, which tends towards the ſun and prevents them from running out in rectilinear tangents, is, at different diſtances from the ſun, inverſely as the ſquares of thoſe diſtances.

In order to prove this we will ſhew, that at A and P, or at the greateſt and leaſt diſtances of the revolving body from S the central body, the centrifugal force is inverſely as the ſquares of the diſtances: and from hence it will follow that the centripetal force likewiſe, which is equal to the other, muſt be varyed in the ſame proportion. It was proved, in propoſition 165, that when bodies revolve in equal circles, that is, in circles which in equal parts have the ſame curvature, the centrifugal force is as the ſquare of the velocity with which the body moves. Now at A and P the greateſt and leaſt diſtances from S the curvature of the ellipſis in equal parts is equal. Therefore the centrifugal force, when the body is at A, is to the centrifugal force of it when at P, as the ſquare of its velocity at A to the ſquare of its velocity at P. But, by propoſition 161, the velocity of the revolving body, when it is at A, is to its velocity, when it is at P, as the diſtances inverſely. From whence it follows that the ſquare of its velocity at A is to the ſquare of its velocity at P as the ſquares of the diſtances inverſely. But the centrifugal forces are as the ſquares of the velocities. Therefore the centrifugal forces at A and at P are likewiſe as the ſquares of the diſtances inverſely. And ſince the centripetal force is equal to the centrifugal force, it muſt be varyed by the ſame law; and conſequently at A and at P the centripetal force muſt be inverſely as the ſquares of the bodys diſtances from the point S, towards which it is impelled.

The application of propoſition 167 to the caſe of bodies moving in elliptical orbits may occaſion a difficulty here, which we ſhould take care to clear up. It is there aſſerted that the central forces, of which the centrifugal force is one, are in elliptical orbits directly as the ſquare of the revolving bodys velocity and inverſely as its diſtance from the focus, towards which it tends, that is, as the ſquare of the velocity divided by the diſtance. Now here without any regard to the diſtance from S we ſay, that the centrifugal force is as the ſquare of the velocity. But the reader will remember that propoſition 167 relates to bodies that are in ſuch parts of their orbits as are ſimilarly placed in reſpect of the focus. Whereas we are now ſpeaking of the ſame body in different parts of the ſame orbit A and P, which, becauſe one is the greateſt and the other the leaſt diſtance from S, are not ſimilarly placed in reſpect of S. Therefore what was ſaid in propoſition 167 about the centrifugal force cannot be ſo applyed here as to be made the foundation of any objection.

By the foregoing propoſition we prove that the centripetal force, by which the planets that move round the ſun are retained in their reſpective orbits, is, at different diſtances from the ſun, inverſely as the ſquares of thoſe diſtances. For the ſquares of the periodical times of the planets that move round the ſun are found by obſervation to be to each other reſpectively as the cubes of their diſtances. Thus the ſquare of Mercurys periodical time bears the ſame proportion to the ſquare of Jupiters; that the cube of Mercurys diſtance bears to the cube of Jupiters. And in the foregoing propoſition we determined the law by which the centripetal force is varyed, wherever this analogy or proportion obtains. But that propoſition will never demonſtrate that the centripetal force, by which the moon is retained in the orbit that it deſcribes round the earth, is, at different diſtances from the earth, inverſely as the ſquares of thoſe diſtances: becauſe this analogy or proportion cannot be applyed where there is only one body revolving round the ſame center, which is the caſe of the moon revolving round the earth. For the nature of the analogy requires that there ſhould be the ſquares of two different periodical times, and the cubes of the diſtances of two different bodies to compare with one another. And ſince we cannot ſay the ſquare of the moons periodical time is to the ſquare of the periodical time of ſome other body revolving round the earth, as the cube of the moons diſtance is to the cube of that other bodys diſtance, becauſe there is no other body beſides the moon, that does revolve round the earth; it follows that we cannot apply that analogy to the caſe of the moon, but muſt make uſe of ſome other argument to prove that the centripetal force, which retains it in its orbit, is, at different diſtances from the earth, inverſely as the ſquares of thoſe diſtances. The propoſition now before us ſupplies us with ſuch an argument. For ſince it is found, by obſervation, that if the moons motion was not to be diſturbed by the ſun, it would deſcribe an immoveable ellipſis ſuch as ADPG round the earth at S, or in one of the focuſes, it follows that the centripetal force, by which it is impelled towards the earth, is, at different diſtances, inverſely as the ſquares of thoſe diſtances. And, ſince it has been proved that the force of gravity decreaſes and encreaſes by the ſame law at different diſtances from the earths center, it follows farther that the force, which impells the moon towards the earth and by ſo impelling it retains it in its obit, may, as far as yet appears, be the force of gravity or the ſame force that makes heavy bodies deſcend.

173. *When a body moves in an elliptical orbit and the centripetal force impells it towards one of the focuſes, the greateſt diſtance of the revolving body from that focus is called the higher apſis, and the leaſt diſtance is called the lower apſis.*

Thus Plat. VI. fig. 1. A is the higher apſis and P is the lower.

174. *A line drawn from one apſis to the other is called the line of the apſes.*

Such is the line ASP, which is the longer axis of the orbit deſcribed by the revolving body.

175. *If, in the departure of the revolving body from the focus where the centripetal force acts, this force decreaſes faſter than the ſquares of the diſtances from that focus encreaſe; or if in the approach of it to the focus, the centripetal force encreaſes faſter than the ſquares of the diſtances decreaſe; the orbit will be a moveable ellipſis, and the line of the apſes will turn round upon the focus in the ſame direction in which the body revolves.*

For the better underſtanding this propoſition, we will firſt explane what is meant by a centripetal force that in the departure of a body from the focus, towards which that force impells it, decreaſes faſter than the ſquares of the diſtances encreaſe, and will ſhew that a force, which decreaſes too faſt in the departure of the revolving body, will likewiſe encreaſe too faſt in the approach of it to the point S or focus, towards which that body is impelled. Whilſt a body moves round in the orbit ADPG Plat. VI. fig. 1. it is ſometimes nearer to S and ſometimes more remote from it: ſuppoſe four different diſtances of the body in four different parts of the orbit, that are to each other in the proportion of the following numbers, 1, 2, 3, 4, the ſquares of theſe numbers are 1, 4, 9, 16, and the reciprocals of theſe ſquares are $\frac{1}{1}$, $\frac{1}{4}$, $\frac{1}{9}$, $\frac{1}{16}$. And any centripetal force, which at the different diſtances 1, 2, 3, 4, is inverſely or reciprocally as the ſquares of theſe diſtances, decreaſes in the ſame proportion with theſe fractions $\frac{1}{1}$, $\frac{1}{4}$, $\frac{1}{9}$, $\frac{1}{16}$, which are the reciprocals of the ſquares of the diſtances 1, 2, 3, 4. The force of gravity in departing from the earths center decreaſes in this proportion. So that if we call the force 1 which impells the moon towards the earth, when its diſtance from the earth is called 1, then, if the moon was at the diſtance 2 from the earth or at twice the former diſtance, the force of gravity, that impells it towards the earth, would be leſs than before; nor does it decreaſe only as the diſtance itſelf, but as the ſquare of the diſtance encreaſes: thus at the diſtance 2 the force of gravity is not $\frac{1}{2}$ of what it is at the diſtance 1, but only $\frac{1}{4}$. So likewiſe at the diſtance 3 this force is not $\frac{1}{3}$ of what it is at the diſtance 1, but only $\frac{1}{9}$. And at the diſtance 4, the gravity, is reduced to $\frac{1}{16}$ of what it is at the diſtance 1. But if the force of gravity in the departure of the moon or any other heavy body from the earths center, decreaſes in the proportion of $\frac{1}{1}$, $\frac{1}{4}$, $\frac{1}{9}$, $\frac{1}{16}$, whilſt the diſtances of the moon encreaſe in the proportion of 1, 2, 3, 4; it is evident that in the approach of the moon towards the earth again, whilſt the diſtances decreaſe in the proportion of 4, 3, 2, 1, the force of gravity muſt neceſſarily encreaſe in the proportion of $\frac{1}{16}$, $\frac{1}{9}$, $\frac{1}{4}$, $\frac{1}{1}$. A force of this ſort, which at different diſtances from the center is inverſely as the ſquares of thoſe diſtances, will, as has been ſaid before, make a revolving body deſcribe round any point S, towards which that body is impelled, an ellipſis ADPG having S in one of its focuſes.

The different diſtances of the revolving body from S, as we juſt ſuppoſed them, are 1, 2, 3, 4. The cubes of theſe diſtances are 1, 8, 27, 64. The reciprocals of theſe cubes are $\frac{1}{1}$, $\frac{1}{8}$, $\frac{1}{27}$, $\frac{1}{64}$. And a force, which in the departure of the body from S at the diſtances 1, 2, 3, 4, decreaſes in the proportion of $\frac{1}{1}$, $\frac{1}{8}$, $\frac{1}{27}$, $\frac{1}{64}$, is as the reciprocals of the cubes of the diſtances, or decreaſes as the cubes of the diſtances encreaſe. Now it is plane that the fractions $\frac{1}{1}$, $\frac{1}{8}$, $\frac{1}{27}$, $\frac{1}{64}$, decreaſe faſter than thoſe mentioned in the laſt paragraph, that is, faſter than $\frac{1}{1}$, $\frac{1}{4}$, $\frac{1}{9}$, $\frac{1}{16}$. Therefore a force, which in the departure of the revolving body decreaſes as the cubes of the diſtances encreaſe, decreaſes faſter then the force of gravity, which decreaſes only as the ſquares of the diſtances encreaſe. The ſame may likewiſe be ſaid of any force, which, though at the diſtances 1, 2, 3, 4, it does not decreaſe ſo faſt as $\frac{1}{1}$, $\frac{1}{8}$, $\frac{1}{27}$, $\frac{1}{64}$, yet decreaſes faſter than $\frac{1}{1}$, $\frac{1}{4}$, $\frac{1}{9}$, $\frac{1}{16}$. Now it is plane that a force, which at the diſtances 1, 2, 3, 4, decreaſes in the proportion of $\frac{1}{1}$, $\frac{1}{8}$, $\frac{1}{27}$, $\frac{1}{64}$, in the departure of the revolving body from the center towards which it is impelled, will at the diſtances 4, 3, 2, 1, in the approach of the body encreaſe in the proportion of $\frac{1}{64}$, $\frac{1}{27}$, $\frac{1}{8}$, $\frac{1}{1}$. But theſe fractions increaſe faſter than $\frac{1}{16}$, $\frac{1}{9}$, $\frac{1}{4}$, $\frac{1}{1}$, that is, ſuch a force will encreaſe faſter than the ſquares of the diſtances decreaſe, or a centripetal force which decreaſes too faſt in the departure of the revolving body will likewiſe encreaſe too faſt in the approach of it.

But before we go on to ſhew that ſuch a force as this will make the body deſcribe a moveable ellipſis, it will be neceſſary to explane what is meant by ſuch an ellipſis. In order to do this we muſt obſerve that the line of the apſes always paſſes through the greateſt and leaſt diſtances. Now ſuppoſe A, Plat. VI. fig. 7. to be the greateſt diſtance of the revolving body from the focus T, B to be the leaſt diſtance, and ATB the line of the apſes. Let the body ſet out from A, in the direction AH, and by ſome centripetal force, which urges it towards T, let it be drawn from the tangent AH and be made to move in an elliptical orbit round T. This orbit would be an immoveable one, if the body ſetting out from A or the greateſt diſtance was to come to the leaſt diſtance by moving to B, that is, by going half round the focus T. But ſuppoſe that the body was not to come to its leaſt diſtance till it had gone more than half round T, but that after it had paſſed by B it was ſtill to approach nearer to T, till it came to D, ſo that D ſhould be the leaſt diſtance when the body ſet out from A. ATB paſſing through A the higher apſis, through T the focus, and through B, that was the lower apſis of the orbit in which the body began to move, was the line of the apſes: but after the body has arrived at its leaſt diſtance, then, becauſe D is this leaſt diſtance, the line of the apſes, which always paſſes through the leaſt diſtance, muſt have changed place, and inſtead of being ATB as it was at firſt, is now got into the ſituation DTC. That is, during the motion of the body in its orbit from the higher to the lower apſis, the line of the apſes has turned round upon the point T in the ſame direction, in which the body was moving. In like manner if the body ſets out from D the lower apſis, and in going half round the focus T comes to the greateſt diſtance at C, then the orbit itſelf would be immoveable during this part of the bodys revolution. But if it muſt go more than half round T before it arrives at its greateſt diſtance, as ſuppoſe this greateſt diſtance to be at G, then, whilſt the body is moving from the lower apſis to the higher, this higher apſis ſhifts its place and moves on before the body, that is, the line of the apſes, and conſequently the whole orbit along with this line, does likewiſe in this part of the bodys revolution turn a little way round upon the point T in the ſame direction in which the body itſelf is moving. So that this part of the motion is performed in part of an ellipſis whoſe higher apſis is at G and its lower apſis about F.

Having thus explaned, what is meant by the terms of the propoſition, we may now go on to ſhew the truth of it by proving that if a centripetal force in the departure of a revolving body from the center, towards which it is impelled, decreaſes faſter than the ſquares of the diſtances encreaſe, or in the approach of the body towards that center encreaſes faſter than the ſquares of the diſtances decreaſe, the orbit deſcribed in this caſe may be conſidered as a moveable ellipſis, the line of whoſe apſes keeps turning round upon the focus, towards which the body is impelled, in the ſame direction in which the body itſelf is moving.

We have ſeen that if a body ſets out from A to move round the focus S in the ellipſis ADPG, Plat. VI. fig. 1. then in ſome part of its motion between A and P, ſuppoſe at D, it has ſuch a velocity as, if the centripetal and projectile forces were perpendicular to each other, would make the body move in a circle round S at the diſtance DS. And we have ſeen beſides that the centripetal force, which makes a body revolve in ſuch an orbit, muſt, at different diſtances from S, be inverſely as the ſquares of thoſe diſtances. Now if the centripetal force, in the approach of the body towards S, encreaſes faſter than the ſquares of the diſtances decreaſe, this force will be every where greater in proportion than ſuch a force as would make the body deſcribe the orbit ADPG. Therefore though the bodys velocity when it is at D, would be ſufficient to make it deſcribe a circle round S, if the centripetal force encreaſed only as the ſquares of the diſtances decreaſe, yet ſince the centripetal force does by the ſuppoſition encreaſe faſter and ſo is every where greater in proportion, the velocity, that is requiſite to make the body deſcribe a circle round S, muſt be greater than what it has at D. Becauſe the greater the centripetal force is ſo much greater in proportion muſt be the velocity of the body in order to make it revolve in a circle. From hence then it follows that the body muſt move or deſcend farther than D before it can acquire ſuch a velocity. And the farther the body muſt move from A before it acquires ſuch a velocity as would be requiſite to make it revolve in a circular orbit round S, ſo as to keep always at the ſame diſtance from S, the farther it muſt neceſſarily move from A, before it can have acquired ſuch a velocity as will be ſufficient to make it begin to depart from S. Now if the centripetal force had encreaſed as the ſquares of the diſtances decreaſe, the body in moving half round S, or in deſcending from A to P, would have acquired velocity enough to make it begin to depart from S, or P would have been its leaſt diſtance. Therefore if the centripetal force encreaſes faſter, the body muſt move more than half round S or muſt deſcend farther than from A to P before it will have acquired the requiſite velocity, that is, before it will begin to depart from S or be arrived at the leaſt diſtance. And ſince when the centripetal force encreaſes faſter than the ſquares of the diſtances decreaſe, the body ſetting out from A muſt move more than half round S before it comes to its leaſt diſtance or lower apſis, it follows from what has been ſaid already in explaning the propoſition, that the lower apſis muſt have moved forwards, or that the line of the apſes muſt have turned round upon the focus, towards which the centripetal force is directed, whilſt the body has been moving from the higher apſis to the lower: and conſequently the orbit deſcribed will be ſuch an one as is repreſented in fig. 7. where the body ſetting out from A, which is the higher apſis, does not arrive at the lower apſis till it has got to D, which is at the diſtance of more than half the orbit from A.

In like manner, fig. 1. if the body ſets out from P, and the centripetal force decreaſes as the ſquares of the diſtances encreaſe, the velocity becomes ſuch, when the body arrives at G, as is requiſite to make it deſcribe a circle round S at the diſtance SG ſo as neither to depart farther from S nor approach nearer to it. But if the centripetal force decreaſes faſter than this, or is every where leſs in proportion, then the body muſt aſcend farther than from P to G, before its velocity, which was ſo great at P as to make it depart from S, is far enough diminiſhed to ſuffer the body to keep revolving round S ſo as to keep always at the ſame diſtance from it. Now if the body muſt aſcend farther than G before its velocity is ſufficiently diminiſhed to ſuffer it to revolve round S ſo as always to keep the ſame diſtance from it, then it muſt neceſſarily aſcend farther than A, before this velocity will be ſmall enough to permit it to approach towards S, that is, the body muſt move farther than from P to A or more than half round, before it will arrive at its greateſt diſtance or higher apſis. And conſequently the orbit will be ſuch an one as is repreſented fig. 7. where if B is the lower apſis from whence the body begins to move the higher will not be at A, but at C, or if D is the lower apſis the higher will not be at C but at G; ſo that the orbit revolves round T in the ſame direction that the body itſelf revolves.

Or otherwiſe. If a body that is impelled towards S, Plat. VI. fig. 1. and revolves round it in an elliptical orbit having S in one of its focuſes; the centripetal and projectile forces make a right angle with each other at A, or when the body is at the higher apſis or greateſt diſtance from S. And this angle is again a right one, when the body is at P the lower apſis or leaſt diſtance. Whilſt the body is deſcending from A towards P this angle diminiſhes till it becomes the leaſt of all at D, and then encreaſes again till it becomes a right one at P. In like manner, whilſt the body aſcends from P towards A, this angle encreaſes till it becomes the greateſt of all at G, after which it decreaſes till it is again reduced to a right one at A. This has been proved, in propoſition 152, and, in propoſition 172, we have ſhewn farther, that when a body moves in this manner the centripetal force at different diſtances from S is inverſely as the ſquares of thoſe diſtances.

But if in the approach of the body towards S the centripetal force encreaſes faſter than in this proportion, that is, if it is every where greater in proportion, the body will be more urged towards S; and conſequently the angle made by the projectile and centripetal forces will be every where leſs. Now if the centripetal force encreaſed as the ſquares of the diſtances decreaſe, this angle would be a right one when the body had deſcended from A to P; and when this angle in the deſcent of the body is a right one, the body is then at its lower apſis or leaſt diſtance from S and not before. But if the centripetal force encreaſes faſter than in this proportion, this angle is every where leſs. Therefore if the centripetal force encreaſes faſter than in this proportion, the angle at P will be leſs than a right one; and conſequently the body muſt move farther than P, or more than half round S, before this angle becomes a right one, that is, before it arrives at its leaſt diſtance from S, or at the lower apſis. And becauſe the longer axis of the elliptical orbit or the line of the apſes always paſſes through the leaſt diſtance, if the leaſt diſtance has ſhifted place and gone forwards from P, the line of the apſes muſt have turned round upon the focus T, towards which the body is impelled, in the ſame direction in which the body itſelf is moving.

In like manner, if we conſider P as the leaſt diſtance and the body as ſetting out from thence; ſince the centripetal force, whilſt the body departs from S, decreaſes faſter than the ſquares of the diſtances encreaſe, that is, ſince this force is every where leſs in proportion, the body will be every where leſs urged towards S, and conſequently the angle made by the projectile and centripetal forces will be every where greater. Now if the centripetal force decreaſed only as the ſquares of the diſtances encreaſe, this angle would be a right one, when the body had aſcended from P to A, and when this angle in the aſcent of the body is a right one, the body is then at its higher apſis or greateſt diſtance from S and not before. But if the centripetal force decreaſes faſter than in this proportion, this angle is every where greater. Therefore, if the centripetal force decreaſes faſter than in this proportion, the angle at A will be greater than a right one; and conſequently the body muſt move farther than A or more than half round S before this angle becomes a right one, that is, before it arrives at its greateſt diſtance. And becauſe the longer axis of the elliptical orbit, or the line of the apſes, always paſſes through the greateſt diſtance, if the greateſt diſtance has ſhifted place and gone forwards from A, the line of the apſes muſt have turned round, as before, upon the focus T, towards which the body is impelled, in the ſame direction in which the body itſelf is moving.

We ſhall have occaſion to apply this propoſition to the moons motion. For though the centripetal force, which impells the moon towards the earth, would make it deſcribe an immoveable ellipſis having the earth in one of its focuſes, if there was no other force but this to act upon it: yet we ſhall ſee hereafter, that the moons motion is diſturbed by the action of the ſun upon it. For as the moon gravitates towards or is attracted by the earth, and by this force is retained in its orbit; ſo likewiſe it gravitates towards or is attracted by the ſun, and this force diſturbs its motion round the earth. In particular it will appear in aſtronomy that though the force of gravity, by which the moon is impelled towards the earth, is, at different diſtances, inverſely as the ſquares of thoſe diſtances, and conſequently would make the moon revolve round the earth in an immoveable ellipſis: yet at the new and full of the moon the force with which the ſun acts upon it cauſes its gravity towards the earth to decreaſe in the departure of the moon, and to encreaſe in the approach of it, faſter than in the inverſe proportion of the ſquares of the diſtances. From whence it will follow that, at the new and full, the moons orbit will be a moveable ellipſis, ſuch an one as is repreſented, Plat. VI. fig. 7. the longer axis of which turns round upon T, or upon the focus where the earth is placed, in the ſame direction in which the moon moves.

176. *If in the departure of the revolving body from the focus, where the centripetal force acts, this force decreaſes ſlower than the ſquares of the diſtances from that focus encreaſe; or if in the approach of it to the focus the centripetal force encreaſes ſlower than the ſquares of the diſtances decreaſe, the orbit will be a moveable ellipſis, and the line of the apſes will turn round upon the focus in the contrary direction to that in which the body revolves.*

We will make uſe of the ſame method in this propoſition that we did in the laſt. Firſt, by a centripetal force, which is inverſely as the ſquares of the diſtances, we mean ſuch an one as decreaſes in the ſame proportion that the ſquares of the diſtances encreaſe, or encreaſes in the ſame proportion that thoſe ſquares decreaſe. Suppoſe for inſtance that the moon in revolving round the earth is ſometimes nearer to it and ſometimes farther from it. Then if at any diſtance, which we will call 1, the force of gravity is likewiſe called 1, or $\frac{1}{1}$; at the diſtance 2 the force of gravity would be $\frac{1}{4}$; at the diſtance 3 it would be $\frac{1}{9}$; and at the diſtance 4 it would be $\frac{1}{16}$. For theſe diſtances encreaſe in the proportion of 1, 2, 3, 4, and the ſquares of theſe diſtances are 1, 4, 9, 16. Therefore any force, ſuch as that of gravity, which is inverſely or reciprocally as the ſquares of theſe diſtances, will be as the reciprocals of 1, 4, 9, 16, that is, as $\frac{1}{1}$, $\frac{1}{4}$, $\frac{1}{9}$, $\frac{1}{16}$. In like manner when the moon is approaching towards the earth and the diſtances decreaſe in the proportion of 4, 3, 2, 1, the ſquares of theſe diſtances are 16, 9, 4, 1, the reciprocals of theſe ſquares are $\frac{1}{16}$, $\frac{1}{9}$, $\frac{1}{4}$, $\frac{1}{1}$. Therefore the force of gravity, which is inverſely or reciprocally as the ſquares of the diſtances, at the ſeveral diſtances 4, 3, 2, 1, encreaſes in the proportion of $\frac{1}{16}$, $\frac{1}{9}$, $\frac{1}{4}$, $\frac{1}{1}$. But ſuppoſe in the departure of the moon from the earth the centripetal force was inverſely, not as the ſquares of the diſtances, but as the diſtances themſelves. For inſtance, ſuppoſe at the ſeveral diſtances 1, 2, 3, 4, the force which impells the moon towards the earth was not as $\frac{1}{1}$, $\frac{1}{4}$, $\frac{1}{9}$, $\frac{1}{16}$, or as the reciprocals of the ſquares of the diſtances, but as $\frac{1}{1}$, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, or as the reciprocals of the diſtances themſelves. I imagine it is evident that ſuch a force as this decreaſes ſlower than the force of gravity does. Becauſe the fractions $\frac{1}{1}$, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, decreaſe ſlower than $\frac{1}{1}$, $\frac{1}{4}$, $\frac{1}{9}$, $\frac{1}{16}$. But the ſame force, which in the departure of the moon from the earth decreaſes in the proportion of $\frac{1}{1}$, $\frac{1}{2}$, $\frac{1}{3}$. $\frac{1}{4}$, will in the approach of it encreaſe in the proportion of $\frac{1}{4}$, $\frac{1}{3}$, $\frac{1}{2}$, $\frac{1}{1}$. Now as $\frac{1}{1}$, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, decreaſe ſlower than $\frac{1}{1}$, $\frac{1}{4}$, $\frac{1}{9}$, $\frac{1}{16}$. So likewiſe $\frac{1}{4}$, $\frac{1}{3}$, $\frac{1}{2}$, $\frac{1}{1}$, encreaſe ſlower than $\frac{1}{16}$, $\frac{1}{9}$, $\frac{1}{4}$, $\frac{1}{1}$. Therefore a centripetal force, which is inverſely as the diſtances, decreaſes ſlower in the departure of the revolving body and encreaſes ſlower in the approach of it than the force of gravity, or than any other force which is inverſely as the ſquares of the diſtances. And ſo likewiſe by the ſame way of reaſoning it may be ſhewn that any force which decreaſes too ſlowly in the departure of ſuch a body will encreaſe too ſlowly in the approach of it.

Having premiſed this we are next to conſider what is meant by an ellipſis that revolves upon one of its focuſes in the contrary direction to that in which the body itſelf moves. Let the body ſet out from A, Plat. VI. fig. 8, that is, let the projectile force act in the direction AH, whilſt the centripetal force conſtantly impells it towards T. A is the higher apſis; or greateſt diſtance, and if the body muſt move half round T, or from A to B, before it comes to the lower apſis or leaſt diſtance, then the orbit would be an immoveable ellipſis having T in one of its focuſes. But if the body ſetting out from A comes to the lower apſis or leaſt diſtance at D, or at any other point before it arrives at B, that is, before it has gone half round T; then this leaſt diſtance or lower apſis muſt have ſhifted place ſo as to meet the body. For ſince the longer axis or line of the apſes always paſſes through the greateſt and leaſt diſtances, and ſince, when the body ſet out from A, this point A was the greateſt diſtance, B, or the oppoſite point, through which the line ATB paſſes, muſt have been the leaſt diſtance. But the body is at the leaſt diſtance at D before it arrives at B. Therefore this leaſt diſtance or lower apſis muſt have ſhifted from B to D, whilſt the body was moving downwards from A; that is, the line of the apſes muſt have turned round upon the point T in the contrary direction to that in which the body moves, ſo as to meet the body before it has got half round. Or if we conſider the body as ſetting out at firſt from B the lower apſis; then if it moves half round T before it comes to the higher apſis or greateſt diſtance, and A is this higher apſis; the orbit is then an immoveable ellipſis. But if, when it ſets out from B, it goes on in the curve BC inſtead of BA, and is at its greateſt diſtance, when it arrives at C, or before it has gone half round the focus T, then becauſe the line of the apſes was in the poſition ATB, when the body was at the leaſt diſtance from T, and is in the ſituation CTD, when it arrives at its greateſt diſtance, it is evident that this line of the apſes muſt have turned round upon the point T ſo as to meet the body, that is, it muſt have moved in a direction contrary to that in which the body is moving. We have thus conſidered the nature of ſuch a moveable ellipſis as is deſcribed in the propoſition, by ſuppoſing the body to ſet off from A the greateſt diſtance and to arrive at its leaſt diſtance when it comes to D: and then by ſuppoſing it to ſet off from its leaſt diſtance at B and to arrive at its greateſt diſtance when it comes to C. The reader will perceive that ſuch a motion as is here ſuppoſed is not a continued one: for if D is the leaſt diſtance, it cannot move on and ſet off from B, ſo as that B ſhall be the leaſt diſtance too; becauſe this would be making two leaſt diſtances. But what has here been ſaid will ſerve to explane the motion of a body in ſuch an ellipſis, when it is a continued one. For if it ſets out from A, and arrives at its leaſt diſtance at D before it has gone half round T, and after this, inſtead of going on in the orbit DC ſo as to move half round T before it comes at its greateſt diſtance, if it moves along in the curve DG, and arrives at its greateſt diſtance before it has gone half round T, the orbit is a moveable ellipſis, and the longer axis of it, or line of the apſes, turns round upon the focus towards which the body is impelled, in a contrary direction to that in which the body itſelf is moving.

We are now to prove, if a body moves in an elliptical orbit, and the centripetal force encreaſes and decreaſes ſlower than in the inverſe proportion of the ſquares of the diſtances from the focus towards which the body is impelled, that then this ellipſis will be a moveable one, and the longer axis or line of the apſes will move the contrary way to the body.

If a body ſets out from A, Plat. VI. fig. 1. and moves in an elliptical orbit by the action of a centripetal force which conſtantly impells it towards S; and if this centripetal force, as the revolving body approaches S, encreaſes in the ſame proportion that the ſquares of the diſtances from S decreaſe; then at D this body will have ſuch a velocity as is requiſite to make it move round S in a circle having DS for its ſemidiameter. But ſuch a centripetal force is every where greater in proportion than one which encreaſes ſlower than the ſquares of the diſtances decreaſe. And the greater the centripetal force is, ſo much greater velocity is required to make a body revolve in a circle. Therefore though the body muſt deſcend from A to D in order to acquire a ſufficient velocity to make it move in a circle round S, if the centripetal force encreaſes as faſt as the ſquares of the diſtances decreaſe; yet if the centripetal force encreaſes ſlower and ſo is every where leſs in proportion, it need not deſcend ſo far as D in order to acquire ſuch a velocity. Now the velocity, that would cauſe the body to move in a circle, is one that balances the centripetal force in ſuch a manner as to keep the body conſtantly at the ſame diſtance from the center. And the velocity which a body revolving in an elliptical orbit has at its leaſt diſtance is ſuch an one as is ſufficient to make the body depart farther from the center, towards which it is impelled. But when the centripetal force encreaſes ſlower than the ſquares of the diſtances decreaſe, the revolving body will acquire a ſufficient velocity to make it move in a circle before it arrives at D; and conſequently it need not move ſo far in order to acquire a ſufficient velocity to make it depart from the center S, as it muſt have done if the centripetal force had encreaſed as faſt as the ſquares of the diſtances decreaſe. But if the centripetal force had encreaſed as faſt as the ſquares of the diſtances decreaſe, the velocity acquired by deſcending from A to P or by moving half round S would have been ſufficient to make the body begin to depart from S, and P would have been the leaſt diſtance. Therefore if the centripetal force encreaſes ſlower than the ſquares of the diſtances decreaſe, the body will acquire a ſufficient velocity to make it begin to depart from S before it has deſcended ſo far as from A to P, that is, the body will be at its leaſt diſtance from S before it has gone half round: and conſequently the orbit deſcribed by the body muſt be ſuch an one as is repreſented fig. 8. where the body ſetting out from A does not move half round or to B in order to come to its leaſt diſtance, but is at this leaſt diſtance or at the lower apſis when it has deſcended from A to D.

In like manner if the body ſets out from P, fig. 1. and the centripetal force, whilſt it departs from S, decreaſes as faſt as the ſquares of the diſtances encreaſe, the body, when it is at G, will have ſuch a velocity as in proper circumſtances would make it deſcribe a circle round S always keeping at the ſame diſtance GS from the point S. And when the body arrives at A having gone half round S its velocity would be ſo far diminiſhed as to ſuffer it again to approach towards S. But if the centripetal force decreaſes ſlower, then it muſt be every where greater in proportion, and conſequently the velocity requiſite to make the body move in a circle ſo as to keep always at the ſame diſtance from S muſt be greater than what it has at G. And conſequently this centripetal force, which is every where greater in proportion than one that decreaſes as faſt as the ſquares of the diſtances encreaſe, will cauſe the body to begin to approach towards S before the velocity is ſo ſmall as it is at A. But the body is at its greateſt diſtance from S, when it begins to approach it. Therefore a centripetal force, which decreaſes ſlower than the ſquares of the diſtances encreaſe, will cauſe that the revolving body ſetting out from P ſhall arrive at its greateſt diſtance or higher apſis before it arrives at the oppoſite point A, that is, before it has gone half round S: and conſequently the orbit deſcribed by a body, which is acted upon by ſuch a centripetal force, will be ſuch an one as is repreſented fig. 8. where the body ſetting out from D the lower apſis does not go half round T or to C in order to come to the higher, but moves in ſuch a curve as DG, which is part of an elliptical orbit, whoſe lower apſis is about F and its higher lies nearer to D than the point C does.

Or otherwiſe. When a body revolves in the ellipſis ADPG, Plat. VI. fig. 1. the angle that the projectile force makes with the centripetal force at A is a right one. This angle decreaſes till the body arrives at D; where it is the leaſt of all, and then encreaſes till it becomes a right one again at P. And as the body goes on from P and aſcends towards A, this angle keeps encreaſing till it becomes the greateſt of all at G, and then decreaſes ſo as to be a right one when the body is returned to A. Now the leſs a body is urged towards S by the centripetal force in any part of its orbit, ſo much greater is the angle which the centripetal and projectile forces make with each other in that part. For inſtance, at the point C the centripetal force urges the body towards S in the line CS, whilſt the projectile force impells it in the tangent C*c*; and the leſs the body is urged in the direction CS ſo much farther C*c* will recede from CS, or ſo much greater will be the angle made by the lines CS and C*c*. This angle is a right one, when the body is at A, and is a right one again, when the body has gone half round S or is come to P the oppoſite point to A, provided the centripetal force encreaſes as faſt as the ſquares of the diſtances decreaſe, by propoſition 152. But if the centripetal force encreaſes ſlower, then it is every where leſs in proportion, and conſequently, ſince the body is every where leſs urged towards S, the angle made by the centripetal and projectile forces will be every where greater as the body deſcends, and therefore will be a right one before the body arrives at P or before it has gone half round S. But, when this angle is a right one, the body is then at its leaſt diſtance or at its lower apſis. Therefore if the centripetal force encreaſes ſlower than

PLATE VI.

Fig. 1.

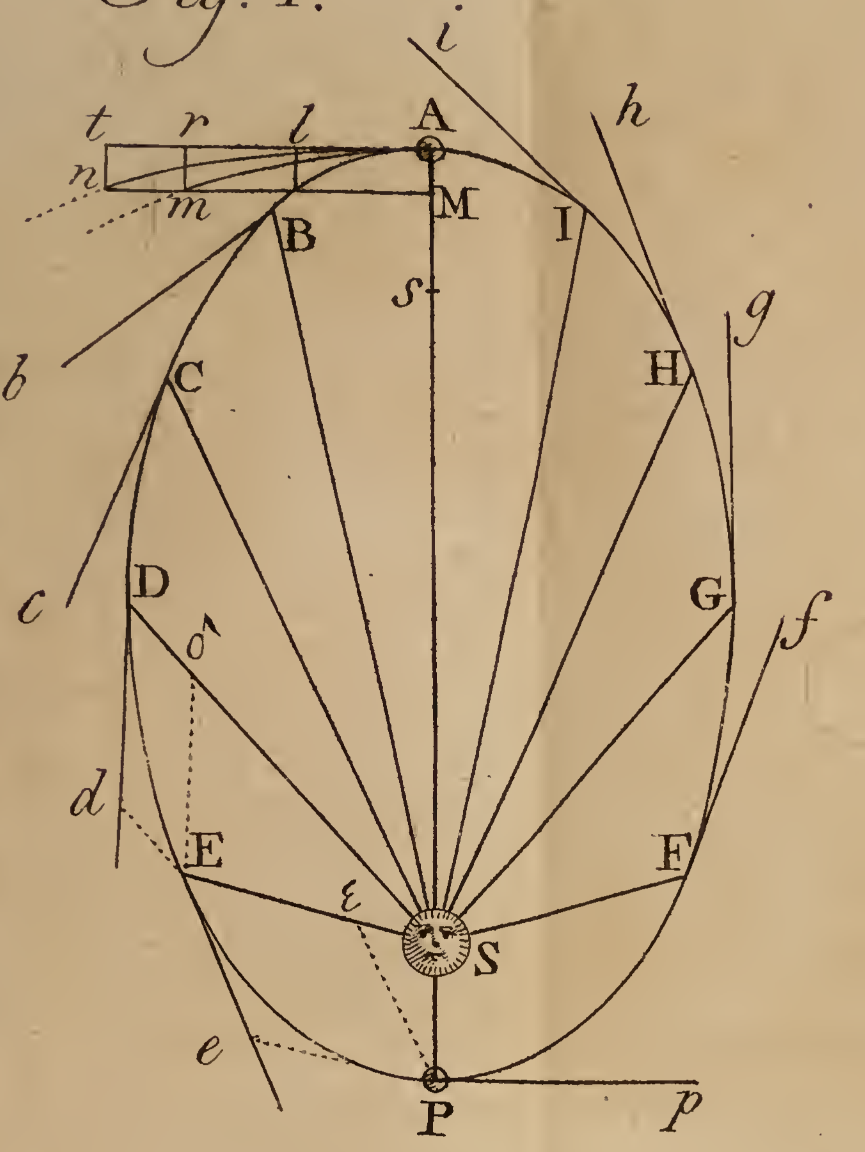


Fig. 2.

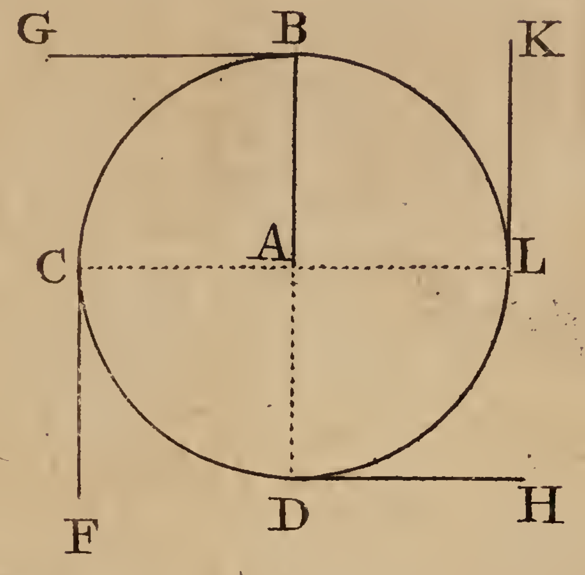


Fig. 3.

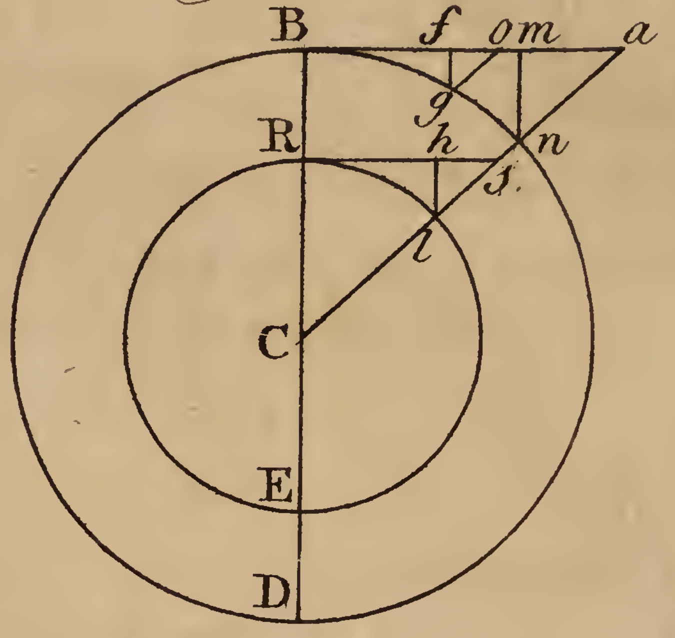


Fig. 4.

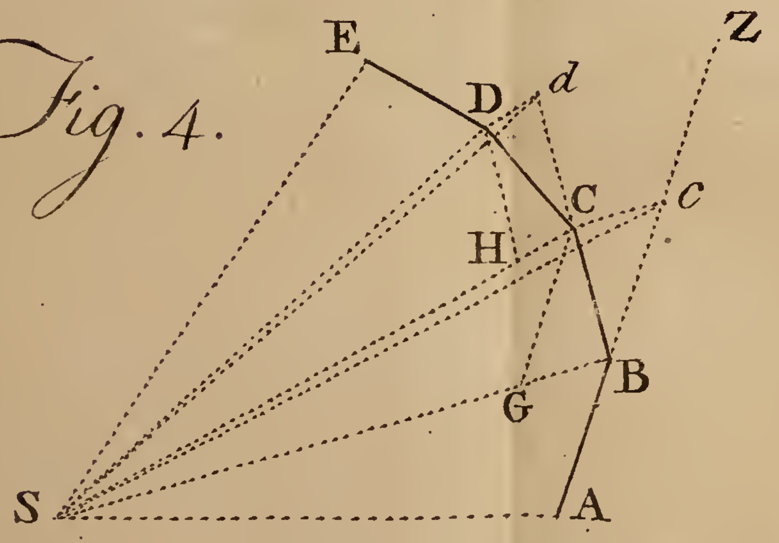


Fig. 5.

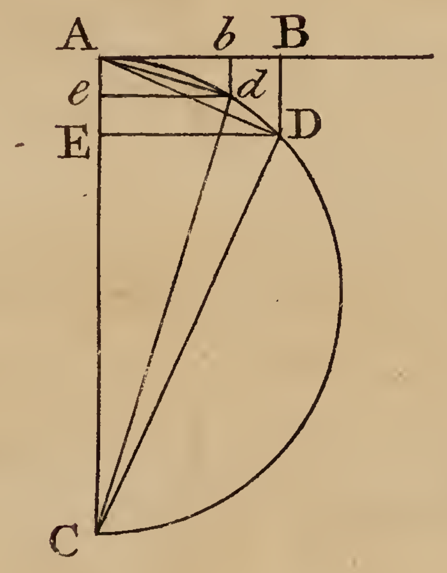


Fig. 6.

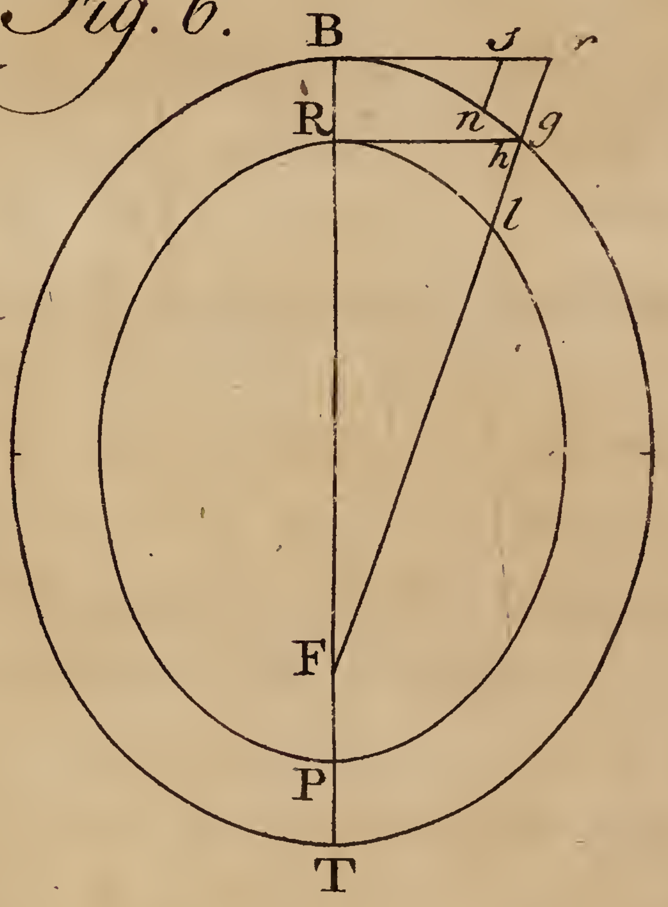


Fig. 7.

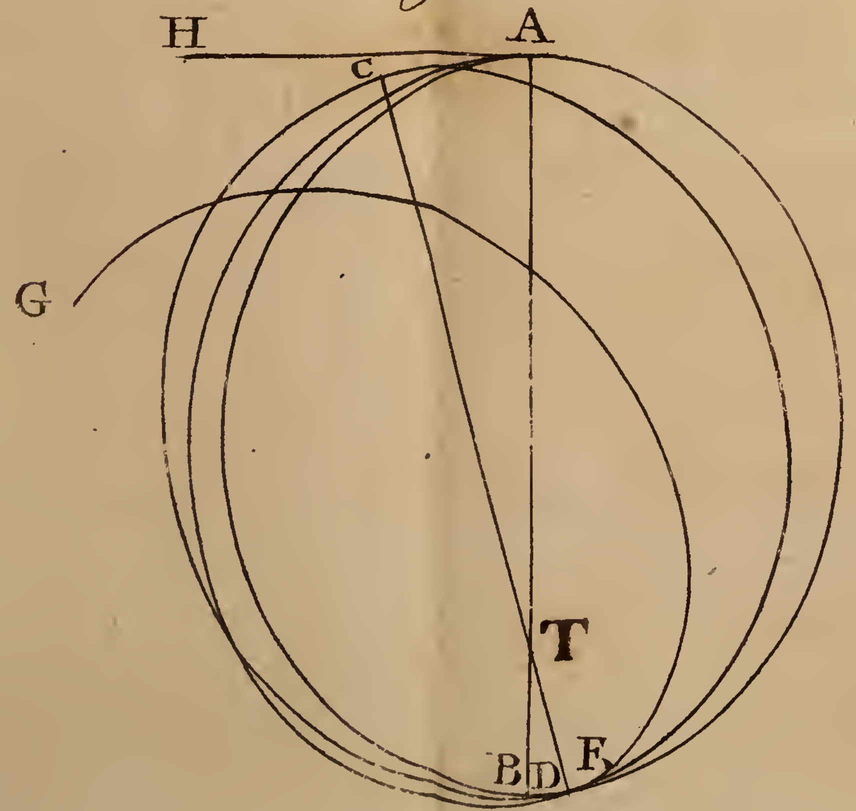
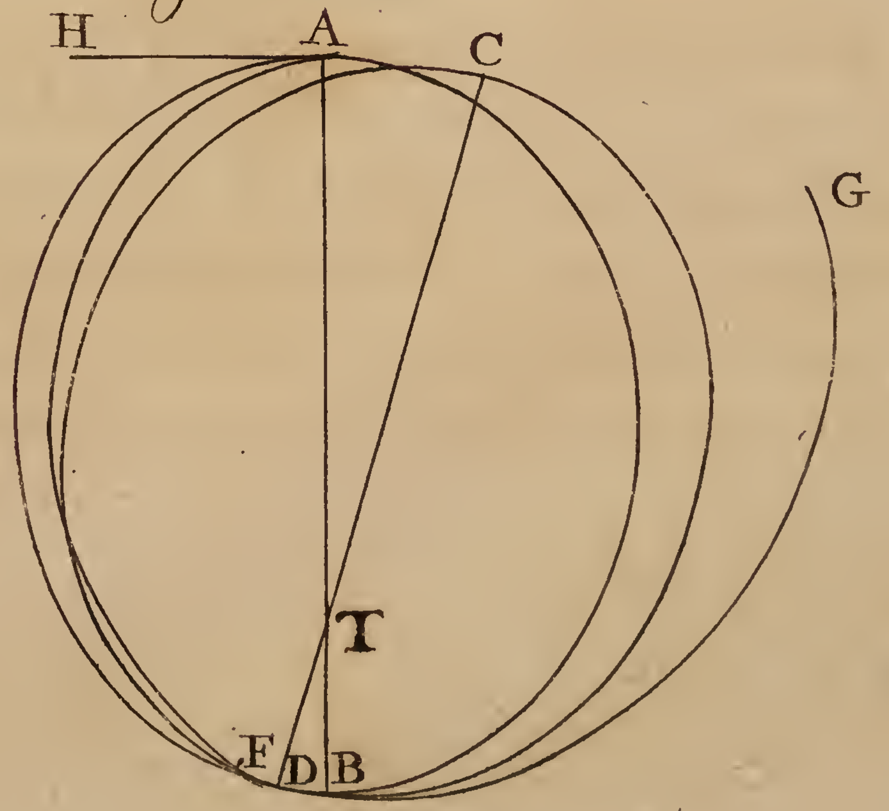


Fig. 8.



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than the ſquares of the diſtances decreaſe, the body ſetting out from its higher apſis will arrive at its lower before it gets half round S; and conſequently the line of the apſes muſt have turned round upon S in a contrary direction to that in which the body is moving, ſo that the lower apſis may meet the body.

In like manner, when the body is at P the lower apſis, and begins to aſcend towards A, the angle made by the centripetal and projectile forces, which is a right one at P, will be a right one again when the body arrives at A or at the point oppoſite to P; provided the centripetal force decreaſes as faſt as the ſquares of the diſtances encreaſe. But a centripetal force, which decreaſes ſlower than this, will as the body aſcends from P be every where greater, and conſequently the body will be every where in its aſcent more urged towards S. For which reaſon the angle made by the centripetal and projectile forces will be every where leſs. Therefore it will be reduced to a right one before the body arrives at A, that is, before it has moved from P to the oppoſite point or half round S. But if the body ſetting out from P the lower apſis gets to the higher apſis before it has moved half round S, this higher apſis muſt have come to meet the body, that is, the line of the apſes muſt have turned upon the point S in a contrary direction to the motion of the body.

This propoſition we ſhall likewiſe apply in aſtronomy to the moons motion. For at the quarters, the diſturbing force of the ſun cauſes the centripetal force, which impells the moon towards the earth, to decreaſe and encreaſe ſlower than in the inverſe proportion of the ſquares of the moons diſtances from the earth. Upon which account the moons orbit at the time of the quarters is to be conſidered as a moveable ellipſis, whoſe longer axis turns upon the earth, placed in one of its focuſes, in a direction contrary to that in which the moon itſelf moves.

I hope that what has been ſaid about the motion of bodies both in immoveable and moveable elliptical orbits will be uſeful to ſome of my readers, and that they, who are maſters of a better method of proving theſe propoſitions, will excuſe the many errors and want of exactneſs in theſe tedious and prolix attempts to explane what I have found to be very little underſtood by ſuch perſons as are unacquainted with the deeper parts of mathematics.

The end of Mechanics.

A SYSTEM OF NATURAL PHILOSOPHY.

OPTICS.

CHAP. I.

Definitions and lemmas.

THE cauſe and nature of viſion are properly the ſubject of that part of philoſophy, which is called by the name of optics. But as light is the cauſe of viſion the word *optics* is commonly uſed in a more extenſive ſenſe; and every thing in philoſophy is looked upon as a part of optics, which relates to the nature and qualities of light.

If we uſe the word *optics*, in the ſtricter ſenſe of it, for the theory of viſion; the ſcience of optics is divided into two parts: one part is called dioptrics, and the other catoptrics. The laws of refraction, and the effects, which the refraction of light has in viſion, are the ſubject of dioptrics. The laws of reflection, and the effects, which the reflexion of light has in viſion are the ſubject of catoptrics.

Thus, when the rays of light paſs through a convex lens or burning glaſs, they are refracted or bent from a ſtrait line. And it is a part of dioptrics to ſhew, both in what manner they are bent, and likewiſe what effect this bending of them will have, if we look at an object through ſuch a lens. When the rays of light come ſtrait forwards from any object that we are looking at, and are not refracted at all, till they come to the eye; yet as they paſs into the eye, they are refracted; and viſion itſelf depends upon this refraction. Therefore, as the rays of light are refracted by the eye in order to make us ſee objects, we find that in treatiſes of dioptrics the ſtructure of the eye, the effects that it has upon light, and the whole theory of plane viſion, or viſion with the naked eye, are uſually explaned.

When the rays of light ſtrike upon a poliſhed ſurface of glaſs or metal they are thrown back from it or reflected. And it is the buſineſs of catoptrics to ſhew both in what manner this reflexion is made and what effect it will have upon an eye that looks in the glaſs.

Theſe few inſtances may ſerve to bring the reader acquainted with the meaning of the words *dioptrics* and *catoptrics*. The diſtinction itſelf will be of no great uſe to us. For it is ſcarce worth our while to ſeparate theſe different branches of optics from each other, and to keep the dioptrical part quite diſtinct from the catoptrical. Indeed there are many propoſitions in optics where both parts are mixed; and many that cannot properly be reduced to either. And therefore in the following treatiſe I ſhall not attend much to the diviſion of this ſcience into theſe two branches but ſhall rather chuſe to follow ſuch a method as the nature of the ſubject leads me to.

Before we enter upon the ſcience itſelf, there are ſeveral terms to be explaned; ſuch as the reader may be ſuppoſed unacquainted with, as they are not much uſed in common life and are many of them almoſt peculiar to optics. There are likewiſe ſome lemmas to be demonſtrated, that is, ſome propoſitions are to be proved, which are previous to the ſcience now before us, and will clear the way to the demonſtration of thoſe, which follow.

1. *A ray of light is the leaſt particle of light, that can either be intercepted alone, whilſt all the reſt are ſuffered to paſs, or that can be let paſs alone, whilſt all the reſt are intercepted.*

In the firſt parts of optics we ſhall not keep exactly to this definition of a ray of light, but ſhall uſe the word *ray* in a larger ſenſe for a ſingle particle of light as it comes from the ſun. And however ſmall a particle of light is in that ſtate, in which it is emitted by the ſun; yet, when we come in the latter part of optics to examine more accurately into the nature of light, we ſhall find that theſe particles are not the leaſt poſſible, not the leaſt, that can be intercepted alone, whilſt all the reſt are ſuffered to paſs, or that can be let paſs alone, whilſt all the reſt are intercepted. For a ray, as it comes from the ſun, will be found to be made up of many other ſmaller rays, which are in their nature and quality quite different from each other. Every ray, in the ſenſe we ſhall uſe this word at firſt, is properly ſpeaking a bundle of ſeven ſmaller rays. But, till we come to ſhew by what means theſe ſmaller parts may be ſeparated from each other, we ſhall conſider this bundle as if it was but one ſingle ray, or as if it was the ſmalleſt part into which light can be ſeparated.

2. *Every thing is called a medium in optics, that is tranſparent, or that affords a paſſage for the rays of light to go through it.*

Air,

Air, water, glaſs, a diamond, or even a vacuum are any of them a medium; for they are all tranſparent or give light a free paſſage through them.

3. *The inflexion of a ray of light conſiſts in its being bent from a ſtrait courſe, whilſt it is paſſing on in one and the ſame medium.*

If a beam of light or number of rays is let into a dark room through a hole in the window-ſhutter, each ray, of which the beam conſiſts, will move ſtrait forwards or in a right line. But if the edge of a knife be held near the beam, the rays that paſs neareſt to the knife are bent from their ſtrait courſe towards it. If HIF, Plat. VII. fig. 1. is the edge of a knife, near to which a beam of light is to paſs conſiſting of ſeveral rays, ſome of which deſcribe the lines AB, EF, and would, if they were left to themſelves, go on in ſtrait lines BK, FL, theſe rays ſuffer an inflexion at the edge of the knife; that is, they are bent towards it, and are made to deſcribe the lines BD, FG. But in this bending the rays ſtill paſs on, and are not thrown back again or reflected. Nor are they thus bent by paſſing out of one medium into another; for as they were moving in the air, whilſt they deſcribed the lines AB, EF, ſo likewiſe they continue to move on in the air, that is, in the ſame medium, when they are bent from their ſtrait courſe into the lines BD, FG.

4. *The refraction of a ray of light conſiſts in its being bent from a ſtrait courſe, whilſt it is paſſing out of one medium into another.*

If a ray of light deſcribes the line AB, Plat. VII. fig. 2. whilſt it is moving in the air, and at B paſſes out of the air, which is one medium, into a piece of glaſs XYZ, which is a different medium; it will not go ſtrait forwards in the glaſs and deſcribe the line BD, but will be bent in its paſſage at B, and be made to deſcribe the line BC. This bending of the ray in its paſſage out of one medium into another is the refraction of it.

5. *The reflexion of a ray of light conſiſts in its being ſo much bent from a ſtrait courſe as to be turned back again into the ſame medium or ſame part of the medium that it was moving in before.*

If ABDE, Plat. VII. fig. 3. is a thick piece of glaſs and the air is behind the lower ſurface of it DCE, a ray of light, which deſcribed the line AC, inſtead of paſſing into the air, when it comes to C, may be bent back again into the glaſs or the ſame medium it was in before, and be made to deſcribe the line CB. This bending of the rays courſe is called the reflexion of it.

But as this is a leſs uſual ſort of reflexion, we may inſtance in another. If DCE is the ſurface of a common looking-glaſs, and a ray of light deſcribes the line AC in the air, and ſtrikes upon the glaſs at C, it is bent in its courſe at C ſo much as to throw it back again into the ſame part of the air that it was moving in before, that is, into the air, which lies before the glaſs. This bending of the ray is the reflexion of it.

By this time we muſt have ſeen how the inflexion of light differs both from the refraction and the reflexion of it. When a ray is inflected, its courſe is bent, and yet the ray continues all the while in one and the ſame medium. When it is refracted, its courſe is bent by paſſing out of one medium into another. When it is inflected, it is bent but a little, and paſſes on from the part of the medium that lies before the inflecting ſubſtance to the part that lies behind it. When it is reflected, it is bent ſo much as to be thrown back again into the ſame part of the medium, that it was moving in before.

In what refraction differs from reflexion is plane likewiſe from the foregoing definitions. For in refraction the ray paſſes out of one medium into another, whereas in reflexion it is thrown back again and continues in one and the ſame medium.

6. *The angle of incidence is that, which is contained between the line deſcribed by the incident ray and a line at the point of incidence drawn perpendicular to the ſurface, on which the ray ſtrikes.*

When the ray deſcribes the line AB, Plat. VII. fig. 2. and falls on the refracting ſurface **XBZ**; at B the point of incidence, or point where the ray falls, draw BH perpendicular to XZ: then the angle ABH contained between AB and BH is the angle of incidence. Or if the ray falls in the direction AC on the reflecting ſurface DE, Plat. VII. fig. 3. then drawing OC, at the point of incidence, perpendicular to DE, the angle ACO, which is contained between AC and CO, is the angle of incidence.

7. *The refracted angle is that, which is contained between the line deſcribed by the refracted ray, and a line perpendicular to the refracting ſurface at the point where the ray paſſes through that ſurface.*

When a ray of light that has deſcribed AB, Plat. VII. fig. 2. paſſes out of air into glaſs through the ſurface XZ, and in paſſing is refracted, ſo that, inſtead of going ſtrait forwards in the line BD, it is bent into and made to deſcribe the line BC; if the perpendicular HB is continued to G; the angle CBG is contained between BC the line deſcribed by the refracted ray and BG a perpendicular to the refracting ſurface at the point B. And this is called the refracted angle.

8 *The*

8. *The angle of refraction is that, which is contained between the direction of an incident ray, and the direction of the ſame ray after it is refracted.*

AB, Plat. VII. fig. 2. is the line which the incident ray deſcribes; and BD, or the line AB continued beyond the refracting ſurface, is in the ſame direction in which the incident ray moved. BC is the direction of the refracted ray. The angle DBC, which is contained between theſe two lines, is the angle of refraction.

9. *The angle of reflexion is that, which is contained between the line deſcribed by a reflected ray and a line perpendicular to the reflecting ſurface at the point of reflexion.*

If a ray, which at its incidence deſcribed the line AC, Plat. VII. fig. 3. is reflected from the point C into the line CB, and OC is drawn at this point C perpendicular to DE the reflecting ſurface; the angle OCB, which is contained between theſe two lines CB and CO, is the angle of reflexion.

10. *Rays of light, that come from one and the ſame point, and, as they go farther from this point, ſpread and ſeparate farther from each other, are called diverging rays.*

If rays of light flow or are propagated from B, Plat. VII. fig. 4. in the directions BD, BA, BC, BE, theſe rays are diverging ones. They come from one and the ſame point B, and as they go farther from this point they ſpread and ſeparate farther from each other. For it is evident, that theſe rays are farther aſunder when they are at D and E, than when at F and G.

11. *Rays of light are called converging ones, when they tend or are directed to one and the ſame point, and, as they approach this point, come nearer to each other.*

If rays are propagated in the lines DB, AB, CB, EB, Plat. VII. fig. 4. ſo as to be all of them directed to B; theſe rays converge. They are directed or tend to one and the ſame point B; and, as they approach this point, they come nearer to each other. For it is evident, that they are cloſer together when at F and G, than when at D and A.

12. *Rays of light are parallel, when the lines, that they deſcribe, are parallel.*

13. *The point from whence diverging rays are propagated is called a radiant.*

Thus B, Plat. VII. fig. 4. is the radiant from which the rays that deſcribe BD, BA, BC, and BE are propagated.

14. *A focus is that point, to which converging rays tend.*

Thus if converging rays deſcribe DB, AB, CB, and EB, Plat. VII. fig. 4. ſo that all of them tend or are directed to B; then B is the focus of thoſe rays.

From what has been ſaid, it appears that, if rays, which diverge from B in the directions BD, BA, BC, and BE, are by any means turned back again in the lines they were deſcribing, they then become converging ones. And conſequently B, which is conſidered as the radiant, when the rays diverge, becomes the focus, when they are made to converge.

15. *Rays, which converge to a focus, when they have met at it and paſs on from thence, become diverging, and the focus is changed into a radiant.*

If two rays deſcribe the lines AC, BC, Plat. VII. fig. 5. and ſo converge to the focus C; when they have met at C, they muſt paſs on from thence ſtrait forwards in the ſame lines that they were deſcribing before, that is, in the lines CE and BD. But rays that deſcribe CE and BD diverge from C as from a radiant. Therefore the ſame point, to which the rays tended at firſt, is the point from whence they are propagated afterwards; or C, which is conſidered as the focus of the rays, when they are on one ſide of it, and are coming to it, is to be conſidered as the radiant, when they are on the other ſide, and are going from it.

16. *That point from whence refracted or reflected rays appear to diverge is called an imaginary radiant.*

If rays diverge from R Plat. VII. fig. 6. in the lines RA, RB, then R is the radiant. But if thoſe rays by paſſing through the ſurface S*s* are refracted into the lines AD and BG, inſtead of going ſtrait forwards in the lines AC, BE; then the rays after refraction diverge from one another juſt as much as they would have done if they had come ſtrait from *r*: and as they appear to have been propagated from *r* or go on as if they had come from thence, this point *r* is likewiſe a radiant. But becauſe the rays did not come from *r*, but are only made by the refraction to proceed as if they had come from thence; *r* though a radiant is not a real one, but, to diſtinguiſh it from a real one, is called an imaginary one.

In like manner, if rays diverge from a radiant A Plat. VII. fig. 7. in the lines A*b*, A*d*, A*e*, and fall upon the ſurface of a looking-glaſs *be*, from whence they are reflected in the lines *bf*, *dg*, *eh*; theſe rays after reflexion proceed as if they had come from the point *a*. This point *a*, which is the radiant from whence the reflected rays appear to diverge, is an imaginary radiant.

The ſame point, which is here called an imaginary radiant, is ſometimes called a virtual focus. The reaſon of this latter name will be explaned hereafter.

17. *When converging rays are turned out of the way by being either refracted or reflected, the point, towards which they tended before refraction or reflexion, is called an imaginary focus.*

It is called a focus, becauſe the rays are directed towards it; but then it is only an imaginary one, becauſe the refraction or reflexion, which they ſuffer, prevents them from meeting there. If two rays deſcribe the lines TM, WN, Plat. VII. fig. 6, ſo as to converge towards *f*; and if, before they arrive at *f*, they paſs through the ſurface S*s* and inſtead of going ſtrait on in their firſt courſe or in the lines M*f*, N*f*, are refracted into the lines MF, NF, then theſe rays, though they tended towards *f*, will never meet there; and *f*, which was their focus before refraction, is an imaginary focus.

In like manner if rays, that converge, deſcribe the lines *fb*, *gd*, *he*, Plat. VII. fig. 7. ſo as to tend towards the point *a*, this point is their focus. But if, before they arrive there, they fall upon the ſurface of a looking-glaſs *be* and are reflected from thence in the lines *b*A, *d*A, *e*A, then they will meet at A, which is therefore their real focus, and the point *a*, towards which they were directed before this reflexion, is their imaginary focus.

18. *A lens is a thin round piece of poliſhed glaſs, which has either both its ſides ſpherical, or elſe one of them ſpherical and the other plane.*

Spherical ſurfaces are of two ſorts; either convex or concave. From whence it is evident, that, as a thin piece of glaſs has two ſides, by multiplying the number of ſides into the different ſorts of ſurfaces, which ſuch a glaſs may have, there may be ſix different kinds of glaſſes. A glaſs may have both ſides convex; or one convex and the other plane; or both ſides concave; or one ſide concave and the other plane; or one ſide convex and the other concave; or both ſides plane. But the laſt ſort of glaſs, that has both ſides plane, is not called a lens; ſo that there are only five ſorts of lenſes.

A glaſs, which is convex on both ſides is called a convex lens. The name *lens* is given it, becauſe it is in the ſhape of a lentil. And from this one glaſs, which is properly called a lens, the name has been transferred to other glaſſes having either one or both ſides ſpherical, though they are not of the ſame ſhape. If a convex lens was to be cut into two equal parts by a plane paſſing through the middle of it, the ſection, or place where it is cut, would appear like *abcd*, Plat. VII. fig. 8. A glaſs which is convex on one ſide and plane on the other, is a plano-convex lens. A ſection of it would be like *ef*. A glaſs, which is concave on both ſides, is called a concave lens; and *gh* is a ſection of it. A glaſs, which is concave on one ſide and plane on the other, is a plano-concave lens, and *ik* is a ſection of ſuch a lens. A glaſs, which is convex on one ſide and concave on the other, is called a meniſcus or moon-ſhaped lens, becauſe a ſection of it *lm* is in the form of a moon or creſcent.

19. *The axis of a lens is a right line, which is conceived as paſſing through the ſubſtance of it, and is perpendicular to both its ſurfaces.*

If any of theſe ſections of the ſeveral ſorts of lenſes between A and B Plat. VII. fig. 8. was to turn round upon the line *cd*, which is drawn through the ſubſtance of each of them, and is perpendicular to both the ſurfaces; the ſection in ſuch a revolution would generate the lens, of which it is a ſection. Thus, for inſtance, if *ef* was to turn round upon *cd*; *ecf* which is a ſegment of a circle would, as it turned round, deſcribe a ſegment of a ſphere; and the ſide towards *c* would be convex. The right line *adb* would in the mean time deſcribe a plane ſurface. Therefore both together would deſcribe or generate a ſolid, which would be convex on one ſide and plane on the other: and ſuch a ſolid is a plano-convex lens. Becauſe the lens, conſidered as a mathematical ſolid, is generated by a revolution of a ſection of it upon the line *cd*; therefore this line is called the axis of the lens.

20. *The poles or vertexes of a lens are the two extremities of its axis.*

In each of the lenſes Plat. VII. fig. 8. *c* and *d* are the two poles or vertexes.

21. *Diverging rays, when they fall upon a lens, form a cone, the apex of which is the radiant, and the baſe is the lens.*

Let *k* Plat. VII. fig. 9. repreſent a radiant, and *ab* a convex lens: then if *ka*, *ke*, *ko*, *kb*, are the lines, which rays of light diverging from *k* and falling upon the lens deſcribe, this whole diverging beam and the lens together are a cone, the apex of which is the radiant *k*, and the baſe is the lens *ab*.

Indeed this figure is not a cone: for upon a plane ſurface ſuch as the paper is, the entire cone could not be well repreſented ſo as to make it anſwer the uſes we ſhall have of it. But *akb* is a ſection of this cone made by a plane paſſing quite through it from the apex *k* to the baſe *ab*. And if the lens is conceived to turn round its axis *cd*, and the other part of the figure *ak* and *bk* to turn round at the ſame time upon the line *kc*; in ſuch a revolution *acbd* would generate a circular lens for the baſe of the cone, and *akb* would generate the cone itſelf. That is, the whole figure *kab* revolving in this manner would deſcribe the ſpace which is taken up by the lens itſelf and by rays diverging in all directions from *k* and falling upon the lens.

In like manner if B, Plat. VII. fig. 10. is a radiant, and rays diverge from it in the directions BR, BD, BS; theſe rays and the lens together form a cone, the apex of which is B, and the baſe RS. Or, if C is a radiant, and the rays, which diverge from it deſcribe the lines CR, CD, CS; theſe rays and the lens form an oblique cone RCS, the apex of which is C, and the baſe RS.

22. *In every beam of light the middle ray is called the axis.*

In a diverging beam, or cone of diverging rays, ſuch as CR, CD, CS. Plat. VII. fig. 10. or BR, BD, BS, the middle ray, or that which deſcribes CD in one of theſe cones, or BD in the other, is the axis of the cone, and is therefore called the axis of the beam. And from hence in a beam of parallel light the middle ray is likewiſe called the axis of it. Thus if rays were proceeding in the parallel lines *b*R, *x*D, *b*S, the middle ray, or that which deſcribes *x*D, is the axis. And in the parallel beam *a*R, *z*D, *a*S, the middle ray *z*D is the axis.

23. *Rays are ſaid to fall directly upon a lens, if their axis coincides with the axis of the lens: but if their axis is oblique to that of the lens, they are ſaid to fall obliquely.*

The diverging beam BR, BS, BD, Plat. VII. fig. 10. falls directly upon the lens RS. For though all the rays which compoſe that beam are not perpendicular to the lens, yet the middle ray or axis BD is. Therefore ſince the axis of the lens is likewiſe perpendicular to its ſurface, the middle ray, as it paſſes through the lens, deſcribes this axis, that is, the axis of the beam coincides with the axis of the lens.

But CR, CS, CD, is an oblique beam, for the middle ray or axis CD is oblique to the ſurface of the lens, and conſequently is oblique likewiſe to a line drawn perpendicular to this ſurface at D, which line is the axis of the lens.

24. *When a beam of light paſſes through a lens, if the refraction of the lens makes the rays converge, theſe refracted rays form a cone, whoſe baſe is the lens and whoſe apex is the focus.*

When the rays of light, which at their incidence deſcribe *ka*, *ke*, *kc*, *ko*, *kb*, Plat. VII. fig. 9. have paſſed through the convex lens *ab*, and are made to converge in the lines *af*, *nf*, *df*, *rf*, *bf*, ſo as to meet in the focus *f*; all theſe refracted rays taken together make up a cone, having the lens *ab* for its baſe, and the focus *f* for its vertex. The whole cone is not indeed expreſſed in this figure; for *ab* is not the whole lens or baſe, but only a ſection of it; and conſequently the figure repreſents only ſuch a ſection of the converging rays, or refracted cone, as would be produced by a plane paſſing through the middle of its baſe and through the middle of the cone itſelf ſo as to ſplit it quite up to its apex.

Or, if the incident rays are not diverging but parallel; as they would be, if they deſcribed the lines *b*R, *x*D, *b*S, Plat. VII. fig. 10; then likewiſe the refracted ones being made to converge form a cone of the ſame ſort having the lens RS for its baſe, and the focus B for its apex. The rays *a*R, *z*D, *a*S, which are parallel at their incidence and fall obliquely, form an oblique cone RAS after they are refracted.

25. *When a beam of light is made to converge by being refracted, the incident rays and refracted cone taken together is called a pencil of rays.*

The incident beam *akb*, Plat. VII. fig. 9. and the refracted one *afb* taken together make a pencil. So likewiſe, when the incident rays are parallel, and deſcribe the lines *b*R, *x*D, *b*S, Plat. X. fig. 10. they and the refracted cone RBS taken together make a pencil. This is called a pencil, becauſe, if B the focus is received upon a paper, it will paint the image or figure of that point from whence the rays came which are united in that focus. Thus if there were ſeveral beams of light, the rays of one beam, deſcribing *a*R, *z*D, *a*S, the rays of another deſcribing *b*R, *x*D, *b*S, and the rays of a third *c*R, *y*D, *c*S, theſe rays, when refracted, would form three cones RAS, RBS, RCS. And theſe cones with their reſpective incident beams make three pencils. Now ſuppoſing theſe three beams to have proceeded from a croſs at a great diſtance from the lens, one from the higheſt point of it, another from the middle, and the third from the loweſt point; the focuſes of theſe pencils, received upon a paper at ABC would paint the image of theſe three points in the croſs. And if the rays proceeding from each point of the croſs were refracted in the ſame manner, they would all of them together paint the entire image of the croſs ABC. The manner how this effect is produced will be explaned in its proper place. It is here mentioned only to ſhew the reaſon why the incident rays of any beam and their refracted cone taken together ſhould be called a pencil.

26. *The ſines of ſmall angles are nearly proportional to the angles themſelves.*

Upon C Plat. VII. fig. 11. as a center deſcribe the circle ABED. If the angle ECG is double the angle ECF; then the arc EG, which ſubtends the former, will be double the arc EF, which ſubtends the latter. And in all caſes whatever, whether the angles are great or ſmall, the arcs that ſubtend them are greater or leſs in the ſame proportion that the angles are; a double angle is ſubtended by a double arc, a triple angle by a triple arc, and ſo on. But the ſines of angles, when they are great ones, are not proportional to the angles. For though a double angle is ſubtended by a double arc, yet it has not a double ſine. In the angle ECG from G the extremity of one leg draw GK perpendicular to the other, and GK, which is the ſine of this angle, is planely not double F*p*, which is the ſine of the angle ECF; notwithſtanding the angle ECG, of which GK is the ſine, is double the angle ECF, of which F*p* is the ſine. For if FI is drawn parallel to EC, then F*p* and KH are equal, becauſe they are oppoſite ſides of a parallelogram Euc. b. I. prop. 34. and it is evident that KH is more than half KG, or that KG is not double KH, and conſequently is not double F*p*. This is the caſe in great angles: but in ſmall ones it is otherwiſe. For if the angle ECO, is double ECL and conſequently the arc EO double the arc EL, the ſine O*n* will likewiſe be very nearly double to the ſine L*m*; in the ſame manner a triple angle would have nearly a triple ſine, and ſo on; the ſines encreaſing, whilſt the angles are ſmall, in a proportion ſo near that of the angles themſelves, that it may be looked upon as the ſame proportion without any ſenſible error.

It is plane that if the arcs and ſines coincided, they muſt always encreaſe in the ſame proportion, becauſe they would always encreaſe together and always be equal to each other. Now in great angles the ſines and arcs differ very much or are far from coinciding; as GK differs much from GE, and F*p* from FE. But then, ſince F*p* is nearer coinciding with FE than GK is with GE, it follows that F*p* will differ leſs from FE than GK does from GE, and conſequently that F*p* does not bear the ſame proportion to FE, that GK bears to GE. Therefore neither does Fp bear the ſame proportion to GK, that FE bears to GE; Euc. book V. prop. 16. or the ſines are not in the ſame proportion with the arcs, and conſequently are not in the ſame proportion with the angles, which thoſe arcs ſubtend. But in ſmall angles the ſines and arcs nearly coincide; and conſequently differ little from each other. As On and OE are nearly equal, and ſo are Lm and LE. Therefore On and OE will encreaſe in nearly the ſame proportion, and likewiſe Lm and LE: and On bears nearly the ſame proportion to OE that Lm does to LE, or, On is to Lm as OE to LE nearly; Euc. b. V. prop. 16. that is, the ſines are nearly proportional to the arcs and conſequently to the angles, which thoſe arcs ſubtend.

The angles we ſhall have occaſion to ſpeak of in optics are commonly very ſmall ones, and therefore we may for the future conſider the ſines as the meaſure of them, or may look upon the ſines as proportional to the angles.

27. *The divergency of rays is meaſured by the angle contained between the lines, which the rays deſcribe.*

The divergency of rays is conſidered as quantity, or as being capable of greater and leſs. Rays have a greater divergency as the lines of their direction contain a greater angle; and a leſs divergency as thoſe lines contain a leſs angle. The rays which deſcribe the lines BD and BE Plat. VII. fig. 4. diverge juſt as much more than thoſe, which deſcribe BA and BC, as the angle DBE is greater than the angle ABC.

The reader muſt obſerve that the divergency of rays is meaſured by the angle, which the lines of their direction contain, and not by the diſtance between thoſe lines. For though the rays, that deſcribe BD, and BE are at a greater diſtance from each other, when they are at D and E, than when at F and G, yet their divergency at D and E is no greater than at F and G: becauſe the angle DBE, and FBG is the ſame. So likewiſe Plat. VIII. fig. 1. the rays, which come from the radiant D and deſcribe DA and DB, diverge more than thoſe, which come from the radiant E and deſcribe EA and EB, though the diſtance between theſe rays when they are at A and B is the ſame. For the angle ADB is greater than the angle AEB; and theſe are the angles, which meaſure their divergency.

28. *If the diſtance between diverging rays, that come from different radiants, is the ſame; then the diſtances of the radiants are inverſely as the divergency of the rays, which come from them.*

If there are two different radiants D and E Plat. VIII. fig. 1. and the rays diverging from D deſcribe the lines D A, D B, and thoſe from E deſcribe the lines E A, E B, ſo that the diſtance between the former rays, or AB, is the ſame as the diſtance between the latter, then thoſe rays, which come from the nearer radiant D have a greater divergency than thoſe, which come from the remoter radiant E. And we are to ſhew that CD, or the diſtance of D, is juſt as much leſs than CE, or than the diſtance of E, as the angle ADB, or the divergency of the rays proceeding from D, is greater than the angle AEB, or, than the divergency of the rays proceeding from E. That is, juſt in the ſame proportion as the diſtance CD is leſs than the diſtance CE the divergency of the rays proceeding from D is greater than the divergency of the rays proceeding from E; or the diſtance of the radiants are inverſely as the divergency of the rays.

Now ADC is half ADB, and likewiſe AEC is half AEB. And what can be proved of half the angles of divergency, may be inferred to be true of the whole angles. I ſay therefore, that ADC is as much greater than AEC as CD is leſs than CE, or that, ADC is to AEC as CD to CE inverſely, or as CD to CE inverted; that is, upon the whole ADC is to AEC as CE to CD: for CE and CD are CD and CE inverted.

At E draw EF ſo as to make the angle FEC equal to the angle ADC. Then it is plane that ADC is juſt as much greater than AEC, as FEC is greater than AEC: becauſe ADC and FEC are equal by the conſtruction. But if theſe angles are ſmall they are very nearly in the proportion of the ſubtenſes FC and AC, by propoſition 26; that is FEC or ADC is juſt as much greater than AEC as FC is greater than AC. We are therefore to enquire how much FC is greater than AC. Now ADC is equal to FEC by the conſtruction, and ADC is the external angle and FEC is the internal oppoſite angle on the ſame ſide made by the line CE croſſing the two lines AD and FE. Therefore AD and FE are parallel, Euc. b. I. prop. 28. And becauſe AD and FE are parallel, therefore CAD is equal to CFE, or the external angle made by a line CF croſſing theſe two parallel ones is equal to the internal oppoſite angle on the ſame ſide. Euc. b. I. prop. 29. Therefore there are two angles in the triangle FEC reſpectively equal to two angles in the triangle ADC; and conſequently the third angle in one muſt be equal to the third angle in the other. Euc. b. I. prop. 32. corol. 2. That is; theſe two triangles FEC and ADC are equiangular, from whence it follows that their ſides are proportional. Euc. b. VI. prop. 4. FC to AC as EC to DC. We find therefore that EC is juſt as much greater than DC as FC is greater than AC; or as the angle FEC or its equal the angle ADC is greater than the angle AEC; or as half the divergency of rays that proceed from D is greater than half the divergency of rays that proceed from E. But EC, which is proportional to half the divergency of rays from D, is the diſtance of the radiant E, and DC, which is proportional to half the divergency of rays from E, is the diſtance of the radiant D. Therefore theſe diſtances of the radiants are as half the divergency of the rays proceeding from them inverted.

But ſince half the angles of divergency are inverſely as the diſtances of the radiants, therefore twice thoſe halves or the whole angles of divergency muſt be in the ſame proportion: twice ADC or ADB muſt be as much greater than twice AEC or than AEB, as ADC is greater than AEC. But ADC is to AEC as the diſtances of D and E inverſely. Therefore ADB and AEB are likewiſe as the diſtances of D and E inverſely.

29. *If the diſtance between converging rays, that tend to different focuſes, is the ſame; then the diſtances of the focuſes are inverſely as the convergency of the rays, which tend to them.*

If AD, BD, Plat. VIII. fig. 1. are the lines deſcribed by rays converging to the focus D, and AE, BE the lines deſcribed by other rays converging to the remoter focus E; ſo that the diſtance AB between the former rays and the latter is the ſame; it is evident, that the rays, which tend to D the nearer focus, converge more than thoſe, which tend to E the remoter focus, or that the angle ADC is greater than the angle AEB. Theſe angles, or the convergency of the rays whoſe directions contain theſe angles, are inverſely as the diſtances of the focuſes D and E, or as thoſe diſtances inverted. That is the angle ADB, or the convergency of rays tending to D, is as much greater than the angle AEB, or than the convergency of rays tending to E, as EC the diſtance of the latter focus is greater than DC the diſtance of the former focus.

That the angle ADB bears the ſame proportion to the angle AEB, that EC bears to DC has been already proved in the laſt propoſition; and this is all that is neceſſary to prove the truth of this.

30. *If rays proceed from a radiant at an infinite diſtance, their divergency is nothing, and the rays are conſidered as parallel ones.*

Let F, Plat. VIII. fig. 2. be a radiant from whence rays diverge in the directions FDB, FCA. The diſtance between theſe rays at D and C is DC, and at B and A is BA. But BA is greater than DC; therefore the rays, the farther they go from F, are farther aſunder from one another; and conſequently are diverging ones. This is the caſe when the radiant F is at a finite diſtance. But if the diſtance AF of the radiant F is infinite, then DC will be equal to BA, or however the difference between them will be infinitely ſmall; ſo that when the rays are at B and A the diſtance between them will be no greater, than when they were at D and C; or theſe rays proceed in lines which may be looked upon as parallel ones.

Draw AB perpendicular to FA, and CE parallel and equal to AB, and complete the parallelogram ACEB by drawing EB. The triangles FBA and BDE are ſimilar. For the angle BAF is equal to the angle DEB; both becauſe they are oppoſite angles in the parallelogram ACEB, Euc. b. 1. prop 34. and becauſe by the conſtruction they are right ones. The angle BFA is likewiſe equal to the angle DBE: becauſe FA and BE are parallel, and theſe two angles are the alternate angles made by the right line FB croſſing theſe two parallel ones. Euc. b. I. prop. 29. And conſequently, ſince there are two angles in one of theſe triangles reſpectively equal to two angles in the other, the third angle FBA in one is equal to the third angle BDE in the other. Euc. b. I. prop. 32. corol. 2. But equiangular triangles have their ſides proportional. Therefore ED bears the ſame proportion to EB, that AB bears to AF. Euc. b.VI. prop. 4. Now if AF was infinite, the finite line AB would be ſo ſmall in reſpect of AF as to bear no proportion to it; and conſequently when AF is infinite ED will bear no proportion to EB. For as AB to AF, ſo is ED to EB. But EB is a finite line, whatever the length of AF is. And ſince ED is ſo ſmall, when AF is infinite, as to bear no proportion to this finite line EB, it muſt be infinitely ſmall. But ED is the difference between CD and AB. Therefore, when AF is at an infinite diſtance, the difference between CD and AB being infinitely ſmall, theſe two lines may be conſidered as equal and conſequently DB and CA may be conſidered as parallel. So that if F is a radiant at an infinite diſtance, the rays, that come from thence and deſcribe lines ſuch as DB, CA, are parallel to each other.

This propoſition may likewiſe be deduced from prop. 28. For, ſince the divergency of rays is inverſely as the diſtance of the radiant, it follows, that, when the diſtance of the radiant is infinitely great, the divergency of the rays muſt be infinitely ſmall. And, when the divergency of rays is infinitely ſmall they may be conſidered as parallel.

This is the caſe of rays that come from one and the ſame point in the ſun. I ſay from one and the ſame point, becauſe the whole body of the ſun is not a ſingle radiant but is made up of many; and the rays which come from its center are not parallel to thoſe which come from its circumference. But all rays which come from the center of it are parallel to one another: becauſe the diſtance of the radiant in this caſe, when compared with the diſtance between any two rays, may be taken for infinite.

finite. In like manner all the rays, which come from one and the ſame point either in the edge of the ſun or in any part of its face are parallel for the ſame reaſon.

31. *If rays tend to a focus at an infinite diſtance, their convergency is nothing, and the rays are conſidered as parallel ones.*

Let F, Plat.VIII. fig. 2. be a focus to which rays tend in the directions AF and BF: then if the diſtance of F is infinite the lines BD and AC, which the rays deſcribe are parallel to one another. This has already been proved in the foregoing propoſition.

32. *The ſides of plane triangles are to one another as the ſines of the angles which they ſubtend.*

By plane triangles we mean triangles, whoſe ſides are right lines. Thus the triangle ABC, Plat.VIII. fig. 3. is a plane one. Now in this triangle the ſide AB ſubtends or is oppoſite to the angle ACB: and the ſide AC ſubtends the angle ABC. Set one leg of the compaſſes on C and with the opening CB ſtrike the arc BF, then continue AC to H and from B draw BE perpendicular to CH. This line BE is the ſine of the angle BCE: and it is likewiſe the ſine of the angle ACB. For ACB is the complement of BCE to two right ones; and the complement of any angle to two right ones has the ſame ſine with the angle itſelf. Or otherwiſe. BE is the ſine of ACB; becauſe the ſine of an angle is a line drawn from the extremity of one leg perpendicular to the other leg. Now B is the extremity of one leg, and no line can be drawn from B perpendicular to the other leg AC, unleſs AC is continued in the direction CH. Therefore a line drawn from B perpendicular to AC continued, that is, the line BE is the ſine of the angle ACB. Then making B the center, with the ſame opening BC of the compaſſes as before, ſtrike the arc CG and from C draw the line CD perpendicular to BD: this line CD is the ſine of the angle ABC. Now theſe two ſides AB and AC are to each other in the ſame proportion as the ſines of the angles ACB and ABC, which theſe ſides ſubtend. That is AB is to AC as BE to CD.

The triangle BAE is ſimilar to the triangle CAD. For in the triangle BAE the angle at E is a right one, by the conſtruction: and ſo is the angle at D in the triangle CAD. And the angle at A is common to both of them. Therefore there are two angles in one of theſe triangles reſpectively equal to two angles in the other: and conſequently the third angle in one or EBA is equal to the third angle in the other or to DCA. Euc. b. I. prop. 32. corol. 2. From whence it follows that the ſides of theſe triangles, which ſubtend the equal angles, are proportional, AB to AC as BE to CD. That is, AB and AC, two ſides of ABC, are in the ſame proportion to one another with BE and CD the ſines of the angles, which thoſe ſides ſubtend.

CHAP. II.

Of light, and the cauſe and laws of its refraction.

33. *The progreſs of light is not inſtantaneous, but the rays of it take up ſome time to paſs from one place to another.*

THE Carteſians are of a different opinion and maintain, that light is propagated from a luminous body to any diſtance however great in an inſtant. But the truth of our aſſertion is amply proved by obſervations, that have been made upon the ſatellites or moons that move round the planet jupiter. Let A be the ſun, Plat. VIII. fig. 4. ORPS the earths orbit, F the planet jupiter, and NGH the orbit of its innermoſt moon or ſatellite. Now as jupiter is an opake body it will caſt a ſhadow oppoſite to the ſun; and let FGH repreſent this ſhadow. If the ſatellite, whilſt it is deſcribing its orbit, paſſes through this ſhadow from G to H, it will be eclipſed as long as it continues in the ſhadow. For ſince it has no light of its own, and ſhines only by reflecting the ſuns light, when it is in any part between G and H the opake body of jupiter prevents the ſun from ſhining upon it: and therefore the ſatellite will receive no light, which it might reflect, and upon this account is eclipſed to a ſpectator in the earth which we will ſuppoſe to be at B. When the ſatellite comes to H it emerges out of the ſhadow, and will again become viſible to a ſpectator at B. This innermoſt ſatellite is 42 $\frac{1}{2}$ hours in going round its orbit. That is, if it is at any time ſeen at H, in 42 $\frac{1}{2}$ hours it will have gone round and be returned to H again. Therefore if the earth was to be at reſt at B, ſince the ſatellite emerges every time it comes to H, a ſpectator at B on the earth would ſee 30 emerſions in the time of 30 revolutions or in $30 \times 42\frac{1}{2} = 1275$ hours time. And the 30th of theſe emerſions would be ſeen at the end of theſe 30 revolutions or at the diſtance of 1275 hours from the firſt of them. But in 1275 hours the earth inſtead of ſtanding ſtill at B will be gotten in its orbit to C. And conſequently the light from the ſatellite, after it emerges, has farther to go, before it gets to the earth, than it would have had, if the earth had continued at B. MC is the difference of theſe two diſtances. So that if the propagation of light is inſtantaneous, it will take up no time at all to deſcribe MC;

but a ſpectator at the earth will ſee the 30th emerſion, though the earth is at C, at the ſame time that he would have ſeen it, if he and the earth had continued all the while at B. But in fact it is found that, when the earth is at C, this 30th emerſion will not be ſeen at the end of the 30 revolutions or 1275 hours after the firſt; but ſome time, as ſuppoſe 10 minutes, later. Therefore, ſince this emerſion is ſeen 10 minutes later, when the earth is at C, than when it is at B, the firſt ray of light which makes the ſatellite viſible after its emerſion is 10 minutes longer in coming to C than it would be in coming to B. And conſequently light is not propagated from any one place to any other in an inſtant: for it takes up 10 minutes to deſcribe the line MC, which is the difference between the two diſtances of the earth from the ſatellite at the two places B and C.

In like manner if the earth was at D, a longer time will be taken up for the ſatellite to enter the ſhadow at G 30 times, than would be, if the earth was nearer to it as at E. Or the laſt light which comes from the ſatellite when it enters the ſhadow, will be longer in paſſing to D than to E. For the ſatellite does not diſappear, or is not ſeen to enter the ſhadow, or is not loſt ſight of, till this laſt light is come to the ſpectators eye. And conſequently, ſince this entrance happens later when the earth is at D than it does when the earth is at E, this laſt light is longer in paſſing to D than to E.

By computing from obſervations of this ſort, it is found, that if tables of the times of entrance and emerſion of this or any other ſatellite are calculated for the middle diſtances of the earth from the ſatellite, that is, if tables are made for finding the time that an emerſion will happen, when the earth is at O, or an entrance, when the earth is at P, theſe emerſions and entrances will happen about 7 minutes ſooner than they ought to do by the tables, if the earth is at R, or at its leaſt diſtance from the ſatellite, and about 7 minutes later than by the tables, if the earth is at S, or at its greateſt diſtance from the ſatellite. Therefore theſe emerſions and entrances are ſeen at the earth about 2×7 or 14 minutes ſooner, when it is at R, than when it is at S. Conſequently a ray of light takes up about 14 minutes to deſcribe the line RS; and about half 14 or 7 minutes to deſcribe half RS or AS. But RS is a diameter of the earths orbit, and AS is a ſemidiameter of the earths orbit or the diſtance of the earth from the ſun. Therefore a ray of light is about 7 minutes in deſcribing a ſemidiameter of the earths orbit or in coming from the ſun to the earth.

The diſtance of the earth from the ſun is ſo great that, if a cannon ball was to move at the rate of 500 feet in every ſecond of time, it could not come from the ſun to the earth in leſs than 25 years. This compariſon ſufficiently ſhews the incredible velocity of light. For at this rate the velocity of light will be above 10 million times greater than that of a cannon ball.

34. *The rays of light are propagated in right lines.*

This is plane from the ſhadows which opake bodies caſt. For if the rays of light did not deſcribe ſtrait lines, they would bend round any opake body, and would prevent it from caſting any ſhadow. It is plane likewiſe from lights finding no paſſage through bended tubes; and from the appearance of it as it paſſes through any little hole into a room filled with duſt or ſmoke.

Since the rays of light are conſtantly propagated in right lines from luminous bodies; for the future, whenever we have occaſion to repreſent a ray, we ſhall do it by a line reaching from the luminous body to the body illuminated; that is, we ſhall ſpeak of the line, which a ray deſcribes, as if that was the ray itſelf.

35. *Light is a body, and the ſeveral rays, or particles of it, are as well ſucceſſive in the ſame line, as contemporary in different lines.*

We have ſeen already, in prop. 34. that light is propagated in right lines. Whereas if it was only an impulſe of a luminous body upon ſome ſubtil fluid, it would paſs through bended tubes as well as ſtrait ones, and would bend round an opake body ſo as to prevent it from caſting a ſhadow; in the ſame manner that ſound, which is owing to vibrations excited in a fluid, is propagated through bended tubes, and when any obſtacle is placed between the ſounding body and the ear, it goes round that obſtacle and is heard on the other ſide. But ſince there is no middle opinion, and light is allowed to be either a real body or elſe an impulſe upon ſome ſubtil fluid; having ſhewn that the latter of theſe opinions is falſe, we may conclude that the former is true.

That light conſiſts of parts is evident; becauſe in the ſame place you may ſtop that which comes one moment, and let that paſs which comes preſently after; and ſince that light, which is ſtopped, cannot be the ſame with that which is ſuffered to paſs, there muſt be different parts. And as they come one after the other, they are ſucceſſive. Again, you may ſtop the light in one place, and may at the ſame time let it paſs in another; and becauſe, as before, that which is ſtopped cannot be the ſame with that which paſſes, theſe are different parts; and ſince ſome parts are ſtopped at the ſame time that others paſs, thoſe parts, which are ſtopped, are contemporary with thoſe, which are let paſs.

36. *The rays of light are exceeding ſmall.*

Since the velocity of a ray of light is above ten million times greater than that of a cannon ball; if a ray of light weighed only one grain it would have near 200 times the moment that a cannon ball of ten pound weight, has when it is firſt diſcharged from the cannon, and if it weighed only $\frac{1}{200}$ of a grain the moment of the ray would be equal to that of the ball. But if ſo ſmall a particle as $\frac{1}{200}$ of a grain would have ſo great a moment, the rays of light muſt be inconceivably ſmall; or elſe upon account of their great velocity their moment would be too great for ſo tender an organ, as the eye is, to bear the continual impulſes of them.

If a hole is made with a pin in a piece of paper, a man looking through that hole ſees all the objects that are before him, let them be ever ſo many. And ſince theſe objects and each point of them becomes viſible by means of rays of light coming from thence to the eye; theſe rays muſt be extremely ſmall to be able ſo many of them to paſs through a pin-hole, without interrupting each others paſſage.

The reader will wonder the leſs at this extreme minuteneſs of the rays of light, when he recollects that matter is infinitely diviſible, and that though we cannot find out means of ſeparating it into very minute particles, yet ſtill in its own nature it is ſeparable without end. How far it is actually ſeparated by a power infinitely ſuperior to ours, by the power of the Author of nature, is ſeen in many inſtances, that we are well acquainted with, or however that we hear of very frequently. If a candle is lighted in the night-time at the top of any high building, it may be ſeen at the ſame inſtant any where at the diſtance of half a mile all round. Now in any one inſtant there is no ſenſible waſte made of the candle and yet in one inſtant rays or particles of light enough muſt fly off from the candle to fill the whole ſpace, in any part of which the light of it could be ſeen, and that ſpace is a ſphere whoſe ſemidiameter is half a mile. We have other inſtances of the minuteneſs of matter beſides that of light. A piece of muſk does not in any length of time loſe its ſmell. And as there muſt be always effluvia fly off from it, whilſt it retains its ſmell, theſe effluvia muſt be extremely ſmall, becauſe notwithſtanding they are conſtantly thrown off for innumerable years, the weight of the muſk is not ſenſibly diminiſhed. Leeuwenhoek by the help of microſcopes diſcovered what very minute particles of matter are capable of animal life. For the leaſt ſort of thoſe animals, which he found in pepper-water, were 1000,000,000

times

times leſs than a grain of ſand. We might have amuſed ourſelves a little farther with ſpeculations of this ſort, and might have conſidered how ſmall the entrails and particularly the heart of theſe animals muſt be; if the late diſcovery of the polype had not made us doubt whether entrails and a heart are neceſſary to the life of an animal.

37. *If a ray of light paſſes obliquely out of a rarer medium into a denſer, it will be refracted towards a perpendicular.*

That light is attracted by other bodies, appears from what has been ſaid in prop. 5. of mechanics: where we obſerved that, if the rays of it, come through the hole of a window-ſhutter into a dark chamber, and a knife is held near the hole with its edge towards the rays, thoſe which paſs very near it are bent down towards it. The force, which thus bends them, whatever it is or to whatever cauſe it is owing, we call attraction. It is a force that we are ſure exiſts in nature, and if it is ſufficient to explane the refraction of light, the rules of true philoſophy will allow us to make uſe of it, and do not require us to look out for any other. But it is not iron only that attracts light in this manner. The edge of a thin plate of any other metal, or of a piece of broken glaſs, produces the ſame effect. Now this attraction will be ſo much ſtronger as there are more particles within any given ſpace. Every particle attracts, and conſequently the more there are of them the greater effect they muſt neceſſarily produce. From hence it follows that a ray of light is more attracted by a denſe medium than it is by a rare one. Thus water attracts light more than air does, and glaſs more than water, and a diamond more than any of them.

Suppoſe then a ray of light to paſs obliquely out of a rarer medium X Plat. VIII. fig. 5. into a denſer one Z through the plane ſurface *ab*, and let *ro* be the incident ray. Thus for inſtance on the ſide X of the line *ab* let there be air, and on the ſide Z glaſs. The ray *ro*, when it gets into the glaſs, will not go ſtrait forwards in the direction *og*, but will be refracted towards a perpendicular, that is, towards a line perpendicular to the ſurface *ab*: *cok* is ſuch a line, and the ray inſtead of going on in the direction *og* will be bent into *os*, a line nearer to *ok*, or making a leſs angle with *ok* the perpendicular than *og* does.

The ray is more attracted by the glaſs than by the air: and for this reaſon it will be accelerated or made to move faſter as it enters the glaſs. For though, when it gets into the glaſs, it is attracted equally all ways, and therefore will have no alteration made in its velocity by ſuch attraction, ſince theſe equal and contrary forces will deſtroy one another; yet as it is paſſing out of the air into the glaſs at *o* it is attracted more towards the glaſs Z than towards the air X. And becauſe the ray is moving towards Z, at the ſame time that it is more attracted thither than it is the contrary way, this attraction conſpires with the rays motion, and conſequently will encreaſe its velocity. Therefore the ray moves faſter, when it gets into the medium Z or the glaſs, than it did, whilſt it was in the medium X or in the air. But the ray, whilſt it was in the air, moved in the line *ro*, which is oblique to the attracting ſurface *ab*. And as the glaſs attracts the ray directly towards it, or in a line perpendicular to its ſurface, it follows that, if we reſolve the oblique motion of the ray *ro* into two others, one of them as *rc* parallel to the ſurface *ab* and the other as *rb* or $co = rb$ perpendicular to it, the parallel motion *rc* of the ray is not affected by this attraction; for an attraction, which acts in the line *rb* or *co*, can neither accelerate nor retard the motion of a ray in the line *rc*. The change of velocity, which the ray ſuffers, is made in the perpendicular part of its motion *co*. Take therefore *ok* greater than *co*, and *oi* equal to *bo* or to *rc*. Then, ſince the rays parallel velocity is not altered by paſſing out of one medium into the other, as *rc* was its parallel velocity in the air, $oi = rc$ will be its parallel velocity in the glaſs. But the perpendicular velocity becomes greater in the glaſs than it was in the air; and as the line *co* repreſented this velocity, whilſt the ray was in the air, let *ok* repreſent it, when in the glaſs; that is, let the ray be ſuppoſed to be accelerated in the proportion of *ok* to *co*, or let its velocity be as much greater in the glaſs than it was in the air as *ok* is greater than *co*. Now the ray, whilſt it was in the air, deſcribed the line *ro*, and this oblique motion might be conſidered as if it was produced by the action of two forces, one in the line *rc*, the other in the line $rb = co$, which are the two ſides of a parallelogram, whoſe diagonal is *ro* the direction of the ray; by prop. 20. of mechanics. One of theſe forces acts parallel to the ſurface *ab* and the other perpendicular to it. The parallel part of this motion is not altered by the paſſage of the ray out of one medium into the other. Therefore, when the ray enters the glaſs it is ſtill acted upon by a force parallel to the ſurface *ab* and equal to *rc*. This force may then be repreſented by $oi = rc$. But the other part of the rays motion, which is in a line perpendicular to the ſurface *ab*, is accelerated, as the ray paſſes out of the air into the glaſs; ſo that if *co* was this part of the motion, whilſt the ray was in the air, *ok* a line longer than *co* muſt repreſent it, after the paſſage of the ray into the glaſs. Therefore when the ray enters the glaſs it may be conſidered as acted upon by two forces *oi* and *oh*, and conſequently will deſcribe the diagonal *os* of a parallelogram, whoſe two ſides are *oi* and *oh*; by prop. 16. of mechanics. Now the parallelogram *oiks* has one ſide *oi* equal to one ſide *rc* in the parallelogram *rcho*: but the ſide *ok* in the former is longer than the ſide *co* in the latter: and theſe ſides *ok* and *co* are perpendicular to the ſurface *ab*. But in all parallelograms, where one ſide is given, the longer the other ſide is the leſs is the angle which the diagonal makes with it. Therefore, ſince *rc* and *oi*, are given, the diagonal *os* will make a leſs angle with the ſide *ok* than the diagonal *ro* does with the ſide *co*. But *os* is the direction of the refracted ray, and *oh* is the perpendicular; *ro* is the incident ray, and *co* the perpendicular. Therefore the refracted ray makes a leſs angle with the perpendicular, or is nearer to it, than the incident ray; that is, the ray by paſſing out of air into glaſs or out of a rarer medium, which attracts it leſs, into a denſer, which attracts it more, is refracted towards a perpendicular.

This propoſition is not only proved by reaſoning in this manner from the cauſe of refraction, but is likewiſe confirmed by matter of fact. Let SB, Plat. VIII. fig. 6. the ſide of a veſſel SABD be perpendicular to the bottom BD: ſuppoſe the veſſel to be empty and the ſun to ſhine upon the ſide SB, ſo as to caſt the ſhadow of this ſide upon the bottom; and let BD be the length of this ſhadow, that is, ſuppoſe the ſhadow to reach from B to D. Now this ſhadow is terminated by a ray IS, which paſſes by the edge of the veſſel S: for all rays that paſs above it fall upon the ſide AD; and all that paſs below it are intercepted by the ſide SB. Let the veſſel then be filled with water up to the top of it SA; and, though all things elſe continue as they were, the ſhadow will become ſhorter and will reach only from B to C. Therefore the ray IS, which went ſtrait forwards from S to D and terminated the ſhadow at D, when the veſſel was empty, deſcribes the lines SC, when the veſſel is full and terminates the ſhadow at C. Therefore this ray is refracted, when the veſſel is full, by paſſing out of the air, where it deſcribed IS, into the water, where it deſcribes SC. This refraction, which is made by the paſſage of the ray out of a rarer medium into a denſer, is towards a perpendicular. For BS is perpendicular to the bottom of the veſſel, and conſequently BS continued, or SP, is perpendicular to the ſurface of the water. Therefore ISP is the angle of incidence. But BSD is equal to ISP, becauſe they are vertical angles made by two right lines croſſing each other; Euc. b. I. prop. 15. ſo that BSD as well as ISP is the angle, which the incident rays direction makes with the perpendicular. But BSC is the angle, which the refracted ray makes with the ſame perpendicular; and

BSC is leſs than BSD. Therefore the direction of the refracted ray is nearer to the perpendicular than the direction of the incident one.

Or otherwiſe. The length of the ſhadow BD, when the veſſel is empty, ſubtends the angle BSD, which is equal to ISP or to the angle of incidence: and the length of the ſhadow BC, when the veſſel is full of water and the ray paſſes out of the air into this water, ſubtends the angle BSC, which is the refracted angle. But the length of the ſhadow is leſs when the veſſel is full. Therefore the refracted angle is leſs than the angle of incidence; or the refracted ray makes a leſs angle with the perpendicular than the incident one does.

38. *If a ray of light paſſes obliquely out of a denſer medium into a rarer, it will be refracted from a perpendicular.*

If an oblique ray of light *ſo*, Plat. VIII. fig. 5. paſſes out of glaſs into air through the ſurface *ab*, that is, if there is a denſer medium on the ſide Z of *ab*, and the ray paſſes out of it obliquely through the ſurface *ab* into a rarer medium on the other ſide X; the ray will not go ſtrait forwards in the line *ot*, but will be refracted into the line *or*, ſo that *cor* the refracted angle, or angle, which the refracted ray *or* makes with the perpendicular *oc*, will be greater than *ſpk* the angle of incidence, or angle which the incident ray *ſo* made with the perpendicular *ok*.

The ray is more attracted by the glaſs than by the air, or more by the denſer medium Z, whatever that medium is, than by the rarer one X. Upon this account when it comes to *o*, and is paſſing out of the glaſs into the air, it is not equally attracted all ways, for the air, into which it is paſſing, attracts it leſs than the glaſs, out of which it is paſſing. Therefore the ray will be drawn more backwards than it is forwards: and this difference of attraction will retard its motion, ſo as to make it move ſlower in the air than it did in the glaſs. But the oblique motion of the ray in the glaſs, or *ſo* may be reſolved into two others, one of them *ſk* parallel to the glaſs, the other *ko* perpendicular to it; by propoſition 20. of mechanics. The parallel motion of the ray in the line *ſk* can ſuffer no alteration as to the velocity from an attraction, which acts in the line *ok*, as the attraction does by which the ray is more drawn towards the glaſs Z than towards the air X. But this attraction will retard the perpendicular motion *ko*. From hence then it follows that taking *oh* equal to *ſk* or *io*; becauſe *ſk* was the parallel motion of the ray in the glaſs, *oh* will be its parallel motion when it enters the air at *o*: for the velocity of this motion undergoes no alteration, and therefore may always be repreſented by a line of the ſame length. But the perpendicular motion *ko* is retarded by the rays paſſing out of the glaſs into the air. Therefore, taking *on* equal to *ko*, if *ko* or *on* repreſented the perpendicular velocity before the ray was retarded, ſome line ſuch as *oc*, which is ſhorter than *on*, muſt repreſent it afterwards. And, ſince the ray, when it is paſſing out of one medium into the other at *o*, may be conſidered as acted upon by two forces *ob* and *oc*, theſe forces together will make it deſcribe *or* the diagonal of a parallelogram, of which *ob* and *oc* are the ſides. The ray had indeed deſcribed the diagonal of a parallelogram, whilſt it was in the glaſs, but then *ks* and *ko* were the ſides. And ſince the ſide *ob* in one of theſe parallelograms is equal to the ſide *ks* in the other, but *oc* in the former is ſhorter then *ko* in the latter; it follows that *or* will make a greater angle with *oc* than *ſo* does with *ko*; becauſe in all parallelograms where one ſide is given, the ſhorter the other ſide is the greater is the angle which the diagonal makes with it. Now the angle *cor*, which *or* makes with *oc*, is that which the refracted ray makes with the perpendicular, or is the refracted angle; and the angle *ſok*, which *ſo* makes with *ko*, is that which the incident ray makes with the perpendicular, or is the angle of incidence. Therefore, when a ray paſſes out of glaſs into air or out of a denſer medium into a rarer, the refracted ray will be farther from the perpendicular than the incident one, that is, the refracted angle will be greater than the angle of incidence, or the ray will be refracted from a perpendicular.

We may apply this propoſition to a common inſtance, which will both illuſtrate the meaning and prove the truth of it. If a veſſel NBAL, Plat. VIII. fig. 7. is empty, and any ſmall object, ſuch as a ſhilling, is laid upon the bottom of it at C, when the ſpectators eye is at R, he will not be able to ſee the ſhilling. For this, like any other object, can only be ſeen by means of rays of light, which come from it to the eye. And ſince all rays deſcribe right lines, when no particular cauſe prevents them, by prop. 34, and no right line can be drawn from C to R but what muſt paſs through the ſide NA of the veſſel, no ray can be propagated from C to the eye at R, becauſe the ſide of the veſſel is in the way and will intercept any ray that was coming from C towards R. But let the veſſel be filled with water, and then, though the eye continues at R and the ſhilling at C, the ſhilling may be ſeen. Now it is evident, that no ray can come ſtrait from C to R any more when the veſſel is full of water than when it is empty; for the ſide AN will alike intercept any ſuch ray in either caſe. Therefore the ray which makes C viſible muſt have been bent or refracted in coming from C to R. This refraction muſt have happened, when the ray coming from C paſſed out of the water into the air: for it will deſcribe a right line in the water, and likewiſe a right line in the air: therefore no bending can happen in any part of the rays paſſage, but where the water and air are contiguous, or at N where the ray paſſes out of one medium into the other. It is farther evident, that the ray, which makes C viſible to the eye at R, could not whilſt it was in the water deſcribe any line below CN, for all rays below this would be intercepted by the ſide of the veſſel NA. Therefore C is made viſible by ſome ray, which in the water deſcribes CN, or ſome line above it; and either of theſe caſes would equally anſwer our preſent purpoſe. Suppoſe it therefore to be the ray CN, which comes to the eye at R, then this ray inſtead of going ſtrait forwards in the line ND, is bent at the point N, or refracted into the line NR: for it comes ſtrait from C to N through the water in the line CN; and it likewiſe comes ſtrait from N to the eye at R through the air; therefore NR is the line, which the ray ſo refracted deſcribes; and the object being ſeen in the direction, which the ray has when it comes to the eye, will appear as if it was at L. That is, the object placed at C, will appear when the veſſel is full, juſt as the ſame object would, if it was placed at L, when the veſſel is empty. But if a ray deſcribes CN, whilſt it is in the water, and, as it paſſes from thence into the air, is refracted into NR, this refraction is from a perpendicular. For, let NP de drawn perpendicular to the ſurface of the water, then ſince CN or CD, which is CN continued, is the direction of the incident ray, when it falls upon the ſurface NB and is about to paſs out of the water into the air, DNP is the angle of incidence; and ſince NR is the refracted ray, PNR is the refracted angle. But PNR is bigger then PND, that is, the refracted angle is bigger than the angle of incidence, or the refracted ray makes a greater angle with the perpendicular than the incident one does. Therefore the ray is refracted from a perpendicular.

39. *All refraction is reciprocal.*

If a ray of light is refracted in paſſing out of one medium into another, then, if that ray was to return back again, it would be refracted juſt as much the contrary way. If the ray *ro*, Plat. VIII. fig. 5. in paſſing out of air into water is refracted into *os*, then ſuppoſing this ray to return to the point *o* in the direction *ſo*, in paſſing out of the water back again into the air it will be refracted into *or*.

This may be expreſſed in another manner, which is this: if the refracted ray *os* becomes the incident ray, then the incident ray *or* will become the refracted one. That is, if a ray falling upon the ſurface *ab* deſcribed the line *ro*, and the line *os* is found, which that ray will deſcribe, when it is refracted; then ſuppoſing a ray to fall in the direction *ſo* upon the ſurface *ab* the line, which the refracted ray will deſcribe, may be known, for it will be *or*.

The ray *ro* is refracted into *os*, as it paſſes out of one of theſe mediums into the other; becauſe it is accelerated and in the latter medium Z moves faſter than it did in the former medium X. But when this or any other ray deſcribes the line *ſo* and paſſes back again out of the medium X into the medium Z, it will be juſt as much retarded as it was accelerated before, and conſequently will be juſt as much refracted the contrary way, and will therefore be bent into *or*.

40. *Where the two mediums are the ſame, the ſine of incidence will always bear the ſame proportion to the ſine of the refracted angle, when the ray paſſes out of one medium into the other; whatever may be the obliquity of the incident ray.*

If X, Plat. II. fig. 5. is air, and Z is glaſs, and a ray of light *ro* paſſes out of one into the other at the point *o*; then ſuppoſing that the angle of incidence *roc* is 50 degrees, and that *rc* the ſine of incidence is to *df* the ſine of the refracted angle as 3 to 2. This which is the proportion of one of theſe ſines to the other at the obliquity of 50 degrees, will likewiſe be the proportion between them at any other obliquity. That is, if the angle of incidence *roc*, inſtead of being 50 degrees, was 70 or 20 degrees, ſtill *rc* would be to *df* as 3 to 2.

The oblique motion of the ray *ro* may be reſolved into two others, ſuch as *rc* parallel to the ſurface *ab* and *rh* or *co* perpendicular to it. This perpendicular motion *co* is accelerated as the ray paſſes out of the air into the glaſs. Take therefore *od* equal to *co* and continue the refracted ray *of* to *s*, ſo that, if *ks* is drawn perpendicular to *ok*, *ks* may be equal to *rc*, and draw *df* the ſine of the refracted angle *kof*. Now the rays perpendicular velocity whilſt it was in the air being *co*, its perpendicular velocity when it is in the glaſs will be *ok*, by propoſition 37; that is, the ray is accelerated in the proportion of *ok* to *od*. But *oks* and *odf* are equiangular triangles: for the angle at *k* in one is equal to the angle at *d* in the other, becauſe they are both of them right ones; and the angle at *o* is common to both; therefore the third angle in one is equal to the third angle in the other. Euc. b. I. prop. 32. corol 2. From whence it follows, that the ſides about the equal angles are proportional *ks* to *df*, as *ok* to *od*. But *ks* is the ſine of incidence, and *df* is the ſine of the refracted angle; and *ok* is the rays velocity in the glaſs, and *od* its velocity in the air.

Therefore the ſine of the angle made with the perpendicular in the air, or *ks*=*rc*, is to the ſine of the angle made with the perpendicular in the glaſs, or to *df*, as the rays velocity in the glaſs is to the rays velocity in the air, or as *ok* to *od*. That is, the ſines of incidence and of the refracted angle are as the rays velocity inverſely. But, ſince the ſine of incidence is to the ſine of the refracted angle as *ok* to *od*, or as the rays velocity in one medium to its velocity in the other; it follows that, if at all obliquities of the incident ray, theſe velocities bear the ſame proportion to one another, then at all obliquities the ſine of incidence will bear the ſame proportion to the refracted ſine. Now where the two mediums X and Z are given, that is, if X is air and Z glaſs, or if X is any medium whatever, and Z is any other medium whatever, the obliquity of the ray makes no alteration in the proportion of *ok* to *od*. For where the mediums are given, their denſities and conſequently the difference of their attractions is given. And, ſince the proportion, which the velocity of the ray in one medium, bears to its velocity in the other, depends upon the difference of their attractions; where this difference is given, the proportion of theſe velocities cannot be altered by changing the obliquity of the ray. So that if the velocity in one medium is to the velocity in the other as *ok* to *od*, and the ſine of incidence to the ſine of the refracted angle as *ks* to *df* in any obliquity of the ray, this proportion will continue the ſame, however the obliquity of the ray may be changed.

The reader will find it of uſe hereafter to remember what this proportion is in the following inſtances.

If a ray of light paſſes obliquely out of air into glaſs; the ſine of incidence *rc* is to the ſine of the refracted angle *fd* as 3 to 2 nearly. Now the angle *fog*, or the angle of refraction is the difference between *roc* or its equal *kog*, which is the angle of incidence, and *kos*, which is the refracted angle. Therefore, ſuppoſing the ſines proportional to the angles, by propoſition 26, the ſine of *fog* the angle of refraction is as the difference between *rc* and *df*, or as 3—2. That is, the ſine of incidence *rc* is to the ſine of refraction as *rc* to *rc*—*df*, or as 3 to 1.

If a ray of light paſſes obliquely out of glaſs into air, the ſine of incidence *fd* is to the ſine of the refracted angle *rc* as 2 to 3, by what has juſt now been laid down, and by propoſition 39. Here *tor* is the angle of refraction, for it is the difference between *fok*, or its equal *cot*, which is the angle of incidence, and *cor*, which is the refracted angle. Therefore conſidering the ſines and angles as proportional, by propoſition 26, the ſine of *tor* is as the difference between *rc* and *db*, or as 3—2. That is, the ſine of incidence *df* is to the ſine of refraction as *df* to *rc*—*df*, or as 2 to 1.

If a ray of light paſſes obliquely out of air into water, the ſine of incidence is to the ſine of the refracted angle as 4 to 3 : and the ſine of incidence to the ſine of refraction as 4 to 4—3=1.

If a ray of light paſſes obliquely out of water into air, the ſine of incidence is to the ſine of the refracted angle as 3 to 4: and the ſine of incidence to the ſine of refraction as 3 to 4—3=1.

41. *Rays of light, which paſs perpendicularly out of one medium into another, ſuffer no refraction.*

The ray *no*, Plat. VIII. fig. 5. which falls perpendicularly upon the ſurface *ab*, will not be refracted in paſſing out of the medium X into the medium Z, but will continue in the ſame direction and will go on in the right line *ok* perpendicular to *ab* on the other ſide.

By the laſt propoſition, *rc* the ſine of incidence, bears the ſame proportion to *df* the ſine of the refracted angle in all obliquities of the incident ray. Now as *ro* is nearer to *no*, or as the incident ray is nearer to the perpendicular, the ſine *rc* muſt be leſs: and conſequently the ſine *df* muſt be leſs in the ſame proportion. That is, the ſine of the refracted angle decreaſes in the ſame proportion that the ſine of incidence decreaſes. But when *ro* and *no* coincide, that is, when the incident ray deſcribes *no* or is perpendicular to the refracting ſurface *ab*, then *rc*, which is the diſtance of *r* from the perpendicular, becomes nothing. And when *rc* is nothing, *df*, which always decreaſes in the ſame proportion with *rc*, will be nothing, or the refracted ray will have no diſtance from the perpendicular *ok*, but will deſcribe this perpendicular. Therefore when the incident ray is perpendicular to the ſurface of the medium, into which it paſſes, the refracted ray, if it may be called ſo, will be perpendicular to the ſame ſurface; or the ray will go ſtrait forward out of one medium into the other, without being bent or refracted in its paſſage.

If a ray of light *no*, that is paſſing out of air into water falls perpendicularly upon *ab* the ſurface of the water, it will paſs ſtrait forwards in the line *od*, and will ſuffer no refraction. For, though the water attracts the ray more than the air does and will conſequently accelerate it, yet becauſe the ray is moving in a line perpendicular to the waters ſurface, this attraction acts in the ſame direction that the ray is moving, and therefore cannot change the direction of the ray or bend it from its courſe, but will only accelerate its motion in the ſame right line *no* continued below the ſurface. In like manner, if the ray *ko* was paſſing out of water into air and was perpendicular to *ab* the ſurface of the water out of which it paſſes, or, which amounts to the ſame thing, to the ſurface of the air into which it paſſes: this ray would be drawn back towards the water by the greater attraction on that ſide. But this attraction, as it acts in a line perpendicular to the ſurface of the water, acts in the ſame direction, in which the ray is moving, and therefore, though it retards the ray, will not bend it from its ſtrait courſe: but the ray will go on in *on*, which is *ko* continued, and as it was perpendicular to the ſurface *ab*, whilſt in the water, it will ſtill be perpendicular to it, when it comes into the air.

CHAP. III.

Of the refraction of light, when it paſſes through a plane ſurface.

42. *When parallel rays paſs obliquely out of one medium into another; they will continue parallel, after they are refracted.*

IF the rays AB and CD, Plat. VIII. fig. 8. are parallel one to another, and paſs obliquely out of air into water through the plane ſurface RZ, theſe rays will not go on in the ſame directions ſo as to proceed ſtrait forwards in BG and DH, they will each of them be refracted towards a perpendicular into BE and DF; by propoſition 37. But BE, DF, or the refracted rays will be parallel to one another as well as the incident ones.

Let KL be drawn perpendicular to the ſurface RZ at the point B. Then ABK is the angle of incidence for the ray AB; and ABZ is the complement of incidence to a right angle. CDZ is likewiſe the complement of incidence for the ray CD. But AB and CD are parallel; and conſequently the external angle CDZ made by the right line or line drawn in a plane ſurface RZ, is equal to the internal oppoſite angle on the ſame ſide ABZ. Euc. b. I. prop. 29. Therefore theſe rays, as their complements of incidence are equal, have their angles of incidence equal. And for this reaſon the refracted angles will be equal, or one ray will be juſt as much refracted as the other is; by propoſition 40. But if the refracted angles are equal, the complements of thoſe angles will likewiſe be equal. Now EBL is the refracted angle for the ray BE and RBE is its complement: in like manner RDF is the complement of the refracted angle for the ray DF. And ſince the external angle RBE, made by the line RZ croſſing the two lines BE and DF is equal to RDF the internal oppoſite angle on the ſame ſide, theſe two lines BE and DF, or the refracted rays, are parallel. Euc. b. II. prop. 28.

The reader will eaſily apply this method of reaſoning in one caſe of parallel rays to all other caſes, where they paſs out of one medium into another through a plane ſurface, whether they paſs out of a rarer medium into a denſer, or out of a denſer into a rarer. For in all caſes the incident rays, if they are parallel, will be equally inclined to a plane ſurface, and conſequently will be equally refracted, which will make them be equally inclined to the ſame plane ſurface after refraction, and for that reaſon they will ſtill be parallel to one another.

43. *Rays, that diverge, will be made to diverge leſs, and rays, that converge, will be made to converge leſs, by paſſing out of a rarer medium into a denſer, through a plane ſurface.*

If the rays RA and RB, Plat. VII. fig. 6. which diverge from the radiant R, paſs out of air into water, or out of any rarer medium Z into any denſer medium X through the plane ſurface S*s*; theſe rays, when they get into the water, will not go ſtrait forwards in the lines AC and BE, but will each of them be refracted towards a perpendicular, by propoſition 37, and will deſcribe AD and BG, that are nearer perpendicular to the ſurface S*s* than AC and BE.

Now AD and BG diverge leſs than AC and BE, that is, than RA and RB; or the refracted rays diverge leſs than the incident ones. IK and RH and all other lines, which are or can be drawn perpendicular to the plane ſurface S*s* are parallel to one another. Euc. b. XI. prop. 8. Therefore lines which are perpendicular to a plane ſurface do not diverge at all: for parallel lines make no angle with each other. But the refracted rays AD and BG are nearer being perpendicular to the ſurface S*s* than the incident ones RA and RB; that is, the refracted rays are nearer being parallel to each other than the incident ones. But ſince rays, if they are parallel do not diverge at all, the nearer they are to being parallel, the leſs they muſt diverge. Therefore the refracted rays diverge leſs than the incident ones.

In like manner, if the rays TM, and WN, which converge towards each other and tend to the focus *f*, paſs out of air into water, or out of any rarer medium Z into any denſer one X, through a plane ſurface S*s*; they will be made to converge leſs, that is, inſtead of going ſtrait forwards in the lines M*f* and N*f*, they will be refracted into ſome ſuch lines as MF, and NF, which make a leſs angle with one another than M*f* and N*f* do.

For the rays paſs out of a rarer medium into a denſer, and therefore are refracted towards the perpendiculars, by propoſition 37. The ſurface, where they are refracted is a plane one; and therefore all lines that can be drawn perpendicular to it either at N, as PO, or at M, or any where elſe are parallel to one another. But ſince the rays are refracted towards perpendiculars, or are nearer being perpendicular to the ſurface S*s* than they were at their incidence, and ſince the perpendiculars are all of them parallel to one another, it follows, that the refracted rays are nearer parallel than the incident ones. Now parallel rays do not converge at all, and conſequently thoſe rays, which are neareſt being parallel, will converge the leaſt. But the refracted rays are nearer being parallel than the incident ones. Therefore the refracted rays converge leſs than the incident ones.

44. *Diverging rays will be made to diverge more, and converging rays will be made to converge more, by paſſing out of a denſer medium into a rarer, through a plane ſurface.*

Let F, Plat. VII. fig. 6. be a radiant, and FM, FN, be rays diverging from it. Then, if theſe rays, at the points M and N, paſs out of water into air, or out of glaſs into air, or out of any denſer medium X into any rarer one Z, through a plane ſurface S*s*; inſtead of going ſtrait forwards in MQ and NV, they will be refracted into MT and NW, ſo as to diverge more, or ſo as to make a greater angle with one another, than the incident rays FM, and FN did.

The rays paſs out of a denſer medium into a rarer, and therefore are refracted from the perpendiculars. But the perpendiculars to a plane ſurface are parallel to one another. Therefore the refracted rays, being farther from the direction of perpendiculars, will be farther from being parallel, that is, will diverge more than the incident ones.

Or otherwiſe, we have ſeen, in propoſition 43, that, if the rays TM, and WN, are the incident ones, and converge to *f*; MF and NF which converge to F, will be the refracted ones. But, by prop. 39, all refraction is reciprocal. Therefore, if theſe rays, FM, and FN, are turned back again, ſo as to diverge from F, and become the incident rays, or if any other rays diverge from thence in the ſame manner; then MT, and NW, will be the refracted ones. Or, ſince, by prop. 43, converging rays are made to converge leſs, or to contain a leſs angle with one another, by paſſing out of a rarer medium into a denſer through a plane ſurface; it follows, from prop. 39, that if theſe rays are turned back again, ſo as to become diverging ones, that paſs out of a denſer medium into a rarer, this contrary refraction will increaſe the angle as much as the other diminiſhed it, or will make the rays diverge more.

If DA and GB, Plat. VII. fig. 6, are rays that converge to *r*, but muſt, be-

before they come thither, paſs through a plane ſurface at the points A and B out of water into air, or out of glaſs into air, or out of any denſer medium X into any rarer one Z; then theſe rays will not go ſtrait forwards in A*r* and B*r*, but will be refracted into AR and BR ſo as to converge more towards each other, or ſo as to make a greater angle with one another after refraction than they did before.

The rays paſs out of a denſer medium into a rarer, and therefore are refracted from the perpendiculars, by prop. 38. But the perpendiculars are parallel to one another, becauſe the ſurface is a plane one. Conſequently the rays after refraction being farther from the direction of the perpendiculars than they were before, will be farther from being parallel to one another, that is, will make a greater angle with one another, or will converge more.

Or otherwiſe. If RA and RB had been the incident rays, then, by prop. 43, AD and BE would have been the refracted ones. Therefore if DA, and BE are the incident rays, AR and BR will be the refracted ones, by prop. 39. Or, ſince, by prop. 43, diverging rays are made to diverge leſs or to contain a leſs angle with one another by paſſing out of a rarer medium into a denſer through a plane ſurface, it follows, from prop. 39, that if theſe rays are turned back again ſo as to become converging ones, that paſs out of a denſer medium into a rarer, or if any other rays converge in the ſame manner, this contrary refraction will encreaſe the angle juſt as much as the other diminiſhed it, or will make the rays converge more.

45. *When diverging rays are refracted at a plane ſurface, the diſtance of the real radiant from that ſurface is to the diſtance of the imaginary radiant, as the ſine of the refracted angle is to the ſine of incidence.*

If R, Plat. VII. fig. 6, is the real radiant, or point from whence the incident rays RA and RB diverge; ſince theſe rays, when they are refracted, diverge leſs, if they paſs out of a rarer medium into a denſer, they will proceed after refraction in AD and BG as if they had come ſtrait from *r*, which is the imaginary radiant, a point more remote than R. Becauſe the leſs the divergency the more remote the radiant, by prop. 28.

Suppoſe, for inſtance, that R is the higheſt leaf of a tree, which ſtands upon the bank of a river, and that rays of light reflected from that leaf deſcribe the lines RA, and RB, whilſt they are in the air, but that at A and B thoſe ſame rays paſs into the water, and are there refracted towards a perpendicular into the lines AD and BG. The refracted rays diverge leſs than the incident ones did; and the eye of a ſpectator will receive them, after this refraction, juſt as if they had come ſtrait from *r*; ſo that the leaf, from whence they came, will appear at *r* inſtead of R, or will appear farther from the ſurface of the water than it really is. And the diſtance of *r* the imaginary radiant will be to the diſtance of R the real radiant, as the ſine of incidence to the ſine of the refracted angle, or as 4 to 3; for this is the proportion, which the ſine of incidence bears to the ſine of the refracted angle. That is, theſe rays will fall upon the eye as if they had come from the top of a tree one part in four higher than it really is.

In order to determine the diſtance of the real and imaginary radiant, we muſt firſt obſerve that the divergency of the incident rays from the perpendicular RH is equal to the angle of incidence. The divergency of the ray RA from the perpendicular RH is the angle ARH; the point of incidence is A, and IK is a perpendicular at the point A; therefore RAI is the angle of incidence. But IK and RH are parallel, becauſe they are both of them perpendicular to the plane ſurface S*s*. Euc. b. XI. prop. 8. Therefore the alternate angles RAI and ARH are equal. Euc. b. I. prop. 29. Secondly the divergency of the refracted rays from the perpendicular *r*H is equal to the refracted angle. The divergency of the refracted ray AD from the perpendicular *r*H is D*r*H, and the refracted angle is DAK. But ſince IK and *r*H are parallel, the external angle DAK is equal to the internal and oppoſite one D*r*H. Euc. b. I, prop. 29. Now the diſtance of the radiant is inverſely as the divergency of the rays, by prop. 28. And the divergency before refraction is to the divergency after it, as the angle of incidence to the refracted angle. Therefore the diſtance of R the real radiant before refraction, is to the diſtance of *r* the imaginary radiant after refraction, as theſe angles inverted, that is, as the refracted angle to the angle of incidence. Or conſidering the angles in the ſame proportion with their ſines, by prop. 26, the diſtance of R from the ſurface, is to the diſtance of *r* from it, as the ſine of the refracted angle to the ſine of incidence.

This demonſtration is general and may be applyed to the caſe of diverging rays, that paſs through a plane ſurface out of water into air, or out of any denſer medium into any rarer. If F was any ſmall object, ſuch as a ſhilling, laid upon the bottom of a veſſel, and the veſſel was to be filled with water: the rays FM and FN, which are reflected from any point in the ſhilling and make it viſible, muſt paſs through the ſurface of the water S*s* into the air, before they come to the eye; and in this paſſage they will be refracted from a perpendicular and be made to diverge more, ſo as to proceed, when they are in the air in the lines MT and NW; by prop. 44. Now the more any rays diverge the nearer is the radiant from whence they come or appear to come, by prop. 28. Therefore theſe rays MT and NW will appear to come from ſome radiant *f* nearer to the ſurface of the water than F is; that is, the ſhilling will appear nearer the ſurface S*s* than it really is; and that in the proportion of the ſine of incidence to the ſine of the refracted angle. Or, ſince the ſine of incidence is to the ſine of the refracted angle out of water into air as 3 to 4, the point *f* will be nearer to the ſurface of the water than F is, in the proportion of 3 to 4; or the ſhilling will appear one part in four nearer to the ſurface than it really is.

For here, as before, LFN, the divergency of an incident ray from the perpendicular, is equal to FNO, the angle of incidence of the ſame ray. And L*f*W, the divergency of this ray from the perpendicular, when it is refracted, is equal to PNW the refracted angle. But, ſince the divergency before refraction is to the divergency after refraction, as the angle of incidence to the refracted angle, or, by prop. 26, as the ſine of incidence to the ſine of the refracted angle, the diſtance of the real radiant F, before refraction, is to the diſtance of the imaginary radiant *f*, after refraction, as theſe ſines inverted, or as the ſine of the refracted angle to the ſine of incidence.

46. *When converging rays are refracted at a plane ſurface, the diſtance of the imaginary focus from that ſurface is to the diſtance of the real focus, as the ſine of the refracted angle to the ſine of incidence.*

If DA and GB are rays tending to *r*, which, before they arrive there, are to paſs out of glaſs into air, or out of any denſer medium into any rarer; then *r* is only an imaginary focus: for though the rays tend thither before refraction, yet they will never meet there: becauſe they will be refracted in their paſſage, and will be made to converge more, by prop. 44. And this greater convergency will bring them to a focus ſooner, or will make them meet at ſome point, as R, which is the real focus, and is nearer to S*s* the ſurface of the glaſs than *r* is, by prop. 29. Now *r*, the imaginary focus, is as much farther from the ſurface than R the real focus, as the ſine of the refracted angle is greater than the ſine of incidence. Or, ſince the ſine of the refracted angle is to the ſine of incidence, out of glaſs into air, as 3 to 2, the diſtance of *r* from the ſurface, is to the diſtance of R, as 3 to 2. Or the real focus is one part in three nearer than the imaginary one. I mention this inſtance, rather than any other, becauſe we ſhall have occaſion hereafter to apply the propoſition particularly to this inſtance; for which reaſon the reader will do well to remember it.

The convergency of any incident ray DA to the perpendicular H*r*, is equal to the angle of incidence of the ſame ray: for DAK is equal to D*r*H, Euc. b. I. prop. 29. And the convergency of this ray to the perpendicular, when it is refracted into AR, is equal to the refracted angle: for ARH is equal to IAR, Euc. b. I. prop. 29. But the diſtance of the focus is inverſely as the convergency of the rays, by prop. 29, that is, the diſtance of the imaginary focus *r* before refraction, is to the diſtance of the real focus R after refraction, inverſely as the angle of incidence to the refracted angle, or as theſe angles inverted, or as the refracted angle to the angle of incidence, or, by prop. 26, as the ſine of the refracted angle to the ſine of incidence.

The ſame demonſtration is likewiſe applicable, when converging rays paſs out of air into glaſs, or out of any rarer medium into any denſer: the only difference is, that when they are to paſs, as in the former caſe, out of a denſer medium into a rarer, they are made to converge more, and the refracted angle is greater than the angle of incidence, and conſequently the focus *r* of the incident rays, or imaginary focus, will be farther from the ſurface S*s* than the focus of the refracted ones. But in this caſe, when WN, and TM converge to *f*, and before they meet paſs out of air into glaſs, they will be made to converge leſs, and the refracted angle will be leſs than the angle of incidence, and conſequently *f*, the focus of the incident rays, or imaginary focus, will be nearer to the ſurface S*s* than F, the focus of the refracted rays, or real focus.

For in this caſe, as in the former, W*f*L, the convergency of the incident ray WN to the perpendicular L*f*, is equal to WNP, the angle of incidence. And NFL, the convergency of the ſame ray to the perpendicular FL, when it is refracted, is equal to FNO, the refracted angle. But the diſtance of the focus is inverſely as the convergency of the rays, by prop. 29. Therefore the diſtance of the focus before refraction is to the diſtance of it afterwards, inverſely as the angle of incidence to the refracted angle; or, as theſe angles inverted, or as the refracted angle to the angle of incidence, or, by prop. 26, as the ſine of the refracted angle to the ſine of incidence.

47. *Rays of light after they have paſſed through a plane glaſs and have been twice refracted in their paſſage, keep the ſame direction in reſpect of one another, that they had at their incidence.*

When rays are to paſs through a plane glaſs *aklb*, Plat. VIII. fig. 9. they firſt paſs out of air into glaſs through the plane ſurface *ak* and are there refracted towards a perpendicular, by prop. 37; and after this they are to paſs again out of glaſs into air through the plane ſurface *lb*, where they will be as much refracted from a perpendicular as they were refracted towards it before, by prop. 38, 40. Therefore the ſecond refraction will always undo what the other has done and will leave the direction of the rays, in reſpect of one another, the ſame after they have paſſed through the glaſs that it was at their incidence upon the ſurface *ak*. If they were parallel at their incidence, they will be parallel after the two refractions; or if they diverged or converged before, they will diverge or converge juſt as much afterwards.

One inſtance in a caſe of diverging rays may clear up this whole matter. Let *c* be a radiant and *cd*, *ce* diverging rays. When they enter the ſurface *ak* they will be made to diverge leſs, by prop. 43, and will proceed within the glaſs in the lines *dg*, *en*. But when theſe rays *dg* and *en* come to the ſecond ſurface of the glaſs *lb*, they paſs out of glaſs into air, and will be made to diverge more, by prop. 44. And ſince this ſecond refraction is juſt equal to the firſt, by prop. 40, and is made the contrary way, it will reſtore the firſt divergency of rays, or *gi* and *no*, when they come out of the glaſs, will diverge juſt as much as *cd* and *ce* did, when they entered it.

CHAP. IV.

Of the refraction of parallel rays, when they paſs through lenſes, which are either convex or concave.

48. *If a beam of parallel rays, which are to paſs out of a rarer medium into a denſer through a convex ſurface, falls directly upon that ſurface, the rays after refraction will converge to their axis.*

LET ABKP, Plat. VIII. fig. 10. be any medium denſer than the medium that ſurrounds it, and let ANB the convex ſurface of this denſer medium be the ſegment of a ſphere, whoſe center is C. Let XN be the axis of a beam of parallel rays, and DE be one of the collateral rays. Suppoſe, for inſtance, that ABKP was a maſs of glaſs, with a convex ſpherical ſurface ANB, and that parallel rays of light DE and XN were paſſing out of the air into this glaſs. The beam is ſuppoſed to fall directly upon the glaſs; therefore the axis or middle ray XN is perpendicu-

lar

PLATE VII.

Fig. 1.

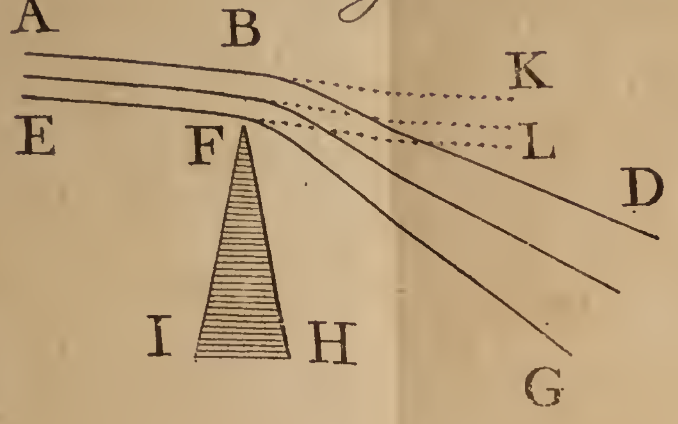


Fig. 2.

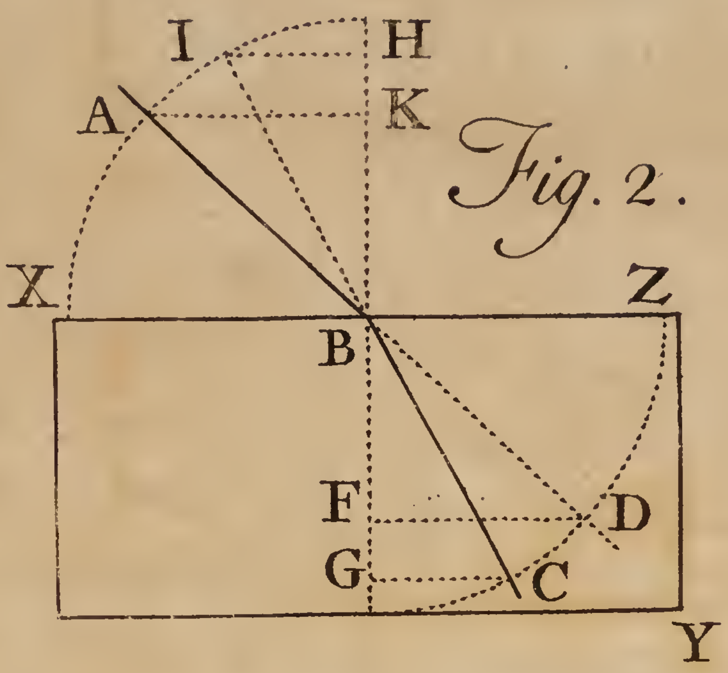


Fig. 3.

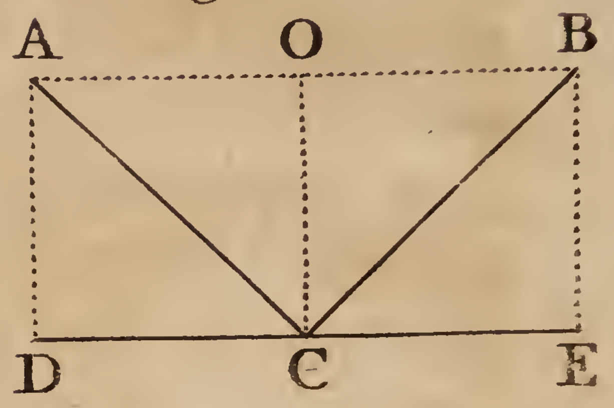


Fig. 4.

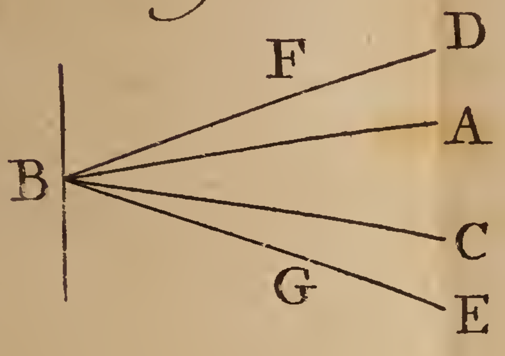


Fig. 5.

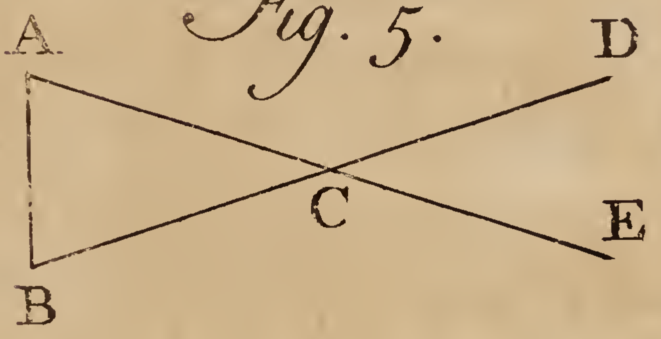


Fig. 6.

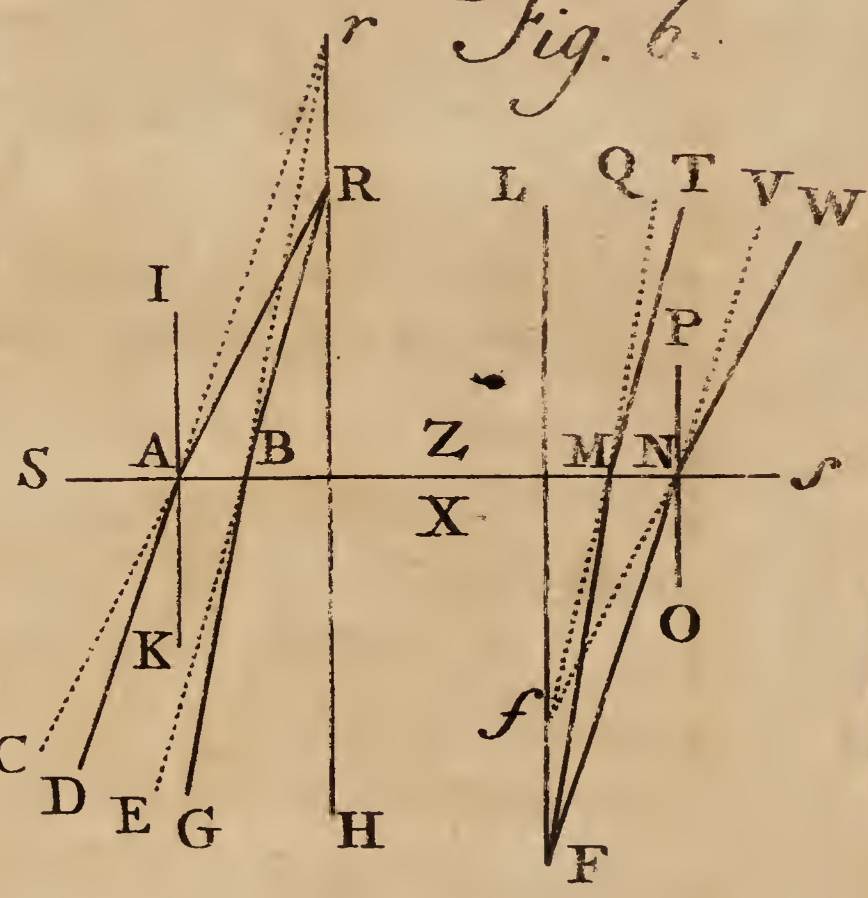


Fig. 7.

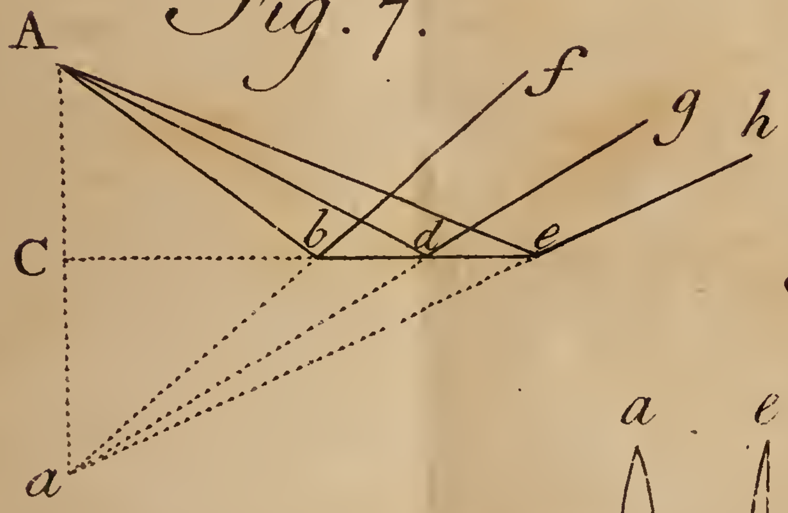


Fig. 8.

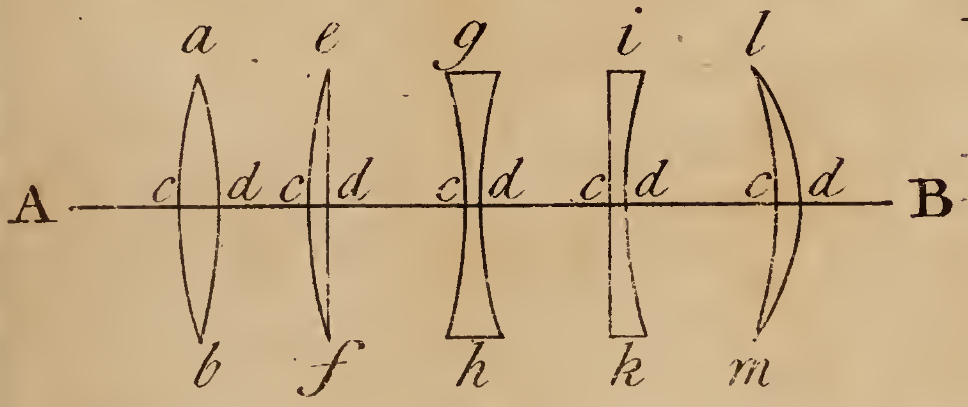


Fig. 9.

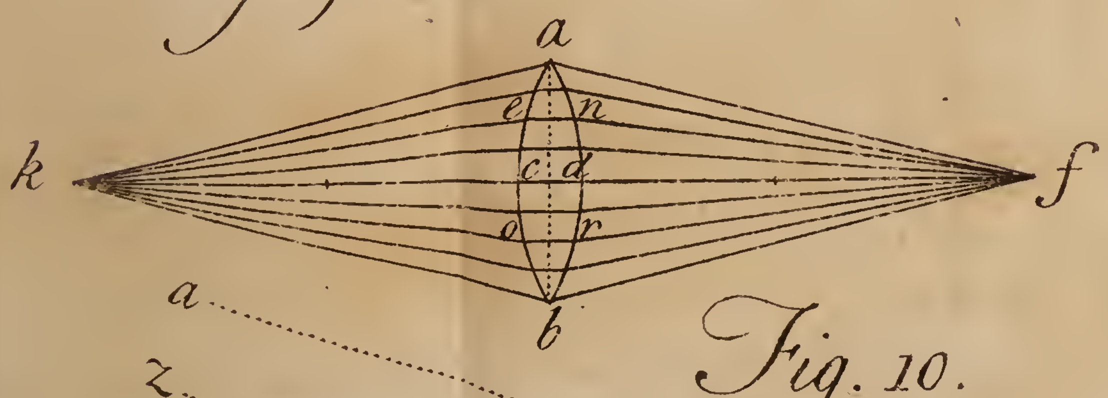


Fig. 10.

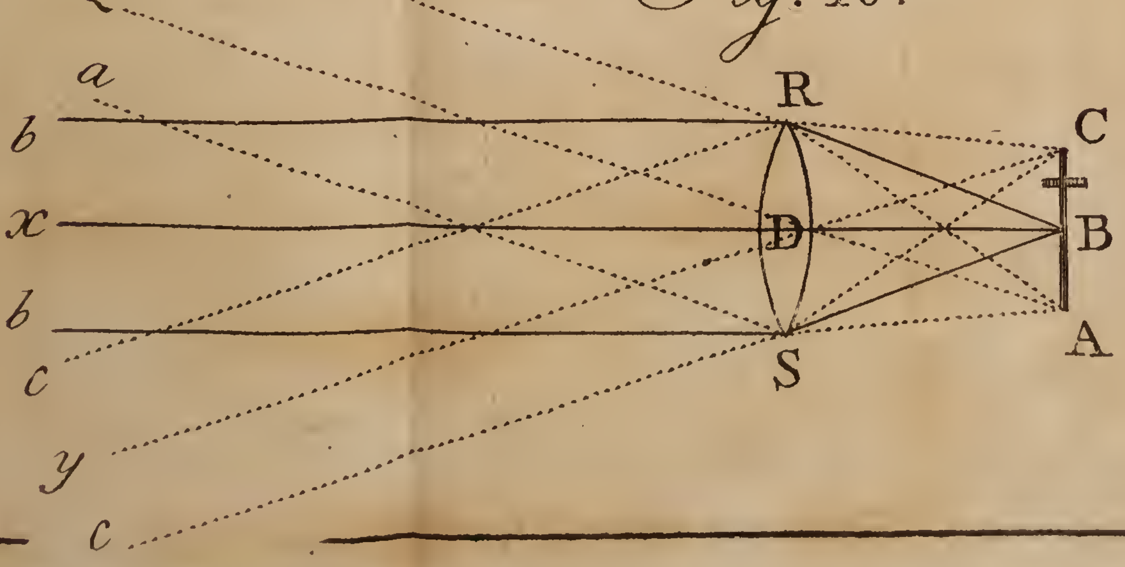
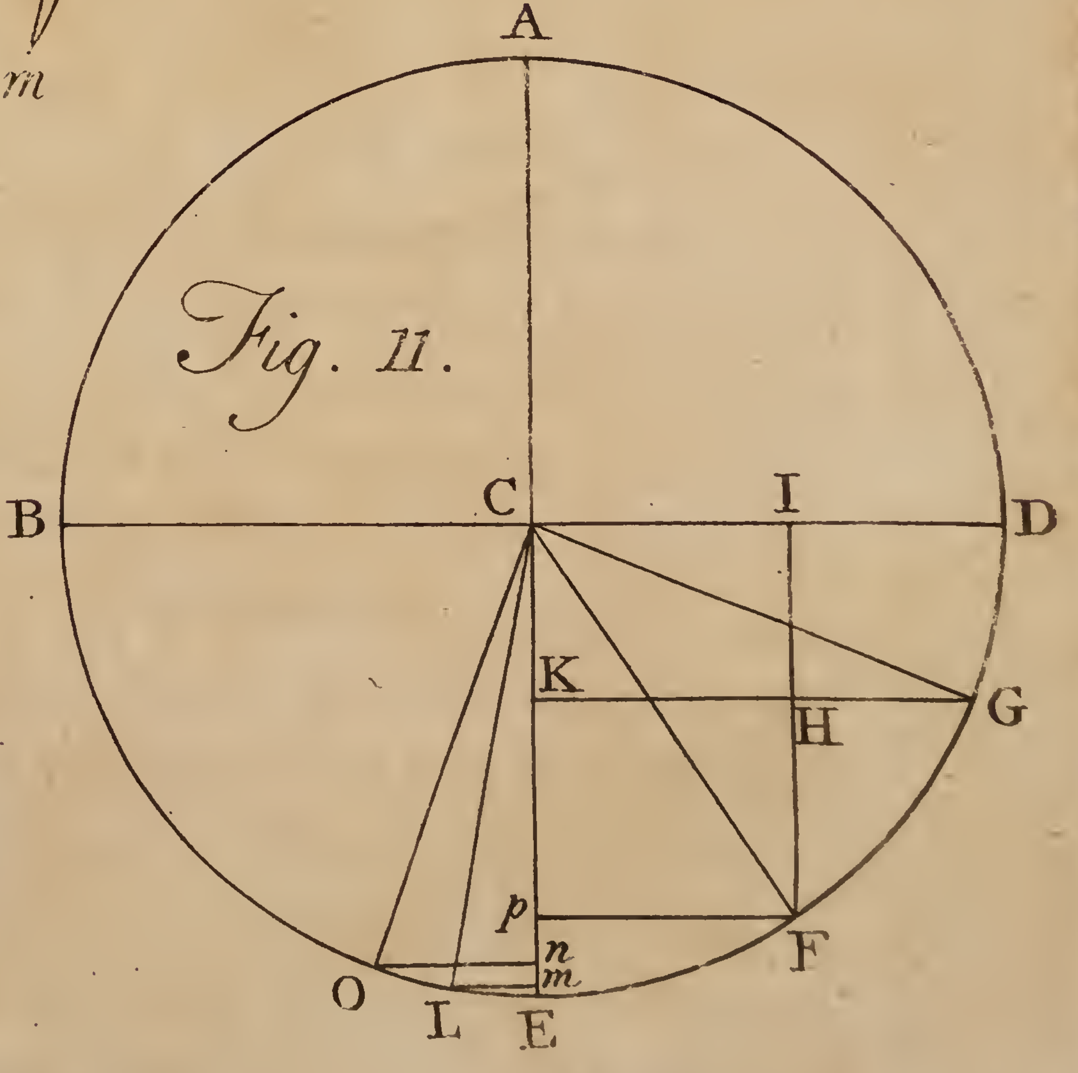
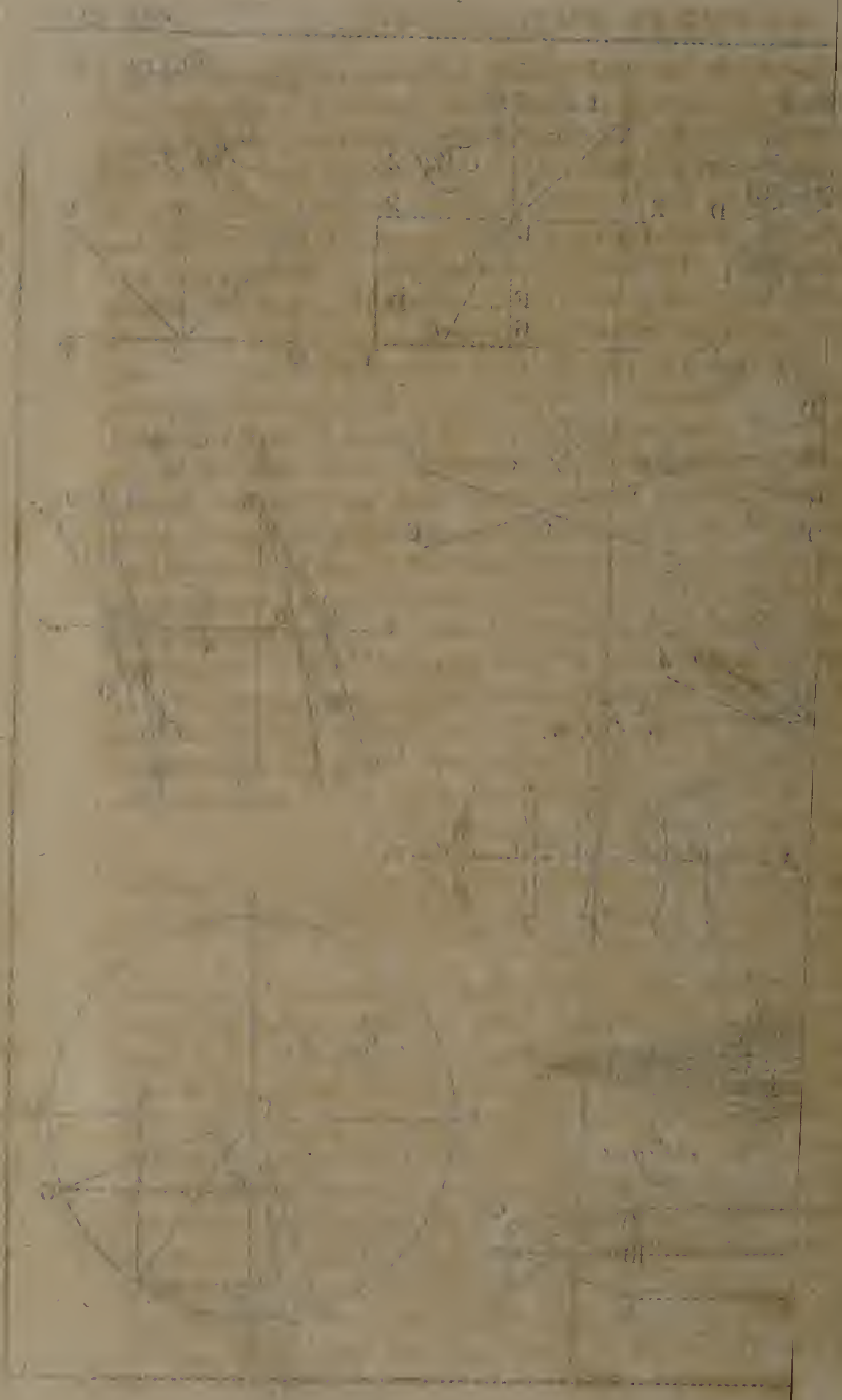


Fig. 11.





lar to the ſurface ANB, and conſequently will enter the glaſs at N without being refracted, and will proceed ſtrait forwards in the direction NCGQ; by propoſition 41. But the ray DE or any other collateral ray, which falls either above the axis, as this does, on the part AN of the ſurface, or below it on the part NB, is oblique to the ſurface. For if through the point E, where the ray DE falls, a line IEC is drawn to C the center of the convexity, this line is perpendicular to the ſurface; and conſequently the ray DE, which does not coincide with IE but makes with it the angle DEI, is oblique to the ſurface: and the angle DEI is the angle of incidence. This ray is to paſs out of a rarer medium into a denſer, and, will be refracted towards a perpendicular, by propoſition 37. But within the glaſs, all lines perpendicular to its ſurface ſuch as EC, MC, NC, converge to C the center of the convexity, which center is in the axis NG. And all the rays, which were parallel, and therefore neither converged nor diverged at their incidence, are refracted towards theſe converging perpendiculars, except the axis XN which paſſes ſtrait forwards. From whence it follows that theſe refracted rays being nearer to the perpendiculars than the incident ones were, muſt converge to their axis. The ray DE in particular, when it gets into the glaſs, does not go on in the line EF, in which it was moving before, but in ſome line EG, which is nearer to the perpendicular EC. But the axis XNGQ paſſes on without being refracted, and all the other rays which fall either above or below it, or on any part of the ſurface except N are made to converge to ſome point G, which is in their axis NG.

49. *If a beam of parallel rays, which are to paſs out of a rarer medium into a denſer through a concave ſurface, falls directly upon that ſurface, the rays after refraction will diverge from their axis.*

If *anb* the ſurface of a denſer medium *kanb* is the concave ſegment of a ſphere, whoſe center is *c*, and parallel rays ſuch as *gn* and *de* are to paſs into this medium, out of a rarer that encompaſſes it; then, if *gn* is the axis of the beam and *de* one of the collateral rays; the ray *gn* will be perpendicular to the ſurface *anb*, becauſe the beam falls directly upon this ſurface, by the ſuppoſition; but *de* or any other collateral ray will be oblique. In particular *de* will be oblique: for at the point *e* the line *cei* drawn from the center is perpendicular to the ſurface, but *de* does not coincide with *ce* but makes with it the angle *dec*. Therefore this ray is oblique and *dec* is the angle of incidence. Suppoſe the medium *kanb* to be glaſs and the medium that encompaſſes it to be air. Then as the rays paſs out of air into glaſs or out of a rarer medium into a denſer, *de* and all the other rays of the beam will be refracted towards perpendiculars, by propoſition 38, except the axis *gn*; for the axis, becauſe it is perpendicular to the ſurface at its incidence, will paſs on ſtrait in the line *nx* without ſuffering any refraction, by propoſition 41. But all the perpendiculars ſuch as *cx*, *cm*, and *ci* diverge within the glaſs from *c* the center, which lies in the axis *gnx*. Therefore the rays, which were parallel to one another at their incidence, when they are refracted towards theſe diverging perpendiculars, muſt likewiſe diverge from ſome point in this axis. The ray *de* in particular, when it gets into the glaſs, does not go ſtrait forwards in the line *ef* parallel to its axis *nx*, but is refracted towards the perpendicular *ei* and deſcribes the line *eh* which diverges from the point *g*, in the axis *gx*. That is the rays *gn* and *de* which were parallel to one another before refraction, will proceed after it in the diverging lines *nx* and *eh*, in the ſame manner that they would have done, if they had come at firſt from a radiant at *g*. What has here been ſhewn of this ray *de* is equally true of all the collateral rays, that fall on any part of the ſurface *anb* either above the axis or below it, or on any ſide of it. The axis alone will proceed ſtrait forwards in the glaſs, and all the other rays will be refracted ſo as to diverge from ſome point *g* in that axis.

This propoſition may be deduced from the foregoing one. And if the reader will attend to the manner of deducing it from thence, he will ſave himſelf much trouble in demonſtrating all the propoſitions which follow relating to concave and plano-concave lenſes.

In propoſition 75 of mechanics, we obſerved that a quantity might change from affirmative to negative by paſſing through nothing; that is, if a quantity has been decreaſing, till it becomes nothing, and then continues ſtill to decreaſe, it will become leſs than nothing or will be negative. We obſerved farther that quantities are called negative only in reference to ſome other quantities to which they are oppoſite, but are in themſelves or without this reference as much poſitive quantities as thoſe which are called affirmative. Both theſe particulars we may now ſhew to be true in ſome other inſtances, which are more immediately applicable to our preſent purpoſe: and one of theſe inſtances will lead us to prove, what may perhaps ſeem a paradox, that a quantity may change from affirmative to negative by paſſing through infinity; that is, if a quantity has been encreaſing till it becomes infinite, and then continues ſtill to encreaſe and ſo becomes greater than infinite, it will then be negative.

With the radius or ſemidiameter *cm*, Plat. IX. fig. 1. ſtrike *emf* the arc of a circle; this arc is convex on the ſide which is towards *l*. Open the compaſſes to the diſtance *dm*, and with the radius *dm* ſtrike *gmh* another arc of a circle, which is likewiſe convex towards *l*; but then it is leſs convex than *emf*, for it approaches nearer to the right line *amb*. From hence it ſeems evident that the longer the radius is, ſo much leſs convex any given arc of the circle will be. The converſe likewiſe of this ſeems evident, that the leſs convex the arc is ſo much longer muſt be the radius: for the more convex arc *emf* has *mc* for the radius; and the leſs convex arc *gmh* has a longer radius as *md*. From hence then we may reaſon in the following manner: as *emf* ſtraitens or grows leſs convex towards *l*, it conſtantly approaches towards the right line *amb*. Thus, if *emf* was a wire bent into the arc of a circle, if the point *m* of the wire was fixed, and you lay hold of the two ends *e* and *f*, by pulling theſe ends towards *l* you would ſtraiten the wire or make it leſs convex towards *l*: for when you had drawn it into the poſition *gmh* it would be ſtraiter or leſs convex than it was before. As this line or wire is ſtill farther unbent, or as the ends *g* and *h* move ſtill farther towards *l* the convexity will keep decreaſing: and when it was quite unbent or was come into the poſition *amb*, that is, when the two ends of it had moved one to *a* and the other to *b*, the line or wire would be quite ſtrait, and therefore its convexity would be nothing. Suppoſe it to keep moving on in the ſame manner as before, that is, ſuppoſe the two ends of it *a* and *b* to move farther towards *l*: this motion had reduced the convexity to nothing before; therefore, if it continues, it will make the convexity leſs than nothing or negative.

But bending the line *amb* by drawing the ends of it *a* and *b* towards *l* ſo as to bring one of them to *i* and the other to *k* will make it concave on the ſide, which is towards *l*. Therefore the line *imk*, which is concave towards *l* is the negative of the line *emf* or of any other line that is convex the ſame way. Now the bending of *imk* conſidered by itſelf without any reſpect to *emf* is as much poſitive as the bending of *emf* or any other line can be. But if we compare them with one another; if as *emf* is convex towards *l*, we aſk what is the convexity of *imk* the ſame way, we ſhall find that the convexity of *imk* towards *l* is leſs than nothing; and conſequently the bending of *imk* compared with that of *emf* is likewiſe leſs than nothing or is negative. Therefore upon the whole, if the bending of a line one way is conſidered as affirmative, the bending of a line the contrary way muſt be conſidered as negative.

But a quantity may change from affirmative to negative, not only by paſſing through nothing ſo as to become leſs than nothing, but likewiſe by paſſing through infinity ſo as to become greater than infinite. For let us obſerve a little what happens to the radius *mc*, whilſt the arc *emf* is unbending itſelf or growing leſs convex. This radius lengthens as the convexity towards *l* or bending of the arc decreaſes. For *mc* is the radius, when the arc is in the poſition *emf*: but when it is grown flatter or is farther unbent or has a leſs convexity and is in the poſition *gmb*, then the radius is *md*. And ſince in the ſame manner, whilſt the convexity of the arc keeps decreaſing, the radius will conſtantly encreaſe; or ſince the radius is inverſely as the convexity of the arc, it follows that when the convexity is nothing or when the arc is quite unbent and coincides with the right line *amb*, the radius will then be the reciprocal of nothing or will be infinitely long. And thus every right line *amb* may be conſidered as a finite or given part of a circle, whoſe radius and conſequently whoſe circumference is infinite. Now, ſince a motion of *e* and *f*, or of the two ends of the arc *emf*, towards *l*, makes the convexity of the arc towards *l* decreaſe and the radius *mc* encreaſe, till the convexity becomes nothing and the radius becomes infinite: the ſame motion of theſe ends towards *l*, after they are paſſed *a* and *b* will make the convexity leſs than nothing, and the radius greater than infinite. But when the convexity towards *l* becomes leſs than nothing, or is negative, the arc is in ſome ſuch poſition as *imk* or is concave towards *l* and the radius is *ml*. And if the radius, when it is on the ſide *c*, is affirmative, then the radius, when it is on the other ſide, muſt be negative. If *mc* is affirmative, *ml* being drawn the contrary way is negative. Therefore ſince the radius by becoming greater than infinite changes ſides, it changes from affirmative to negative by paſſing through infinity.

What has here been ſaid concerning the arc of a circle, the reader will eaſily apply to the ſegment of a ſphere, or to the convex and concave ſurfaces ANB and *anb*, Plat. VIII. fig. 10. 11. For ANB the ſurface of the glaſs is convex towards the air, and its radius NC is in the glaſs. But whilſt ANB flattens or whilſt its convexity decreaſes the radius NC lengthens. Now, when this motion of ANB towards C, by which its convexity is diminiſhed, has continued, till the ſurface is quite flat, then the convexity is nothing and NC is infinite. And for this reaſon a plane glaſs may be conſidered as the ſegment of a ſphere of an infinite radius. But if the ſame motion was ſuppoſed to continue, or if ANB after it is become plane was to ſink ſtill farther in towards C; the convexity towards the air would then become leſs than nothing or negative; that is, the ſurface towards the air would be changed into a concave one *anb*: and the radius would at the ſame time become greater than infinite or would be negative and change ſides: for the center C is in the glaſs when the ſurface of the glaſs is convex and the radius is NC: but when the ſurface is concave the center *c* is in the air and the radius *nc* is drawn the contrary way.

Now as *anb* is the negative of ANB, theſe two glaſſes will have juſt contrary effects: and ſince the convex one makes rays that were parallel at their incidence diverge after they are refracted, the concave one will make their convergency leſs than nothing or negative, that is, it will make them diverge. And ſince the focus, when the convergency is affirmative is at G, or in the glaſs; the focus, when the convergency is negative or when the refracted rays diverge will be at *g* on the contrary ſide or in the air. And if the former focus G, or the focus of converging rays is conſidered as affirmative, the latter focus *g*, that is the focus of diverging rays, or imaginary radiant from whence ſuch rays appear to flow, muſt be negative. This imaginary radiant *g* or negative focus is by ſome writers called a virtual focus.

50. *When parallel rays paſs out of a rarer medium into a denſer through a ſpherical ſurface, the diſtance of the focus whether affirmative or negative, that is, whether real or virtual, will be to the ſemidiameter of the ſphere of which that ſurface is a ſegment, as the ſine of refraction to the ſine of incidence.*

ANB, Plat. VIII. fig. 10. is a convex ſpherical ſurface, and the parallel rays DE and XN, by paſſing out of a rarer medium into a denſer through this ſurface, are made to converge to a focus G, by propoſition 48. This focus is affirmative. And NC the radius of the ſphere, of which the ſurface ANB is a ſegment, will bear the ſame proportion to NG the diſtance of the focus, that the ſine of refraction in this paſſage of the rays out of the rarer medium into the denſer bears to the ſine of incidence. This is one caſe of the propoſition. And when we have ſhewn the propoſition to be true where the ſpherical ſurface of the denſer medium is convex and the focus affirmative, we will then ſhew it to be true, where that ſurface is concave and the focus negative.

Now the ray XN, which is the axis of the beam, and falls perpendicularly upon the ſurface, is not refracted at all, by propoſition 41, but goes ſtrait on in NG the ſame direction continued after it gets into the denſer medium. But DE is oblique to the ſurface, and conſequently, inſtead of going ſtrait forwards in the direction EF, it will be refracted into EG ſo as to converge to the axis NG and to meet it at G, which is the focus. IC is perpendicular to the ſurface at the point E. Therefore DEI is the angle of incidence. But ſince DE and XG are parallel by the ſuppoſition, and IC croſſes them, the external angle DEI is equal to the internal oppoſite angle on the ſame ſide ECN, Euc. b. I. prop. 29. that is, ECN is equal to the angle of incidence. Now the angle ECG is the complement of ECN to two right ones, and upon this account they have both of them the ſame ſine, as was obſerved in explaning propoſition 32. But the ſine of ECN is the ſine of incidence: becauſe ECN is equal to the angle of incidence. Therefore the ſine of ECG is the ſine of incidence. IC and DF, which croſs one another at E make the vertical angles DEI and FEC equal. Euc. b. I. prop. 15. Therefore FEC is equal to the angle of incidence. The refracted ray is EG. Therefore GEC, being the angle which it makes with IC the perpendicular, is the refracted angle. And FEG is the difference between FEC and GEC, or between the angle of incidence and the refracted angle. Therefore FEG is the angle of refraction. Now DF and XG or the directions of the incident rays are parallel by what is ſuppoſed in the propoſition. And conſequently EG, which croſſes them, makes the alternate angles FEG and EGC equal to one another. Euc. b. I. prop. 29. But FEG is the angle of refraction. Therefore the ſine of EGC is the ſine of refraction: becauſe EGC is equal to the angle of refraction. From hence then it appears that, in the triangle EGC, the ſine of the angle at G is the ſine of refraction, and the ſine of the angle at C is the ſine of incidence. But the ſides of plane triangles are to one another as the ſines of the oppoſite angles. Therefore EC, oppoſite to G, is to EG, oppoſite to C, as the ſine of refraction to the ſine of incidence. But EC is equal to NC, for they are ſemidiameters of the ſame ſphere, of which the ſpherical ſurface ANB is a ſegment. And EG is very nearly equal to NG, when E and N are not far aſunder. Therefore NC, the ſemidiameter of the convexity is to NG the diſtance of the focus as EC to EG, that is, as the ſine of refraction to the ſine of incidence.

This demonſtration is univerſal, whatever the two mediums are. But we will apply it particularly to the caſe of rays paſſing out of air into glaſs. In this paſſage of the rays the reader may remember that the ſine of refraction is to the ſine of incidence as 1 to 3. Therefore NC is to NG in this caſe as 1 to 3. Or ſince NC is one ſemidiameter of the ſphere of which ANB is a ſegment; NG the diſtance of the focus, muſt be equal to three times NC and conſequently is 3 ſemidiameters. Therefore if parallel rays paſs out of air into glaſs, through a convex ſpherical ſurface, they are ſo refracted in their paſſage as to be made to converge to a focus, which will be at the diſtance of 3 ſemidiameters of the convexity behind the ſurface.

Perhaps this effect of ſuch a piece of glaſs upon parallel rays may be made out in another manner, and ſo as to give an eaſier general proof.

proof of the firſt caſe in the propoſition. If the ray DE was not refracted at all it would go on, when it got into the glaſs, in the line EF. IED is the angle of incidence, and FEC is equal to it GEC is the refracted angle and FEG is the angle of refraction. Now the reader may obſerve that FEG, or the angle of refraction, is contained between EF the direction of the incident ray, and EG the direction of the refracted one. Suppoſe then, that the angle of refraction was equal to the angle of incidence, what line would the refracted ray deſcribe? FEC is the angle of incidence, EF is the direction of the incident ray; therefore the refracted ray cannot make an angle with EF equal to FEC, or cannot make the angle of refraction equal to the angle of incidence, unleſs it deſcribes EC. Therefore if the angle of refraction was ſo great as to be equal to the angle of incidence, the refracted ray would be perpendicular to the convex ſurface, and the focus would be at C in the center of the convexity. But in fact the angle of refraction is leſs than the angle of incidence, that is, the rays are not ſo much refracted, and conſequently will not converge ſo much as is here ſuppoſed. Therefore the focus will not be ſo near as the center: for the leſs the convergency the more remote the focus, by prop. 29. Now if the angle of refraction was equal to the angle of incidence the focus would be at C or at the diſtance of a ſemidiameter from the ſurface. But in the paſſage of a ray out of air into glaſs, the refracted angle is to the angle of incidence as 1 to 3 or is only $\frac{1}{3}$ of the angle of incidence. Therefore the focus will be 3 times as remote from the ſurface as it would have been if theſe angles were equal, or will be at the diſtance of 3 ſemidiameters from the ſurface.

This is univerſal, whatever proportion the angle of refraction bears to the angle of incidence. For ſince, if they were equal, the focus would be at the diſtance of a ſemidiameter, and ſince the leſs the angle of refraction is, the leſs the refracted rays converge, it follows, by prop. 29, that the leſs the angle of refraction is in proportion to the angle of incidence, ſo much more remote the focus will be in proportion to a ſemidiameter; or as the angle of refraction to the angle of incidence, ſo is a ſemidiameter of the convexity to the diſtance of the focus.

The caſe of this propoſition, which has been already explaned, relates to a convex ſpherical ſurface of the denſer medium, when the refracted rays converge, and the focus is affirmative. The other caſe is, when *anb*, Plat. VIII. fig. 11. the ſpherical ſurface of the denſer medium is concave, and the rays *gn* and *de*, which are parallel at their incidence, are made by the refraction to proceed in the lines *nx* and *eh*, ſo as to diverge from an imaginary radiant, or negative focus *g*. To make what has been ſaid about affirmative and negative quantities more familiar to the reader, before we come to apply it in thoſe inſtances where it will be moſt uſeful to us; we will firſt deduce this caſe of the propoſition from the foregoing one; and then to ſhew the certainty of this ſort of reaſoning we will give a diſtinct demonſtration of it.

When ANB, Plat. VIII. fig. 10. is convex, as the ſine of refraction is to the ſine of incidence, ſo is NC, the ſemidiameter of the convexity to NG the diſtance of the affirmative focus G. Now, if we imagine the ſurface ANB to flatten, the radius NC will increaſe and NG, which is 3 times NC, if the denſer medium is glaſs and the rarer air, will increaſe in the ſame proportion: that is, the planer the ſurface ANB is ſo much more remote will the focus be. If the convexity of the ſurface ANB was to keep decreaſing in the ſame manner till it became nothing, or till the glaſs was quite plane; the radius would then be infinite, and the focus G would be at an infinite diſtance, and conſequently the convergency of the rays would be nothing: that is, if the ſurface of the glaſs was plane the rays, after refraction, would be parallel, as they were at their incidence, and if they are conſidered as converging ones, the focus to which they converge muſt be an infinitely diſtant one. But, if ANB ſinks ſtill farther in, its convexity will then be leſs than nothing, that is, will be negative, or will be changed into concavity, as *anb*, fig. 11. The convergency of the rays likewiſe will be leſs than nothing, or will be negative, that is, the refracted rays will diverge. The radius NC will be greater than infinite, that is, will be *nc*, or will change ſides, and conſequently will be negative. And the focus G will be at a diſtance greater than infinite, and therefore will change ſides and will be at *g*, or will be negative too. This then is the only difference between a convex and a concave ſurface, that the former is affirmative and the latter negative; and the only difference in their effect is, that the affirmative, or convex ſurface will make the refracted rays converge, and will produce an affirmative focus; and the negative, or concave one, will make them diverge and produce a negative focus. And conſequently, ſince as the ſine of refraction is to the ſine of incidence, ſo is NC, the ſemidiameter of the convexity, to NG the diſtance of the affirmative, or real, focus; it follows that as the ſine of refraction is to the ſine of incidence, ſo is *nc* the ſemidiameter of the concavity, or negative convexity, to *ng* the diſtance of the negative focus.

This caſe of the propoſition may be proved diſtinctly in the ſame manner that the other caſe was. If a beam of parallel rays, of which *gn* is the axis, and *de*, one of the collateral rays, is to paſs out of a rarer medium into a denſer, and falls directly upon the concave ſurface of the denſer medium *anb*; then the axis *gn*, being perpendicular to the ſurface, will paſs ſtrait forwards in the direction *gn* continued, or in *nx*, by prop. 41. But the collateral ray *de* falls obliquely upon the ſurface, and conſequently, inſtead of going ſtrait forwards in *ef*, will be refracted towards the perpendicular *ci*, and will deſcribe ſome ſuch line as *eh*, ſo that, after refraction, it will go on as if it had come ſtrait from *g*; or *g* will be the imaginary radiant, that is, the virtual or negative focus, from whence this and any other collateral ray will diverge after refraction. And as the ſine of refraction is to the ſine of incidence, ſo is *nc*, the ſemidiameter of the concavity, to *ng*, the diſtance of this negative focus. *dec* is the angle of incidence; *de* and *gn* are parallel, by the ſuppoſition; therefore the alternate angles *dec* and *ecn* are equal. But *ecg*, which is the complement of *ecn* to two right angles, has the ſame ſine with *ecn*; and that is the ſine of incidence; becauſe *ecn* has juſt been proved to be equal to the angle of incidence. *fei* is equal to *dec*, the angle of incidence; becauſe theſe two angles are vertical ones. Euc. b. I. prop. 15. *hei* is the refracted angle, or the angle, which the refracted ray makes with the perpendicular; and *feh*, which is the difference between *fei* and *hei*, is the angle of refraction. But *ef* or *de* continued to *f*, is parallel to *gn*. Therefore *feh*, the external angle, is equal to *cge*, the internal oppoſite angle on the ſame ſide; Euc. b. I. prop. 29. ſo that the ſine of *cge* is equal to the ſine of *feh*, or to the ſine of refraction. From what has been ſaid it appears, that in the triangle *cge*, the ſine of the angle at *g*, is equal to the ſine of refraction, and the ſine of the angle at *c*, is equal to the ſine of incidence. But the ſides of this triangle, are to each other, as the ſines of the oppoſite angles, by prop. 32. Therefore *ec* is to *eg*, as the ſine of refraction to the ſine of incidence: for the angle at *g* is oppoſite to the ſide *ec*, and the angle at *c* is oppoſite to the ſide *eg*. And, ſince *ce* and *nc* are equal, and *eg* and *ng* are very nearly equal, when *e* and *n* are near to one another; it follows that *nc* the ſemidiameter of the concavity, is to *ng* the diſtance of the negative focus from the ſurface of the denſer medium, as the ſine of refraction to the ſine of incidence.

When rays paſs out of air into glaſs, the ſine of refraction is to the ſine of incidence as 1 to 3. Therefore if the rays are parallel and the ſurface of the glaſs is concave, the ſemidiameter of the glaſs's concavity will be to the diſtance of the point, from whence the refracted rays will appear to diverge, as 1 to 3: that is, *nc* will be $\frac{1}{3}$ of *ng*, or ſince *nc* is 1 ſemidiameter, *ng* the diſtance of the virtual or negative focus will be equal to 3 ſemidiameters of the glaſs's concavity.

The reader will eaſily underſtand the manner of applying the other demonſtration, of the firſt caſe of this propoſition to the caſe now before us, without having it particularly pointed out to him.

51. *Parallel rays that fall directly upon a plano-convex lens, are made to converge by paſſing through it; and the diſtance of the focus, to which they converge, when they are refracted, is equal to a diameter of the lens's convexity.*

This propoſition conſiſts of two different caſes; for the rays may enter the glaſs two ways, either at its plane ſurface or at its ſpherical one.

The firſt caſe is when the rays enter at the plane ſide. If GL, Plat. IX. fig. 2. is a plano-convex lens, and AR, DS, are two parallel rays; DS, which is the axis of the beam, is perpendicular both to the ſurface GL and likewiſe to the ſurface GPL, becauſe the beam is ſuppoſed to fall directly upon the lens. Therefore DS will go ſtrait through the lens in the direction SPF without being at all refracted, by prop. 41. The collateral ray AR is parallel to DS, by the ſuppoſition, and DS is perpendicular to the plane ſurface GL. Therefore AR is likewiſe perpendicular to the ſame ſurface GL. Euc. b. XI. prop. 8. So that AR ſuffers no refraction when it enters the lens at R, but paſſes ſtrait forwards in the direction RB, by prop. 41. But, when it is to paſs out of the lens at B, it is oblique to the ſurface GPL: for ſince C is the center of the convexity, or of the ſphere, of which the ſurface GPL is a ſegment, CO is perpendicular to the ſurface, and conſequently RBE, which is the direction of the ray, is oblique to it. The ray therefore will be refracted from a perpendicular, by prop. 38; for it is paſſing obliquely out of glaſs into air, or out of a denſer medium into a rarer. Now on this ſide of the lens all the perpendiculars, becauſe they are drawn from C the center of the convexity, which is on the other ſide, ſuch as CO and CF, diverge. Conſequently if a ray, that before refraction, was parallel to the axis DF, or that neither diverged from the axis, nor converged towards it, is refracted from theſe perpendiculars, which all of them diverge, the refracted ray, becauſe it is farther from the direction of the perpendiculars than it was at its incidence when it was parallel to its axis, will be made to converge to-

PLATE VIII.

Fig. 1.

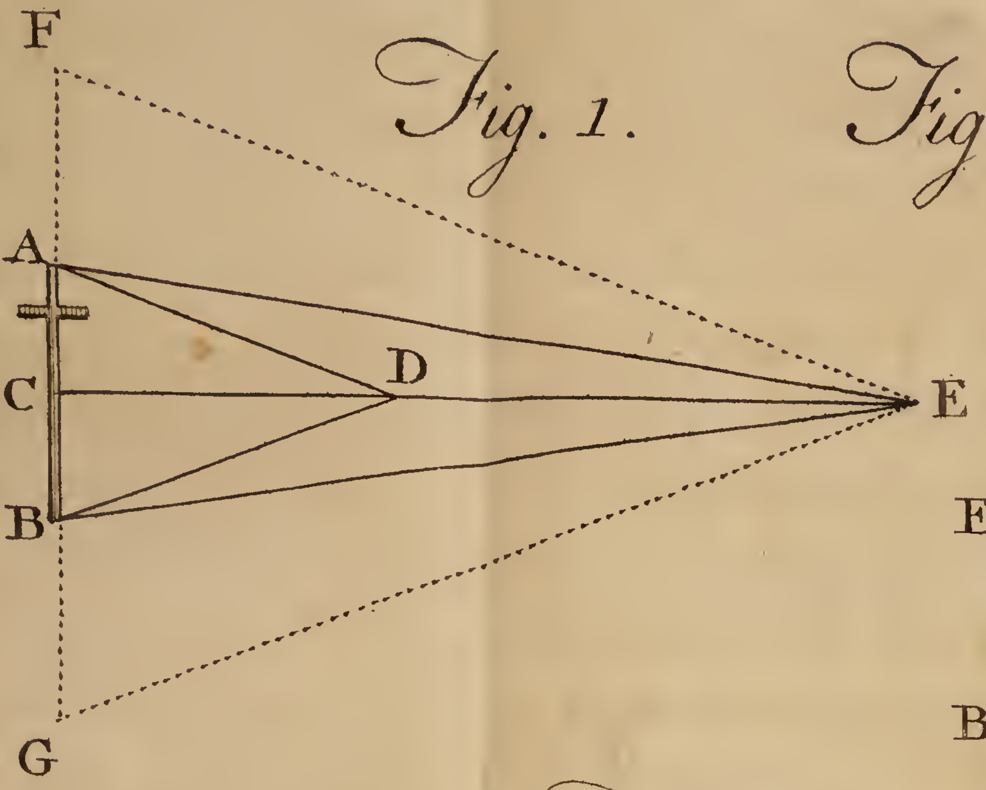


Fig. 2.

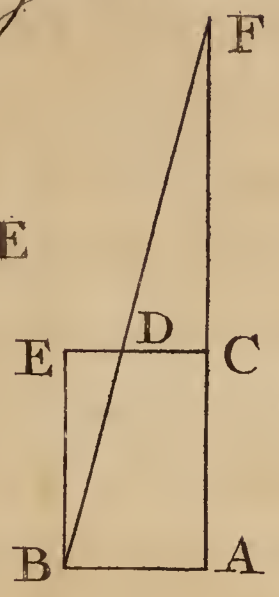


Fig. 3.

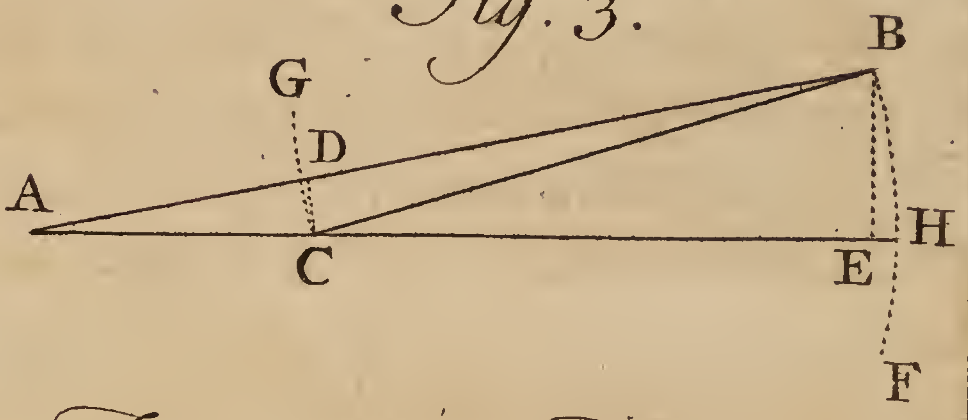


Fig. 4.

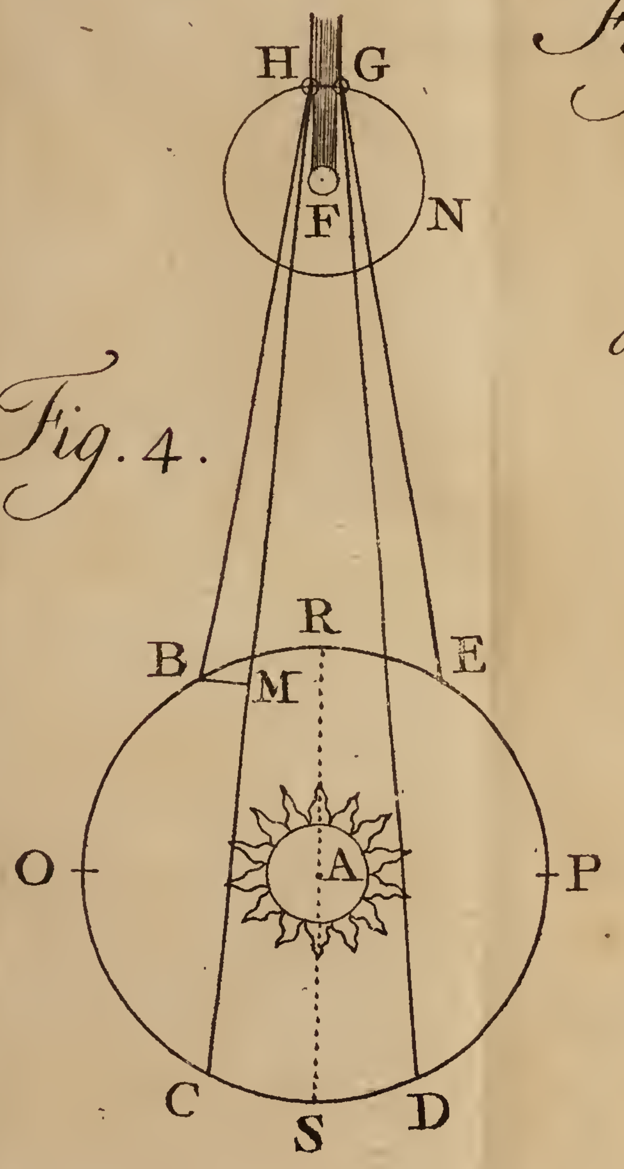


Fig. 5.

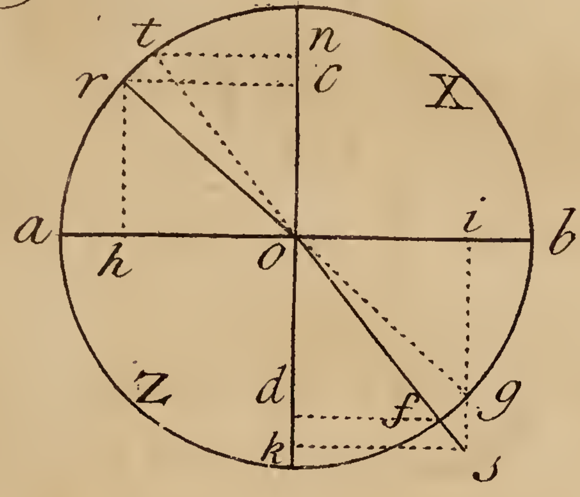


Fig. 6.

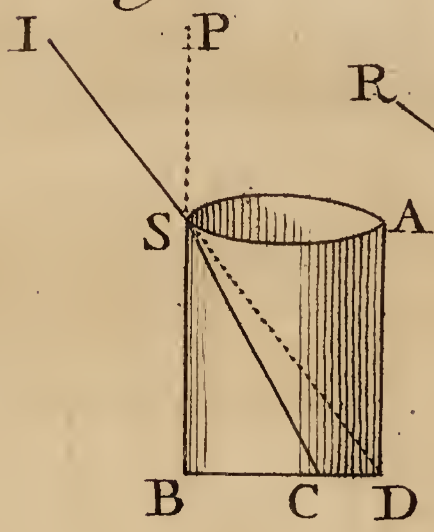


Fig. 7.

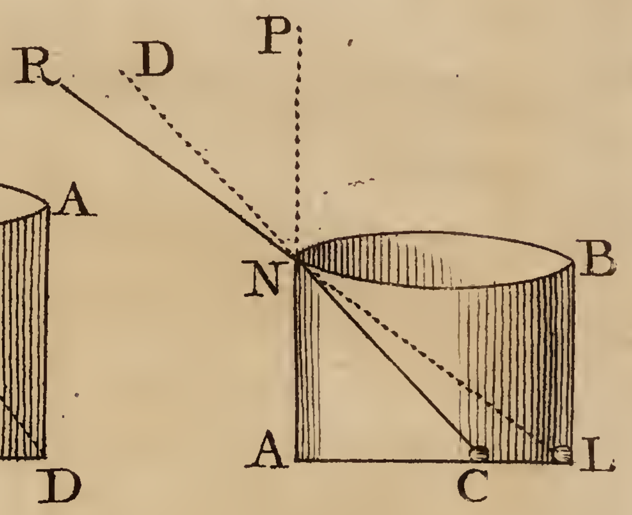


Fig. 8.

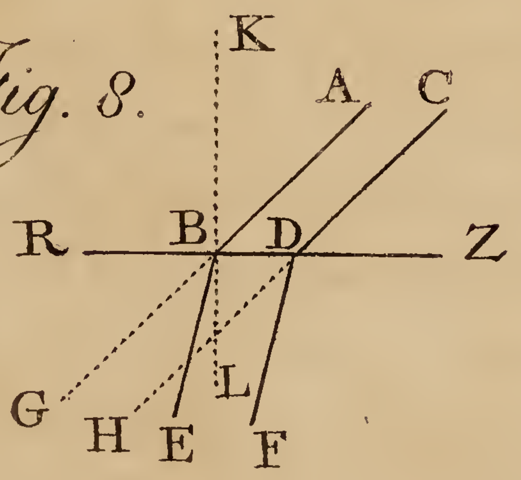
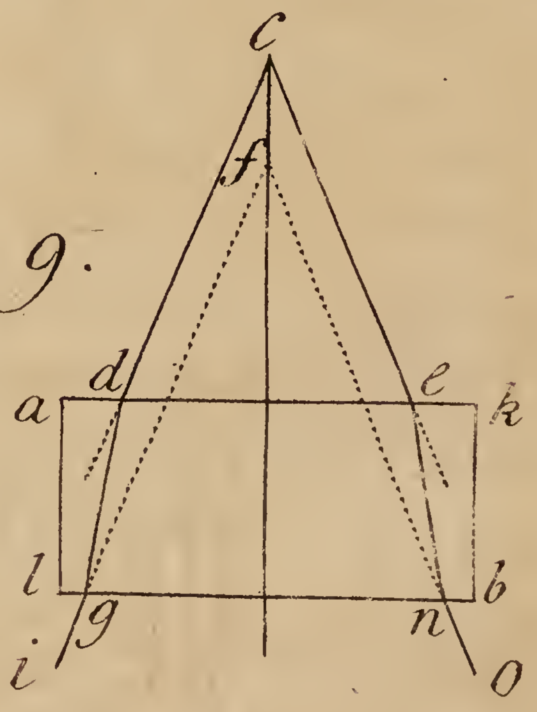


Fig. 9.



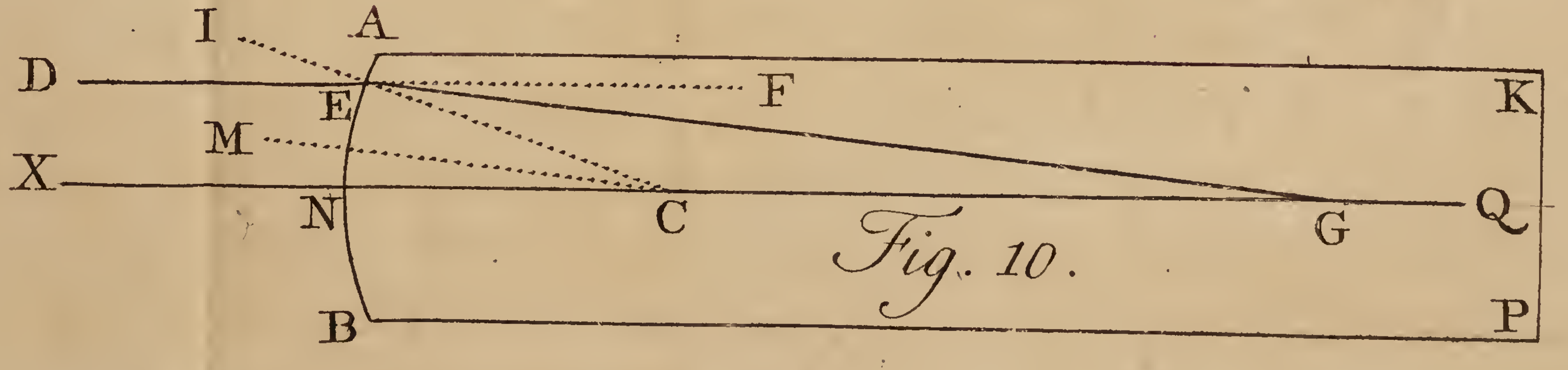
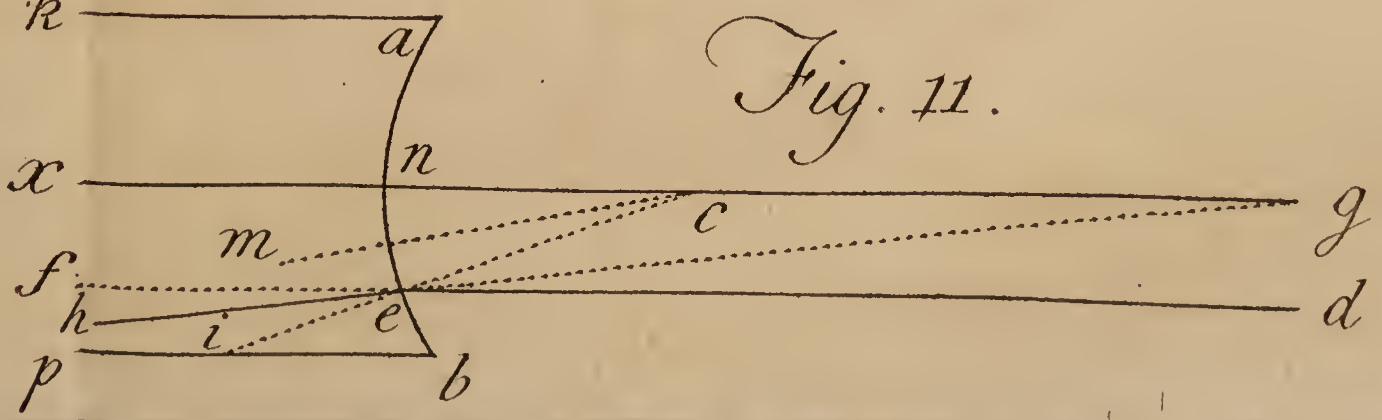


Fig. 10.

Fig. 11.



towards its axis. Thus the ray RB, which if it had gone ſtrait forwards would have deſcribed BE and have made the angle EBO with the perpendicular BO, is refracted from this perpendicular, ſo as to make a greater angle OBF with it. Now if the ray had not been refracted at all, it would have continued parallel to its axis PF, as it was at its incidence; or if it had been refracted towards the perpendicular BO, which diverges from the axis, then the ray itſelf would have diverged from thence. But it is refracted from the perpendicular, and conſequently acquires the contrary direction, or is made to converge towards the axis in the line BF.

PF, or the diſtance of F from the lens, that is, the diſtance of the focus, or point where this or any other collateral ray, when it is made to converge, will meet its axis is equal to twice CP, or to two ſemidiameters or one diameter of the ſphere, of which the lens is a ſegment. The angle of incidence where the ray is refracted is EBO, and AE is parallel to DP, by the conſtruction, therefore OBE or the angle of incidence, which is the external one, is equal to BCF the internal oppoſite one on the ſame ſide. Euc. b. I. prop. 29. OBF is the refracted angle, or angle which the refracted ray BF makes with the perpendicular; EBF, which is the difference between the angle of incidence OBE and the refracted angle OBF, is the angle of refraction. And EBF is equal to the alternate angle BFC. Euc. b. I. prop. 29. From what has been ſaid, it appears, that in the triangle BCF the angle at F is equal to the angle of refraction, and the angle at C is equal to the angle of incidence. But in plane triangles, the ſides are as the ſines of the oppoſite angles, by prop. 32. Therefore BC is to BF as the ſine of the angle at F to the ſine of the angle at C, or as the ſine of refraction to the ſine of incidence. Now the ſine of refraction is to the ſine of incidence, when a ray paſſes out of glaſs into air, as 1 to 2. Therefore BC is to BF as 1 to 2, or BF is twice BC. BC and PC are equal, becauſe they are ſemidiameters of the ſame ſphere. And BF and PF are nearly equal, when B and P are near together, or when the collateral ray AB is near the axis CP. Therefore ſince BF is equal to twice BC, PF, the focal diſtance, is likewiſe equal to twice PC, or to two ſemidiameters or one diameter of the lens's convexity.

In the other caſe of this propoſition the rays enter the lens at the ſpherical ſurface. If EH, Plat. IX. fig. 3. and XP are two parallel rays, of which XP is the axis and EH a collateral ray. This collateral ray will be refracted as it paſſes into the lens ANBD, or out of air into glaſs through a convex ſurface, and will be made to converge to its axis

XS, by propoſition 48. The lens is here made very thick only that the reader may more diſtinctly ſee the direction of the rays, when they are within it; for in the demonſtration this thickneſs muſt be neglected, or we muſt conſider the lens as having little or no thickneſs at all. Now HM the collateral ray converges to its axis PS after this firſt refraction. And theſe converging rays are to paſs out of the glaſs into the air through the ſurface ND, that is, out of a denſer medium into a rarer through a plane ſurface. Therefore they will be made to converge more, by propoſition 44. The firſt refraction, which this ray EH ſuffers when it enters the glaſs, would make it croſs its axis or would produce a focus at the diſtance of 3 ſemidiameters of the glaſs's convexity; ſo that I would be the focus, if the ray was to go ſtrait forwards in the direction HMI, which it acquires by this refraction. But it will be refracted a ſecond time in paſſing out of the glaſs, and will be made to converge more, as has been already proved. So that I will be only an imaginary focus and, by propoſition 29, the real focus will be made nearer than I, or will be at a leſs diſtance than PI, or than 3 ſemidiameters of the convexity. The queſtion is, how much nearer? Now the ray HM and the axis PS converge within the glaſs and paſs out of the glaſs into the air through a plane ſurface. Therefore, by propoſition 46, the diſtance of the imaginary focus or PI will be to the diſtance of the real one as the ſine of the refracted angle to the ſine of incidence. But out of glaſs into air the ſine of the refracted angle is to the ſine of incidence as 3 to 2. And conſequently the diſtance of the real focus where the rays will meet after this ſecond refraction will be only $\frac{2}{3}$ of PI. And PI is 3 ſemidiameters of the lens's convexity. Therefore taking PF equal to $\frac{2}{3}$ of PI or to 2 ſemidiamters F will be the focus.

By this time the reader ſees the difference of the two caſes. If the rays fall firſt upon the plane ſide of the lens they are refracted only once and that is when they are going out of the lens at the convex ſide. But if they fall firſt upon the convex ſide they are refracted twice, once when they enter the lens, and again as they are going out of it at the plane ſide. And in both theſe caſes the focal diſtance is the ſame, for it has been proved in both of them to be equal to 2 ſemidiameters or 1 diameter of the ſphere of which the lens is a ſegment.

Perhaps it may appear ſtrange that the two refractions, which the rays ſuffer in the latter caſe, ſhould not make them converge more or bring them to a focus nearer than the ſingle refraction, which they ſuffer in the former caſe. But this is the truth of the matter, as may thus be made to appear. When the rays fall firſt upon the plane ſide and are refracted ſracted only at the convex one, as they paſs out of the lens; the angle of refraction is to the angle of incidence as 1 to 2, or whatever is the angle of incidence at the convex ſurface the angle of refraction will be half of it. Whereas, when the rays fall firſt upon the convex ſurface, whatever is the angle of incidence there, yet though the rays are twice refracted both the angles of refraction added together will be equal to no more than half of it. For the firſt refraction is made, when the ray is paſſing out of air into glaſs: conſequently the angle of refraction there is only $\frac{1}{3}$ of the angle of incidence. Thus RHE or its equal OHC is the angle of incidence at the convex ſurface; and LHM, which is the angle of refraction, is only $\frac{1}{3}$ of it. The ſecond refraction is made at the plane ſurface when the ray is paſſing out of glaſs into air; conſequently the angle of refraction is $\frac{1}{2}$ the angle of incidence at that ſurface. But what is the angle of incidence at the plane ſurface where the angle of refraction is half of it? HM is the incident ray, MT is the perpendicular, and the angle which MI or MH continued makes with TM, that is, the angle TMI is the angle of incidence. But ſince LO or EH continued and MT are both of them perpendicular to the plane ſurface ND, they are parallel to one another. Euc. b. XI. prop. 8. Therefore TMI the external angle is equal to OHM the internal oppoſite one on the ſame ſide. Euc. b. I. prop. 29. That is, TMI the angle of incidence at the plane ſurface is equal to OHM the angle of refraction at the firſt. Now the angle OHM or the firſt angle of refraction is $\frac{1}{3}$ of the angle of incidence at the convex ſurface. And the angle of refraction at the ſecond ſurface or IMF is $\frac{1}{2}$ TMI, that is $\frac{1}{2}$ OHM, or half $\frac{1}{3}$ of the angle of incidence at the convex ſurface, Therefore both the angles of refraction taken together are $\frac{1}{3}$ and half $\frac{1}{3}$ of the angle of incidence. But a third and half a third of the angle are juſt half the angle. So that both theſe refractions taken together amount to no more than the ſingle refraction, which the rays ſuffer, when they enter the lens at its plane ſurface and are refracted only at their paſſage out of it through the convex one.

52. *Parallel rays that fall directly upon a plano-concave lens are made to diverge by paſſing through it; and the diſtance of the virtual or negative focus, from which they appear to diverge, when they are refracted, is equal to a diameter of the lens's concavity.*

This propoſition as well as the laſt admits of two caſes. For the rays may enter the lens either at the plane ſide or at the concave one.

If *gl*, Plat. IX. fig. 4. is a plano-concave lens, and parallel rays ſuch as *ar* and *ds* fall directly upon the plane ſide of it ſo that *ds* the axis of the beam may be perpendicular to the ſurface; then *ar*, becauſe it is parallel to *ds* will likewiſe be perpendicular to the ſame plane ſurface. Euc. b. XI. prop. 8. And conſequently both theſe rays and all other rays of the ſame parallel beam will paſs into the glaſs without being refracted, by propoſition 41. But at the ſecond or concave ſurface of the lens only *ſc* the axis of the beam, will paſs out unrefracted, becauſe that only is perpendicular to the ſurface. Any collateral ray ſuch as *ar*, when it comes to the concave ſurface, cannot paſs out of it in the right line *be* parallel to its axis *ſc*; for *be* is oblique to the ſurface of the glaſs, and conſequently as the ray paſſes out of glaſs into air, it will be refracted from a perpendicular. *c* being the center of the concavity *obc* is a perpendicular at the point *b* where the ray paſſes out, as *pc* is another perpendicular at the point *p*. But theſe and all other perpendiculars converge to *c*, which is the center of the concavity and is in the axis *pc*. Therefore the ray, which inſtead of paſſing out parallel to its axis is refracted from a perpendicular, will be made to diverge from its axis. The incident rays are parallel, that is, they neither diverge nor converge. They are refracted from the perpendiculars, or when they are refracted they are farther from the direction of the perpendiculars than they were at their incidence. But the perpendiculars converge to the axis. Therefore the refracted rays will diverge from it. The ray *ar* in particular, which comes out of the glaſs at *b*, will not go ſtrait forwards in the line *be*, which makes the angle *ebc* with the perpendicular *bc*, but in ſome line as *bm* that makes the angle *mbc* greater than *ebc* with this perpendicular. Thus the two rays *ar* and *ds*, which were parallel, when they entered the glaſs, will come out of it in the diverging lines *bm* and *pc*; and conſequently have the ſame direction in reſpect of one another as they would have had, if they had come ſtrait from the radiant *f*. This point *f* from whence the refracted rays appear to diverge is called either the imaginary radiant, or the virtual focus, or the negative focus. And *pf* the diſtance of this negative focus from the ſurface of the glaſs is equal to a diameter of the ſphere of which the concave ſurface *gpl* is a ſegment.

That *pf* is equal to twice *pc* might be ſhewn in the ſame manner as in the firſt caſe of the laſt propoſition we ſhewed that PF, fig. 2, is equal to twice PC. But we may prove it likewiſe in another manner by ſhewing that the lens *gpl*, fig. 4. is the negative of GPL, fig. 2, and conſequently that the effect it produces will be equal, only it will produce it the contrary way. Now if GPL was to flatten, or if the convex ſurface was to become planer, the radius PC and conſequently the focal diſtance PF, which is twice the radius, would lengthen; that is, the leſs convex the glaſs is, ſo much more remote will be the focus. Therefore the leſs convex the glaſs is, ſo much leſs the refracted rays converge; for the convergency of the rays is inverſely as the diſtance of the focus, by propoſition 29. And when GPL is quite plane, or when the convexity is nothing, then the lens is only a plane glaſs, and the refracted rays will be parallel, or their convergency will be nothing, or the focal diſtance PF will be infinite, by propoſitions 47, 31. But, if the ſurface GPL moves ſtill farther in the ſame direction or ſinks in, after it is become plane, then its convexity becomes leſs than nothing or negative, and the focal diſtance is greater than infinite or is negative too; that is, it muſt be taken on the contrary ſide. This therefore is the only difference between the two lenſes GPL and *gpl*, the convexity of one is affirmative, and the convexity of the other is negative; the focus of GPL or the affirmative focus is at F on the ſide of the refracted rays, and the focus of *gpl* or the negative focus is at *f* on the ſame ſide with the incident rays; and as the rays are made to converge by paſſing through the lens GPL, the lens *gpl* will produce the contrary effect, or will make their convergency negative or will make them diverge. But this convergency and divergency, this affirmative and negative effect will be reſpectively proportional to the cauſes that produce them. And as the convergency cauſed by the plano-convex lens, or by the lens when it is affirmative, will bring the refracted rays to a focus at the diſtance PF, or at the diſtance of two ſemidiameters of the convexity; ſo the divergency cauſed by the plano-concave lens, or by the lens when it is negative, will make the refracted rays appear to diverge from a negative focus at the diſtance *pf*, or at the diſtance of two ſemidiameters of the concavity.

The ſecond caſe of parallel rays falling directly upon a plano-concave lens is, when they enter the lens at the concave ſurface as *xp*, Plat. IX. fig. 5. which is the axis of a parallel beam and *eh* which is one of the collateral rays. The axis paſſes into the lens on this ſide, and out of it again on the other, without being refracted, by propoſition 41: becauſe as the beam falls directly upon the lens the axis is perpendicular to both its ſurfaces. But *eh*, which is oblique to the ſurface, and all the other collateral rays, as they paſs out of air into glaſs through a concave ſurface, are made to diverge from their axis, by propoſition 49. Thus the ray *eh* will not go ſtrait forwards in the line *hl*, but will be bent into the line *hm* ſo as to diverge from its axis *ps*, as if it had come at firſt from the point *i*. And theſe diverging rays when they paſs out of the lens into the air will be made to diverge more, by propoſition 45. Thus the ray *hm* does not go ſtrait forwards in the line *mk*, but is bent ſtill farther and is made to deſcribe the line *mg*, which diverges from its axis as if it had come at firſt from the point *f*, which is therefore the imaginary radiant, virtual focus, or negative focus of theſe rays after they have paſſed through the lens.

The diſtance of the point *i* from the lens, which is the point from whence the ray *ps* and *hm* appear to diverge after the firſt refraction, is equal to 3 ſemidiameters of the lens's concavity, by propoſition 50. But the ſecond refraction makes the rays diverge more, and makes the imaginary radiant *f* nearer than *i*, in the proportion of the ſine of incidence to the ſine of the refracted angle, that is, in the proportion of 2 to 3 : therefore *pf* is $\frac{2}{3}$ of *pi*, or $\frac{2}{3}$ of 3 ſemidiameters, or is 2 ſemidiameters of the concavity.

The letters of reference are the ſame in fig. 5. and in fig. 3. only the former are ſmall ones and the latter capitals. And with this help the reader, if he thinks it worth his while, may eaſily ſhew that the ſecond caſe of this propoſition is the negative of the ſecond caſe in the foregoing one ; and may apply to the caſe now before us all that was ſaid to prove that the two refractions of a ray, when it falls firſt upon the convex ſide of a plano-convex lens, are only equal to the ſingle refraction, which it ſuffers, when it falls firſt upon the plane ſide.

53. *If parallel rays fall directly upon a lens, which is equally convex on both ſides, they will converge, after they have paſſed through it, to a focus at the diſtance of a ſemidiameter of either convexity.*

The plano-convex lens EFG, Plat. IX. fig. 6. would make parallel rays converge and bring them to a focus at the diſtance of a diameter of its convexity, by propoſition 51. And another plano-convex lens HKI, which has the ſame convexity with the former, or is a ſegment of an equal ſphere, will produce an equal effect : and conſequently, if the rays, after they have paſſed through the firſt lens EFG, fall upon the ſecond HKI, and paſs through that, they will be made to converge as much more as they did, and for that reaſon their focus will be as near again, by propoſition 29. That is, ſince one of theſe lens's bring the rays to a focus at the diſtance of a diameter of the ſphere of which it is a ſegment, both of them together will produce a double convergency in the rays, and therefore will bring them to a focus at half the diſtance, or at the diſtance of a ſemidiameter of the ſphere of which either lens is a ſegment. Now, if the two plane ſides EG and HI of theſe two lenſes were cloſe together, they would make the double convex lens ACBD. Therefore a convex lens of equal convexities on both ſides may be conſidered as two plano-convex lenſes of equal convexities: and as two ſuch lenſes would make rays, that were parallel at their incidence, converge to a focus at the diſtance of a ſemidiameter of the convexity of either; the convex lens, which is made up of both of them, will produce the ſame effect.

The reader may imagine there is ſome difference between two plano-convex lenſes and one double convex one. The difference ſeems to be this; that after the rays are refracted at the ſurface EFG, they will ſuffer a ſecond refraction in paſſing out of the lens through the plane ſurface EG, ſince the firſt lens makes them converge they will fall obliquely upon the ſurface HI and be refracted there a third time, and then again a fourth time, when they paſs out of the ſurface HKI. Whereas in going through the lens ACB, they are refracted only twice, once when they enter it and once when they go out of it. But then this apparent difference makes no real one. For when the rays, that were made to converge within the lens EFG, come out of the glaſs into the air through the plane ſurface EG their convergency is indeed encreaſed; by propoſition 44. But when the rays thus converging paſs out of the air into the other lens HKI through the plane ſurface HI, their convergency is juſt as much diminiſhed; by propoſition 43, 40. Therefore of the four refractions, which the rays ſuffer, when they paſs through two plano-convex lenſes that are not cloſe together, the ſecond and third refractions mutually deſtroy one another; ſo that the whole effect is produced by the firſt when the rays enter the ſurface EFG, and by the fourth when they paſs out of the ſurface HKI, juſt as it would have been if theſe two lenſes had been cloſe together and had made up one double convex one of equal convexities on both ſides ſuch as ACBD.

Burning-glaſſes are double convex lenſes. The ſun is ſo far off, that we may conſider every point or every radiant upon the ſurface of it, as a radiant at an infinite diſtance, and conſequently may ſuppoſe the rays emitted from each point of it to be parallel to one another, without any ſenſible error, by prop. 30. Therefore all the rays of the ſun that fall upon a burning-glaſs, if the convexities of it are equal on both ſides, will, by paſſing through the glaſs, be made to converge, and will be united in a focus behind the lens, at the diſtance of a ſemidiameter of either of its convexities. When all theſe rays are thus brought together, the heat they produce will be very great, and may be ſufficient to ſet fire to a piece of paper, or of touchwood, or even to bodies leſs apt to burn than theſe are, if they are placed in the focus of the lens. This effect of the rays of the ſun when they are thus collected, is the reaſon why that point where they are collected is called the focus: and the name, after it had for this reaſon been given to this point, has been made uſe of as a general one to ſtand for any point where converging rays meet, or to which they tend.

A plano-convex lens will collect the rays of the ſun, and produce a ſufficient heat at the focus to ſet fire to bodies that are placed in it. But the diſtance of the focus, when ſuch a lens is made uſe of, is equal to the diameter of the ſphere, of which the lens is a ſegment.

A convex lens, whoſe convexities are unequal, will ſerve for the ſame purpoſe: and the diſtance of the focus in this caſe ſhall be determined in prop. 55.

The ancients made uſe of refracting burning-glaſſes, as appears from a paſſage in a play of Ariſtophanes, which is called *The clouds*. For Strepſiades, act. II. ſcene 1. tells Socrates, that he had found out an excellent contrivance to defeat his creditors, if they ſhould bring an action againſt him. His contrivance was, that he would get from a jewellers (for thus the ſcholiaſt teaches us to tranſlate the word φαρμακοπώλαις) a certain tranſparent ſtone, which they called ὕαλον, that was uſed for kindling fire; and then ſtanding at a diſtance he would hold it to the ſun, and melt down the wax on which the action was written. The ſcholiaſt calls this inſtrument a round ſtone; but certainly it was not round like a ſphere; for he ſays it was τροχοειδὲς ϖαχὺ round like a wheel and thick. But as from this deſcription it was not a ſphere, ſo neither was it a piece of plane glaſs round at the edges, as appears from the uſe of it; becauſe a circular piece of plane glaſs would not produce the effect, for which this ὕαλος was uſed. For however ridiculous the author of the play intended to make this contrivance of Strepſiades, he has given us a deſcription of the common uſe that this ὕαλος was put to, as well as of the particular uſe which Strepſiades was to make of it. It was ſold at the jewellers and was uſed for kindling fire. But a circular glaſs could not have been applyed to this purpoſe, unleſs both or one of its ſides were convex. I have called it a glaſs, though ὕαλος was a natural ſubſtance and not an artificial one, as the ſcholiaſt informs us. For ὕαλος, though it ſignifyed glaſs at the time when the ſcholiaſt lived, yet the ancients meant by it a tranſparent ſtone, which was properly called κρύον, and was like that artificial glaſs which had been found out when he wrote. The ſubſtance, of which theſe lenſes were made, ſeems not to have been chryſtal but what is now called talc. Though this may be looked upon only as a ſofter ſort of chryſtal. That it was a natural and a ſoft ſubſtance appears from Herodotus, book III. 24. For in deſcribing the manner of burying the dead amongſt the Æthiopians, he ſays, that the dead body was firſt dryed, and then covered with mortar, and when it had been painted as like the deceaſed as could be, it was put into a coffin made of ὕελος, great plenty of which, and ſuch too as is eaſy to work, they dig up in that country. Coffins made of this were, as he tells us, tranſparent, ſo that the corpſe might be ſeen through them. Talc is a ſoft tranſparent ſtone that will eaſily ſplit into thin plates : and therefore a piece of it, thick enough for a lens, will not bear a good poliſh, but when it is made as ſmooth as it can be, it will be full of cracks and ſcratches. Which accounts for the manner in which the ſcholiaſt of Ariſtophanes ſays theſe lenſes made of ὕαλος were uſed. The ſurface of them was firſt oyled and the lens itſelf was warmed. The oyl would fill up the cracks and ſcratches and being of nearly the ſame denſity with the talc itſelf, would by that means make it more tranſparent : as paper, when it is dry, is not near ſo tranſparent, as it is, when it has been wetted with oyl. It is difficult indeed to ſay for what reaſon they warmed the lens, unleſs it was to make the oyl more fluid that it might run into the cracks more eaſily. But perhaps the ancients might imagine, that the rays of the ſun would burn better through a warm lens than through a cold one. And though our modern philoſophers, who are acquainted with the theory of light and refraction, know very well that the heat or coldneſs of the ſubſtance through which the rays are to paſs will make no difference : yet many of thoſe, who make uſe of burning-glaſſes, would even now think it a paradox and would ſcarce be perſuaded to believe that the rays of the ſun may be made to kindle fire by paſſing through cold water.

54. *If parallel rays fall directly upon a lens, which is equally concave on both ſides, they will diverge, after they have paſſed through it, from an imaginary radiant, or negative focus, at the diſtance of a ſemi-diameter of either concavity.*

One plano-concave lens *hki* would make rays, that were parallel at their incidence and fell directly upon it, diverge from a negative focus at the diſtance of a diameter of the ſphere, of which the lens is a ſegment, by prop. 52. Therefore two ſuch lenſes *hki* and *efg*, if the rays were to paſs through both of them, would make the rays diverge as much more, and conſequently, by prop. 28, the negative focus, or imaginary radiant, will be as near again, or at the diſtance of a ſemi-diameter of either concavity. This therefore will be the diſtance of the negative focus, when the rays paſs through a concave lens, having equal concavities on each ſide, ſuch as *acbd* : for this lens *acbd* may be conſidered as the two plano-concave ones *hki* and *efg* with their plane ſides cloſe together.

The reader ſees by this time why a concave, or a plano-concave lens cannot be uſed as a burning-glaſs. The rays of the ſun are made to burn by being made to converge ſo as to be brought together in a focus, that many of them may fall within a narrow compaſs upon ſome point in the thing that is to be burned. But a concave or plano-concave lens makes the rays of the ſun diverge, or ſeparates them farther from one another, inſtead of bringing them cloſer together.

55. *When parallel rays fall directly upon any lens that is convex; as the ſum of the ſemidiameters of both convexities to the ſemidiameter of either; ſo is double the ſemidiameter of the other to the diſtance of the focus, after the rays are refracted.*

We have already ſeen what is the diſtance of the focus, when only one ſide is convex, or when both ſides are equally convex. But ſome lenſes have a greater convexity on one ſide than on the other. The chryſtalline humour of the eye is of this ſort. And the rule here laid down is applicable to all caſes. We will apply it in the inſtance of a lens having equal convexities on both ſides, and to a lens that has one ſide plane: and then from ſhewing the truth of it in theſe two inſtances we may infer the truth of it in all other inſtances.

Firſt in a lens of equal convexities the diſtance of the focus has been ſhewn to be a ſemidiameter of either convexity, in prop. 53. And the focus found by this rule will be at the ſame diſtance. Call the ſemidiameter of one of the convexities s, and then, becauſe the convexities are equal on both ſides, the ſemidiameter of the other convexity muſt likewiſe be equal to s. Therefore the ſum of the ſemidiameters is s+s or 2s. In this caſe the proportion will be as follows. As the ſum of the ſemidiameters of both convexities, or as s+s=2s, is to the ſemidiameter of either convexity, or to s: ſo is double the ſemidiameter of the other convexity, or ſo is 2s, to the focal diſtance. But in this analogy, as s+s to s, ſo is 2s to a fourth proportional, it is eaſy to determine what this fourth proportional muſt be. For ſince the firſt term s+s is equal to the third, or to 2s, the ſecond term s muſt likewiſe be equal to the fourth. Therefore the focal diſtance is s, or one ſemidiameter of the convexity on either ſide.

Secondly, if the lens is a plano-convex one, we cannot apply this rule for finding out the focus, unleſs we conſider the plane ſurface as the ſegment of a ſphere: and, from what has been ſhewn under prop.49.

it appears, that we may conſider it as the ſegment of a ſphere, whoſe ſemidiameter is infinite. Call this infinite ſemidiameter S, and call the ſemidiameter of the ſphere, of which the convex ſide is a ſegment, s. Then the ſum of the ſemidiameters of both convexities, or rather of both ſides, is S+s. Therefore by the rule, as S+s the ſum of the ſemidiameters of both convexities, is to S the ſemidiameter of one of them, ſo is 2s, or double the ſemidiameter of the other, to the focal diſtance of this plano-convex lens. But when the proportion ſtands thus, as S+s is to S, ſo is 2s to a fourth proportional, what will that fourth proportional be? I ſay the fourth term will be equal to the third, or will be 2s. The firſt term is S+s, and the ſecond is S. But theſe two terms are equal to one another. For S is an infinite quantity, and s is a finite one. Therefore ſince this finite quantity s, bears no proportion to the infinite one S; if you add the finite quantity s to the infinite one S, this infinite quantity is not encreaſed by the addition of a quantity ſo ſmall in reſpect of the whole as to be quite inſenſible. Therefore S+s is equal to S: and conſequently, ſince the firſt and ſecond terms S+s and S are equal to one another, it could not be as S+s to S, ſo 2s to the focal diſtance, unleſs this fourth term, or focal diſtance, was likewiſe equal to 2s. But 2s is two ſemidiameters of the lens's convexity. Therefore the diſtance of the focus of a plano-convex lens is rightly determined by this rule: for it is found to be two ſemidiameters, or one diameter of the ſphere, of which the convex ſide of the lens is a ſegment; and this was proved to be the diſtance of it in prop. 51.

But thirdly, this rule for finding the focus either of a convex lens of equal convexities, or of a plano-convex lens may be applyed to any convex lens whatever. For the two caſes already conſidered, may be looked upon as the two extremes. When the convexity on one ſide is equal to the convexity on the other ſide, this is one extreme. And when the convexity on one of the ſides has decreaſed till it becomes nothing, or is quite plane, this is the other extreme. The intermediate caſes are any of thoſe, when one ſide is leſs convex than the other: and as the rule here laid down is true in the two extremes, it will be equally true in all the intermediate caſes. When the convexity of the lens is equal on both ſides, as ACBD, Plat. IX. fig. 6. it appears from what has been ſaid already concerning this rule, that s+s, the firſt term, is double s, the ſecond term, and 2s, the third term, is likewiſe double the focal diſtance. If the ſide ADB of this lens was to ſink in, or to flatten, till at laſt it became quite plane, ſo that the lens was changed into a plano-convex one, ſuch as EFG, then, from what has

been

been ſaid already, it appears that S+s, the firſt term, is equal to S, the ſecond, and 2s, the third term, is likewiſe equal to the focal diſtance. Since therefore in one extreme, the firſt term is double the ſecond, and the third is double the focal diſtance, and when the lens has changed to the other extreme, the firſt term is equal to the ſecond, and the third is equal to the focal diſtance, it follows, that whilſt the lens was changing from one extreme to the other, or in all the intermediate caſes, the firſt and ſecond terms are conſtantly approaching to a ratio of equality, and ſo likewiſe are the third and fourth. And ſince the firſt term becomes equal to the ſecond, and the third equal to the fourth, in the ſame ſtate of the lens, it follows farther, that their reſpective approaches to a ratio of equality have been in the ſame proportion. Therefore in all the intermediate ſtates of the lens, that is, whatever was the convexity of ADB, the firſt term has born the ſame proportion to the ſecond, that the third has to the fourth, or the ſum of the ſemidiameters of both convexities, has always been to the ſemidiameter of either, as double the ſemidiameter of the other, to the focal diſtance.

56. *When parallel rays fall directly upon any concave lens, as the ſum of the ſemidiameters of both concavities, is to the ſemidiameter of either, ſo is double the ſemidiameter of the other to the diſtance of the negative focus.*

This propoſition may be eaſily proved in the ſame manner as the laſt is, the reader will be able to apply that demonſtration without having it repeated. Or indeed, this propoſition is the ſame as the other: for a concave lens is only a lens whoſe convexity is negative, by prop. 49. So that the only difference is, that if, when the lens is convex, the ſemidiameters are conſidered as affirmative, when the lens is concave they will be negative; the refracted rays, which converge in one caſe, will have a negative convergency, that is, will diverge in the other; and the focus, which is on the ſide of the refracted rays, when the convexity is affirmative, will be on the contrary ſide when the convexity is negative; and therefore the focus in the concave lens will be virtual, or negative. But theſe changes make no alteration in the proportion, by which the focal diſtance is determined: for the proportion will be the ſame whether the terms of it are conſidered as affirmative or negative.

57. *The collateral rays of a direct parallel beam, when they are made to converge by a convex or plano-convex lens, do not all of them meet in a focus at exactly the ſame diſtance behind the lens.*

The collateral ray AR Plat. IX. fig. 2. croſſes its axis after refraction, or is brought to a focus at F, which is at the diſtance of a diameter of the lens's convexity, by prop. 51. But any other collateral ray, which falls upon the lens nearer to the axis DS than AR is, will have its focus after refraction, a little farther from the lens than F; and any other collateral ray, which falls upon the lens farther from the axis than AR, will have its focus a little nearer than F. The ray DS is perpendicular to both ſurfaces of the lens, and conſequently is not refracted at all, by prop. 41. Now the nearer any ray falls to DS, the nearer perpendicular, or the leſs oblique, it is to the two ſurfaces of the lens, and conſequently it will be refracted the leſs. But the leſs any ray is refracted, the leſs it will converge; and therefore, by prop. 29, the focus will be ſo much the more remote from the lens. The farther any ray falls from DS, the farther it is from being perpendicular, or the more oblique it is, for which reaſon it will be refracted the more, and will converge the more, and therefore the diſtance of the focus will be leſs. Thus we ſee that all the collateral rays will not have their focus at exactly the ſame point. The focus of the ray DR is at F; but thoſe which fall nearer to the axis will have their focus more remote, and thoſe which fall more remote from the axis will have their focus nearer. But in ſuch collateral rays as are none of them very far from the axis, this difference will be inconſiderable.

If the reader is deſirous of knowing why the demonſtration of propoſition 51, is not univerſally applicable to all other collateral rays, as well as to AR, he may remember that in that demonſtration there was a ſmall error. For we ſuppoſed BF and PF to be equal, though they are not quite, but only nearly, ſo: they cannot be quite equal, unleſs B and P, or R and S, that is AR and DS, were cloſe together. And the farther B is from P, or R from S, or AR the collateral ray from DS the axis, ſo much the longer will BF be in proportion to PF. Now the length of BF is always a given quantity: for it is always twice CB, as appears from the demonſtration of propoſition 51. Therefore the greater proportion this given line BF bears to PF, or the longer BF is in reſpect of PF, ſo much the ſhorter PF muſt be. Euc. b. V. prop. 10. From whence it follows, that the farther B is from P, or R from S, or the collateral ray AR from the axis DS, ſo much the ſhorter is PF, or ſo much leſs is the focal diſtance. But in all ſuch collateral rays as do not fall very far from their axis, or are not very oblique to the ſurface of the lens, the difference between BF and PF being inconſiderable in any of them, the diſtance of the focus is ſo nearly the ſame, that they may all be conſidered as meeting at the ſame point.

The reader will be able to apply this reaſoning, without having it repeated, in the caſe of ſuch a beam of direct and parallel rays as falls firſt upon the convex ſide of a plano-convex lens, or in the caſe of ſuch an one as falls upon either ſide of a double convex lens. For in theſe caſes as well as in that, which we have been explaning, the axis only of the beam is perpendicular to the lens, and paſſes through the lens unrefracted; and thoſe rays, which fall neareſt to the axis, are neareſt perpendicular, or leaſt oblique, and conſequently will be refracted leaſt, and will converge leaſt, and for that reaſon will have their focus moſt remote from the lens. Whereas thoſe rays, which fall upon the lens at any conſiderable diſtance from the axis, are farther from perpendicular, or more oblique, are more refracted, converge more, and conſequently have their focus nearer to the lens.

58. *The collateral rays of a direct parallel beam, when they are made to diverge by a concave, or plano-concave lens, do not all of them diverge from an imaginary radiant, or negative focus, at exactly the ſame diſtance before the lens.*

This is proved, by Plat. IX. fig. 4, in the ſame manner that the foregoing propoſition was proved by fig. 2, only as the lens is here concave, the reader muſt remember that the refracted rays diverge, and the focus is a negative one.

59. *When a beam of parallel rays falls obliquely upon a lens, the diſtance of the focus, if the lens is convex, or of the negative focus, if it is concave; will be nearly the ſame as when the rays fall directly.*

If LPG, Plat. X. fig. 1. which is a ſegment of the ſphere PMFG, is a plano-convex lens, and PO is the axis of it: parallel rays fall directly upon it, if their axis coincides with PO the axis of the lens, or if the collateral rays are parallel to PO. But DN and BE, though they are parallel to one another are oblique to PO, and therefore are collateral rays not of a direct but of an oblique beam. Theſe rays BE and DN, and all the rays of any ſuch oblique beam, when they are refracted, will meet their axis or be united in a focus, at the ſame diſtance behind the lens, that the rays of a direct beam would, that is, by propoſition 51,

at

PLATE IX.

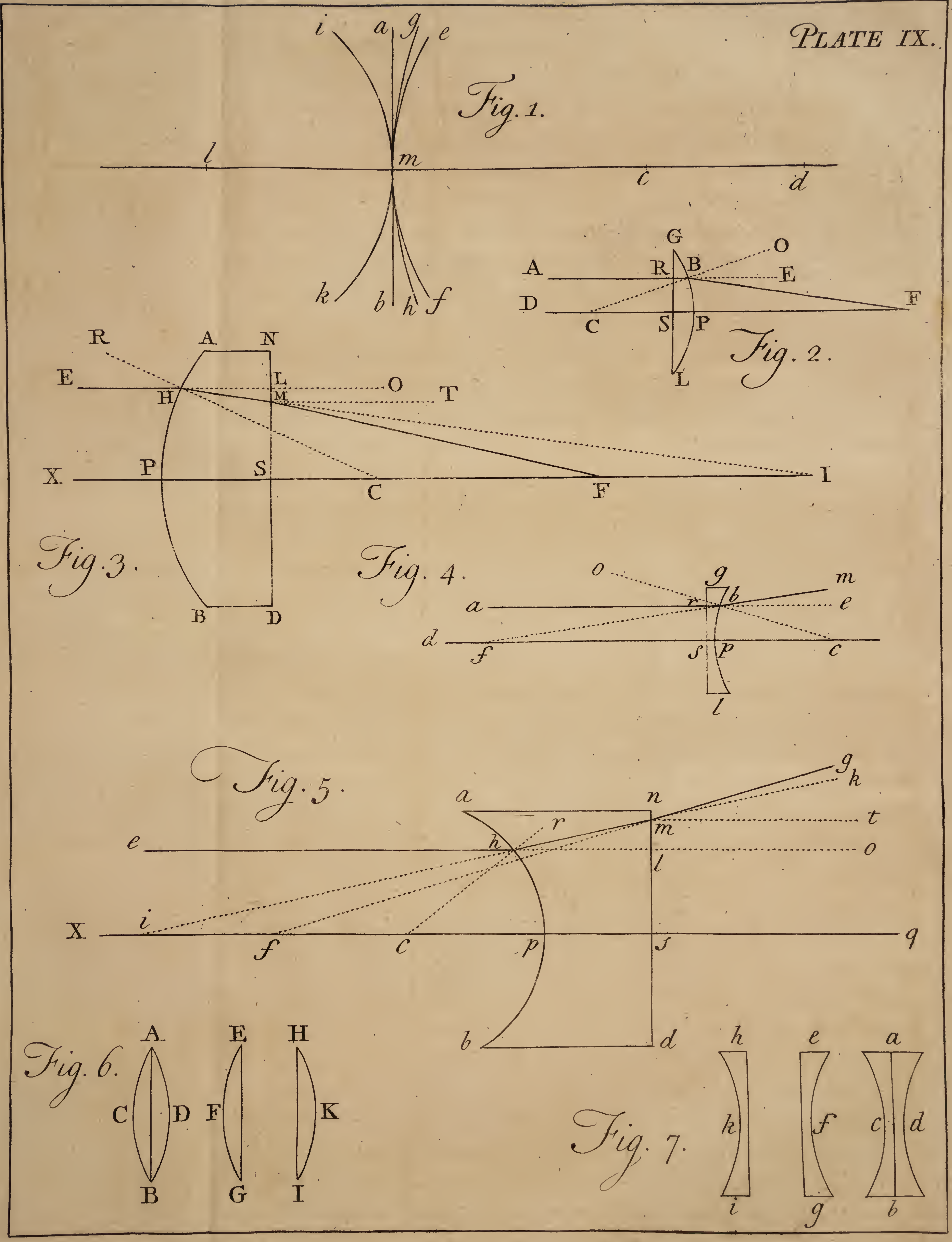


Fig. 1.
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Fig. 2.
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Fig. 3.
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Fig. 4.
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Fig. 5.
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Fig. 6.
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Fig. 7.
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at the diſtance of a diameter of the lens's convexity or of the ſphere, of which the lens is a ſegment.

Suppoſe the pole or vertex of the lens P was placed at R ſo that PO ſhould coincide with RC: then AR, BE, and DN, would be a direct beam; AR or RC, which is AR continued, coinciding with PO would be the axis of the beam, and RE and DN would be collateral rays. In this poſition of the lens the focus will be at F, or at the diſtance of a diameter of the lens's convexity, by propoſition 51, and at this point F the rays BE and DN will meet. But, if the poſition of the lens is changed and the pole or vertex of it inſtead of being at R is at P, then indeed the ray AR will fall beſide the lens; but the other rays BE and DN will ſtill fall upon it; and as the refractive power of the lens is not altered by changing its place, BE and DN will ſtill converge as they did before, and will meet at F. The only alteration is that the rays BE and DN are more oblique than before, and conſequently will be more refracted and will have their focus nearer to the lens; and F is nearer to the lens, when the vertex of it is at P, than when it is at R. But if the obliquity of theſe rays is not very great, that is, if the lens is not far removed the diſtance of F from P will be very nearly the ſame as its diſtance from R; that is, the focus of the oblique beam will be nearly at the ſame diſtance as the focus of the direct one.

The ſame method of reaſoning will demonſtrate this propoſition in any caſe of lenſes either convex or concave, for by altering the poſition of the lens an oblique beam may be changed into a direct one; and when the focal diſtance of the direct beam is known this determines the focal diſtance of the oblique one. For the changing the lens's poſition does not change its refractive power, though it changes the obliquity of the rays. And conſequently, ſince if the poſition is not much changed the obliquity is nearly the ſame, the focal diſtance of a beam that is not very oblique will be nearly at the ſame diſtance as the focus of a direct one.

But we will apply this method to one other caſe: and then from the former caſe and this together we may deduce the truth of the propoſition in all caſes whatever. Let LPN, Plat. X. fig. 2. be a plano-convex lens. In the former caſe we ſuppoſed the rays to fall firſt upon the convex ſide of the lens. But now ſuppoſe AE and BG to be two parallel rays of an oblique beam, which fall firſt upon the plane ſide of the lens. Theſe rays, by the ſuppoſition, are oblique to PO the axis of the lens. But if P the vertex was at the point R then DCR would coincide with PO; and the ray which deſcribes DCR would be perpendicular to both ſides of the lens; and BG and AE, becauſe they are parallel to DR, would likewiſe be parallel to PO, ſo that the beam would be a direct one, and DRS would be its axis. This axis would paſs through the lens unrefracted, by propoſition 41, and BG, AE, which in theſe circumſtances are the collateral rays of a direct beam, would by paſſing through the lens converge to a focus at the diſtance of a diameter of the ſphere PTHR, of which the lens is a ſegment, by propoſition 51. And though by putting the vertex of the lens at P the ray DR does not paſs through it, yet the rays AE, and BG, will meet at the ſame point. For by altering the poſition of the lens nothing is altered, except the obliquity of the rays; and if this is not made very great, their convergency, when they are refracted, and conſequently the diſtance of the focus, will be very nearly the ſame as when the vertex is at R, and the beam falls directly upon the lens.

Now ſince the oblique focus is at the ſame diſtance as the direct one, when the convexity of the ſurface of a plano-convex lens is affirmative; it follows, by what was ſaid under propoſition 49, and 52, that if the convexity was negative or the lens a plano-concave one, the oblique focus would then only become negative or would be before the lens inſtead of being behind it, but ſtill at the ſame diſtance as the negative direct focus.

But a double convex lens is only two plano-convex ones, and a double concave lens is only two plano-concave ones with their plane ſides cloſe together. And ſince the oblique focus whether affirmative or negative is at the ſame diſtance as the direct one, when the lens's are uſed ſeparately, it will likewiſe be at the ſame diſtance, when they are uſed together.

The reader remembers that the axis of an oblique beam is oblique to the ſurface of the lens, and conſequently is refracted. But then the direction of the axis, is not altered by this refraction; for the refraction is ſuch as makes the refracted ray parallel to the incident one. We may explane this matter in two inſtances of an oblique axis, that will make it intelligible in all other inſtances. Let *ef*, Plat. X. fig. 3. be an oblique ray paſſing into the convex lens at the point *f*, and coming out of it again at the point *g*. Now if *da*, which is a tangent to the lens at the point of ingreſs *f*, is parallel to *kc*, which is a tangent to it at the point of egreſs *g*, then the points of contact *f* and *g* are parallel to one another, that is, the points of ingreſs and egreſs are parallel like the two ſides of a plane glaſs; and for this reaſon the refraction which the ray ſuffers at *f* and *g* will not alter its direction more than it would have been altered by paſſing through a plane glaſs. Therefore *gh* the refracted ray will be parallel to *ef* the incident one, by propoſition 47. In like manner *ef* falls upon the concave ſurface of the lens *acbd* at the point *f*, and comes out of the lens at the point *g*, then if *ab* a tangent to the concave ſurface at *f* the point of ingreſs is parallel to the plane ſurface *cd*, this ray paſſes through the lens as it would through a plane glaſs, becauſe the point of ingreſs *f* is parallel to the ſide *cd* as the two ſides of a plane glaſs are parallel to one another. Therefore *gh* the refracted ray will be parellel to *cf* the incident one, by propoſition 47. Such a ray as this there will be in all oblique beams upon whatever lens they fall, and this ray is the axis of the beam. It was proper juſt to mention this matter; but as it will be of no great uſe in our future enquiries, we need not give ourſelves the trouble to determine upon what point of the lens this ray will fall.

CHAP. V.

Of the refraction of diverging rays by a convex lens, and of converging rays by a concave lens.

60. *The principal focal diſtance of a convex or plano-convex lens is the diſtance at which rays, that were parallel at their incidence, are united after refraction. And the principal focal diſtance of a concave or plano-concave lens is the diſtance, from which rays, that were parallel at their incidence, appear to diverge after refraction.*

IF PV, Plat. X. fig. 4. is a diameter of the ſphere of which the plano-convex lens AVB is a ſegment, then if parallel rays were to fall upon this lens, they would be ſo refracted by paſſing through it as to meet in a focus at a diſtance equal to PV, by propoſition 51. And agreeably to this definition the diſtance PV taken on either ſide of the lens is called the principal focal diſtance. In like manner, if *pv*, Plat. X. fig. 5. is a diameter of the concavity of the lens *avb*, then if parallel rays were to fall upon this lens, they would be ſo refracted by paſſing through it as to diverge from a virtual or negative focus at the diſtance *pv*, by propoſition 52. And agreeably to the definition, the diſtance *pv* taken on either ſide of the lens is called the principal focal diſtance. The principal focal diſtance of a lens, that has equal convexities on each ſide, is a ſemidiameter of either convexity: and the principal focal diſtance of the lens, that has equal concavities on each ſide, is a ſemidiameter of either concavity. By propoſition 53, 54.

The only difference in the following propoſitions between a convex and concave lens is this. When in any caſe of a convex lens we have taken the principal focal diſtance on the ſide of the incident rays, in the ſimilar caſe of a concave lens we muſt take the principal focal diſtance negative, that is, we muſt take it on the oppoſite ſide, or on the ſide of the refracted rays.

We ſhall demonſtrate the following propoſitions upon plano-convex and plano-concave lenſes only, but the reader will eaſily ſee that the demonſtrations are univerſal and may be applyed to double-convex and double-concave ones.

We have already ſeen, in the laſt chapter, what effect a convex or plano-convex lens will have upon parallel rays, or rays which come from a radiant at an infinite diſtance, as thoſe may be ſaid to do, which come from the ſun. But ſuppoſe that the radiant was nearer and that the rays, inſtead of being parallel, diverge when they fall upon the lens; as rays would that come from a candle or any other luminous body at no very great diſtance: it ſtill remains for us to enquire what effect the lens would have upon ſuch rays as theſe. Now this enquiry may be divided into three different caſes, according to the different poſition of the radiant in reſpect of the lens's principal focus. For firſt the radiant, the luminous point, or the candle, which is indeed rather many radiants together than one alone, may be in the lens's principal focus, that is at the diſtance PV, Plat. X. fig. 4. Or ſecondly it may be more remote than the principal focus, and be at the diſtance RV, which is greater than PV. Or thirdly, it may be nearer than the principal focus, and be Plat. X. fig. 6. at the diſtance RV, which is leſs than PV. Now parallel rays neither diverge nor converge: but rays in this ſtate are in the middle between divergency and convergency. For if the divergency of rays diminiſhes, they become nearer parallel; and when they are quite parallel their divergency is nothing: conſequently, if they converge, their divergency is leſs than nothing or negative. And ſince convergency is negative divergency, and a concave or plano-concave lens is a negative convex or plano-convex one; it follows that there will be juſt as many negative caſes as there are affirmative ones: that is, there are three caſes of converging rays falling upon a concave or plano-concave lens. Firſt, the converging rays may at their incidence tend to the lens's principal focus; or the imaginary focus may be at the principal focal diſtance, thus Plat, X. fig. 3. when converging rays fall upon the lens *ab* the focus to which they are directed may be *p*, ſo that *pv* the diſtance of this imaginary focus may be equal to a diameter of the lens's concavity or to the principal focal diſtance. Secondly, the incident rays *nb* and *fv* may tend to *r* an imaginary focus at the diſtance *vr*, which is greater than *vp*, or than the principal focal diſtance. Or thirdly, the incident rays *ma* and *xv*, Plat. X. fig. 7. may tend to the imaginary focus *r* at the diſtance *vr*, which is leſs than *vp* or than the principal focal diſtance.

61. *If rays diverge from the principal focus of a convex or plano-convex lens; theſe rays, when they have paſſed through the lens, will be parallel.*

If PF, Plat. IX. fig. 2. is the principal focal diſtance of the lens GL, then rays, as AR and DS, which are parallel at their incidence will, when they are refracted, be united at F. Therefore, by propoſition 41, ſince all refraction is reciprocal, if theſe rays were to be turned back again in the lines FB and FP ſo as to diverge from F, by paſſing through the lens they will be refracted back into their former directions RA and SD ſo as to be parallel. And the effect of the lens upon any other rays diverging from F, or from a radiant at the principal focal diſtance, would be the ſame, all ſuch rays will become parallel, when the lens has refracted them.

So that if a candle was placed at F, or behind any other convex lens at the principal focal diſtance, the rays, which diverge from any one point of the candle, when they fall upon the lens, will be parallel to one another, after they have paſſed through it.

62. *If rays converge to the principal focus of a concave or plano-concave lens; theſe rays, when they have paſſed through the lens, will be parallel.*

If the rays *ar* and *ds*, Plat. IX. fig. 4. are parallel, when they fall upon the plano-concave lens *gl*, they will after they are refracted, go on in the lines *bm* and *pc*, ſo as to diverge from *f* the principal focus. Therefore, ſince all refraction is reciprocal, by propoſition 41, if theſe rays were to be turned back again in the lines *mb*, *cp*, ſo as to converge to *f* the principal focus of the lens, they would be refracted into their former directions *ra* and *ſd* and would become parallel. And whatever rays converge in this manner to the point *f*, or to the principal focus of this or any other lens, will ſuffer juſt ſuch a refraction as will produce the ſame effect in them or will make them parallel to one another.

The chief uſe we ſhall make of this propoſition will be beſt underſtood, when we come to apply it in explaning one ſort of teleſcopes.

63. *If the radiant from whence rays diverge is farther from a convex or plano-convex lens than its principal focus, the rays, after they are refracted, will converge.*

If PV, Plat. X. fig. 4. is the principal focal diſtance of the lens AB, and any radiant point at R is at a greater diſtance than this from the lens; then rays, ſuch as RV and RA, which diverge from thence will be made to converge by paſſing through the lens. If the radiant had been at P inſtead of R, that is, if the radiant had been in the principal focus, the refracted rays would have been parallel, or their divergency would have been nothing, by propoſition 61. But when the radiant is more remote than P, the incident rays diverge ſo much the leſs, by propoſition 28. And conſequently the divergency of the refracted rays will likewiſe be ſo much the leſs. Now when the radiant is in the principal focus the divergency of the incident rays is nothing. Therefore if the radiant inſtead of being at P is at R or more remote than the principal focus, the divergency of the refracted rays will be leſs than nothing or negative, that is, the refracted rays will converge as VF and AF do, and will meet in a focus at F.

That, when the incident rays diverge, the focus after refraction will be more remote than when they are parallel, is evident. For when the radiant is at an infinite diſtance, or when the incident rays are parallel, the divergency before refraction is the leaſt poſſible, and conſequently the negative divergency or convergency after refraction will be the greateſt poſſible; and for this reaſon the focus will be as near to the lens as it can be, by propoſition 29. But, when the radiant is at any diſtance leſs than infinite, the nearer it is ſo much greater is the divergency of the incident rays, by propoſition 28. Therefore ſo much leſs will be their negative divergency, or their convergency, after refraction; and conſequently the focus will be ſo much the more remote, by propoſition 29. From hence it appears that the rays of the ſun will be collected into a focus nearer to the lens than the rays of a candle or any other luminous body placed at R.

64, *When the radiant is more remote from a convex or plano-convex lens than the principal focus, as the difference between the diſtance of the radiant and principal focal diſtance is to the principal focal diſtance, ſo is the diſtance of the radiant before refraction, to the diſtance of the focus after it.*

The diſtance of the radiant R, Plat. X. fig. 4. from the lens AR is RV; the principal focal diſtance is PV; the difference between the diſtance of the radiant and principal focal diſtance, or between RV and PV, is RP. Therefore the propoſition affirms that as RP is to PV, ſo is RV to a fourth proportional, which will be the diſtance of the focus from the lens, after the rays are refracted: and, taking VF this fourth proportional, F will be the point where the refracted rays will concur. The truth of this will appear from the following conſiderations.

If the radiant was in the principal focus or at the diſtance PV, the refracted rays would be parallel; that is, their divergency would be nothing, by propoſition 61. Therefore the refraction of the lens will juſt deſtroy as much divergency as rays have that proceed from a radiant in the principal focus or at the diſtance PV. But the radiant at R is more remote than the principal focus; its diſtance from the lens is RV, which is greater than PV. Now the more remote the radiant is, the leſs the incident rays diverge; for the divergency is inverſely as the diſtance of the radiant, by propoſition 28. Therefore as much greater as RV is than PV, ſo much leſs rays diverge, which proceed from a radiant at R, than rays, which proceed from a radiant at P: or, the divergency of rays falling upon the lens AB, when the radiant is at R, is to the divergency of rays falling upon the ſame lens, when the radiant is at P, as the diſtances RV and PV inverted, that is, as PV to RV. But the refraction of the lens would juſt deſtroy all the devergency of the rays, if the radiant was at P. Therefore the divergency, which the rays have that come from R, is to the divergency, which the refraction of the lens will deſtroy, as PV is to RV. Now PV is leſs than RV, by the ſuppoſition, and conſequently, the rays that come from R, have not ſo much divergency as the lens will deſtroy. Therefore after refraction their divergency will be leſs than nothing, or will be negative. And this negative divergency will be proportional to the difference between the divergency, which the incident rays have, and the divergency, which the lens will deſtroy; or to the difference between PV and RV, that is, it will be proportional to RP. From hence then it follows, that the negative divergency of the rays after refraction is to their divergency before refraction as RP to PV. But as the divergency after refraction to the divergency before it, ſo is the diſtance of the radiant before refraction to its diſtance after refraction, for, by propoſition 28, the diſtance of the radiant is always as the divergency inverſely. That is, as RP to PV, ſo is RV the diſtance of the radiant before refraction to a fourth proportional, which will be the diſtance of the radiant after refraction. But then we muſt remember, that the divergency after refraction is negative, that is, that the rays converge; and conſequently this fourth proportional will be negative, or will be on the contrary ſide of the lens to P, that is, it will be a negative radiant. But as we have frequently obſerved before that if a focus is conſidered as affirmative, a negative focus means a radiant:

diant: ſo here, when we conſider a radiant as affirmative, a negative radiant means a focus. Therefore this fourth proportional, or VF, is the diſtance of the focus. Upon the whole therefore, as RP is to PV, ſo is RV the diſtance of the radiant before refraction to VF the diſtance of the focus after refraction.

Let PV, the principal focal diſtance of the lens AB, be 4 inches. Then the rays of the ſun, or any other parallel rays, that were to paſs through the lens, would be collected into a focus at the diſtance of 4 inches from the lens. But ſuppoſe there was a candle at R, and that RV, the diſtance of it from the lens, is 6 inches. The radiant is then very near to the lens, ſo that the rays, which come from it and fall upon the lens, are not parallel but diverging. Therefore when they have paſſed through the lens, the refraction, which they ſuffer, will not make them converge ſo much, as it would have done, if they had been parallel before, and conſequently the focus will be more remote, by prop. 29. Now the focus of the ſuns rays would be at the diſtance of 4 inches; and how much the focus of the rays coming from the candle will be farther off is found by this propoſition. For the diſtance of the principal focus, or PV=4 inches, the diſtance of the radiant or RV=6 inches. The difference of theſe diſtances RV—PV=RP=6—4=2 inches. Therefore, as RP to PV, or as 2 to 4, ſo is RV to VF, or 6 to 12, which is the focal diſtance. That is, the rays diverging from the candle at R will by the lens be collected into a focus at F, which is 12 inches from the lens.

65. *If rays, when they fall upon a concave or plano-concave lens, converge to an imaginary focus, which is at a greater diſtance from the lens than its principal focus; theſe rays by paſſing through the lens will be made to diverge.*

If the rays *fv* and *na* converge to the point *r*, then *r* is their focus. But if, before they arrive at that focus, they are to paſs through the plano-concave lens *ab*, they will be refracted, and ſo being turned out of the way will never meet at *r*; for which reaſon the focus in theſe circumſtances is called imaginary: the rays are directed towards it, but never meet at it. If this imaginary focus was at *p*, the principal focus of the lens, or at the diſtance *pv*, then the rays, when they have paſſed through the lens, would be parallel, by prop. 62. But if it is at the diſtance *rv*, which is greater than *pv*, or than the principal focal diſtance; then, ſince the imaginary focus is more remote, the convergency of the incident rays is leſs, by prop. 29. Now if the imaginary focus had been at *p*, the refraction of the lens would have made the rays parallel after they had paſſed through it, or would have made their convergency nothing: therefore if the imaginary focus is more remote than *p*, that is, if the incident rays converge leſs, the ſame refraction will make their convergency, when they have paſſed through the lens, leſs than nothing, or negative; that is, the refracted rays, inſtead of converging, as the incident ones did, will have the contrary direction, or will diverge, and will proceed in the lines *vr* and *am*, as if they had come ſtrait from the point *f*, which point is the imaginary radiant, or negative focus, after the rays are refracted.

66. *When the imaginary focus is more remote from a concave lens than its principal focus, as the difference between the diſtance of the imaginary focus and the principal focal diſtance to the principal focal diſtance, ſo is the diſtance of the imaginary focus, before refraction, to the diſtance of the imaginary radiant after it.*

The diſtance of the imaginary focus *r*, Plat. X. fig. 5, from the lens *ab* is *rv*, the principal focal diſtance is *pv*, the difference between *rv* and *pv*, or *rv*—*pv*, is *rp*. Therefore as *rp* is to *pv*, ſo is *rv* to a fourth proportional, which will be the diſtance of the imaginary radiant after refraction. Thus if *vf* is this fourth proportional, the rays *fv* and *na*, which converge to *r* before refraction, will become *vr* and *am*, ſo as to diverge from *f* after refraction.

This propoſition might be proved diſtinctly in the ſame manner as propoſition 64. But inſtead of repeating that demonſtration, we may better ſhew, that every thing in this propoſition is the ſame with every thing in that, only taken on the contrary ſide; that is, this propoſition is only propoſition 64. in negative terms, or fig. 5. is the negative in all reſpects of fig. 4. For firſt, in fig. 4. R is a radiant, from which the incident rays diverge, and fig. 5. *r* is a focus on the contrary ſide of the lens, to which the incident rays converge. Secondly, AB is a plano-convex lens, and *ab* is a plano-concave one. Thirdly, the principal focal diſtance PV is taken on the ſame ſide with the incident rays, and the principal focal diſtance *pv* is taken on the ſide of the refracted rays. Fourthly, the refracted rays AF and VF converge, and the refracted rays *am* and *vr*, or, which is the ſame thing, *fa*, and *fv*, diverge. Therefore fifthly, the focus F is on the ſide of the refracted rays, and the negative focus, or imaginary radiant *f*, is on the ſide of the incident ones. Since therefore, whether diverging rays fall upon a plano-convex lens, or converging ones upon a plano-concave lens, every thing is the ſame, only taken on the contrary ſides, provided the radiant in one caſe, and the imaginary focus in the other, are more remote than the principal focus; the proportions will be the ſame in both caſes; that is, ſince RP is to PV, as RV to FV; *rp* is likewiſe to *pv*, as *rv* to *fv*.

67. *If the radiant, from whence rays diverge, is nearer to a convex, or plano-convex lens, than its principal focus; the rays, after they have paſſed through the lens, will continue to diverge, but leſs than they did before.*

If AB, Plat. X. fig. 6, is the lens, PV its principal focal diſtance and R the place of the radiant, ſo that RV leſs than PV is its diſtance from the lens; any incident rays as RA and RV, which diverge from R before refraction, will diverge leſs after they have been refracted by paſſing through the lens, and will go on in the lines VX and AM, as if they had come at firſt in the lines FA and FV ſtrait forwards from an imaginary radiant F.

Here let us look back to fig. 4. When R is more remote from the lens than P, the refracted rays converge. But if the radiant R was to approach the lens, then, by prop. 28, the divergency of the incident rays would encreaſe, and conſequently their negative divergency, that is, their convergency would be ſo much leſs after they were refracted. Till at laſt, when the radiant R by approaching the lens was got to P, or was at the principal focal diſtance, the divergency of the incident rays would be ſo great that the refraction of the lens would only juſt deſtroy it all and no more; and the negative divergency, or the convergency of the rays after refraction would be nothing, for the refracted rays would be parallel, by prop. 61. Therefore, if the radiant was ſtill to approach the lens and to get nearer to it than the principal focus, as at R, fig. 6. the divergency of the incident rays would by this means be farther encreaſed, by prop. 28, and would be ſo great, that the refraction of the lens would not deſtroy it all. But as this refraction would in any caſe deſtroy juſt as much divergency as the rays would have if the radiant was at P; it follows, that when the radiant is at R, nearer than P, a part of their divergency will then be deſtroyed, and conſequently the rays, though they diverge after refraction, will however diverge leſs than they did at their incidence.

We may obſerve farther, that when the radiant is at R, fig. 4, more remote than the principal focus and the refracted rays converge, the focus is at F, on the ſame ſide of the lens with the refracted rays. When the radiant is at P, the refracted rays are parallel, and therefore, by prop. 31, the diſtance of the focus is infinite. Since therefore as the radiant approaches to the lens AB, the focus departs from it, till the diſtance of the focus becomes infinite, when the radiant is at P; if the radiant was ſtill to approach nearer than P, as ſuppoſe it to be at R, fig. 6, then the diſtance of the focus ſtill encreaſing will be greater than infinite, that is, by what was ſaid under propoſition 49, the focus will then be negative, or will change ſides, and inſtead of being on the ſide of the refracted rays will be on the ſide of the incident ones at F. But ſuch a negative focus is an imaginary radiant. Therefore the rays after refraction, will appear to diverge from the imaginary radiant F.

68. *When the radiant is nearer to a convex, or plano-convex lens, than its principal focus, as the difference between the diſtances of the radiant and principal focus to the diſtance of the principal focus, ſo is the diſtance of the real radiant before refraction to the diſtance of the imaginary one afterwards.*

RV, Plat. X. fig. 6. is the diſtance of the point from whence the rays come, or of the real radiant R, from the lens AB; PV is the diſtance of the principal focus; PV—RV=RP is the difference between them. Therefore the propoſition affirms, that as RP is to PV, ſo is RV to a fourth proportional, which will be the diſtance of the point from whence the refracted rays will appear to come, or of the imaginary radiant. And if FV is this fourth proportional, then the rays, which proceed from the point R before refraction, will go on in the lines AM and VX after refraction, as if they came from the point F.

The demonſtration of this propoſition is the ſame with that of propoſition 64. The letters of reference are the ſame in both; ſo that if the reader wants a diſtinct proof of this propoſition, he need only read that demonſtration and apply it to fig. 6. inſtead of fig. 4. The only difference is, that in this propoſition F is a radiant; whereas in propoſition 64, having firſt conſidered F as a radiant, we afterwards ſhewed that it would be a negative radiant, or that, as it was on the contrary ſide of the lens to the radiant R, it would be a focus.

Or otherwiſe; in propoſition 64. the radiant is beyond the principal focus P, and in the propoſition now before us, it is nearer to the lens than P: ſo that PR in theſe two propoſitions is on the contrary ſides of the principal focus; and PR by changing ſides, makes VF change ſides, as has been ſeen already, in propoſition 67. And as this is the only difference between the two caſes, the proportions will be the ſame in both, PR to PV, as RV to FV.

69. *If the imaginary focus, to which rays converge, is nearer to a concave or plano-concave lens than its principal focus; the rays, after they have paſſed through the lens, will continue to converge, but leſs than they did before.*

Let *ma*, Plat. X. fig. 7, and *xv* be rays converging to *r*, ſo that if there was no lens in the way they would meet at *r*: then *r* is the focus of theſe rays. But if they are to paſs through the plano-concave lens *ab* before they come to *r*, then the refraction, which they ſuffer, will turn them out of the way, and will prevent them from ever coming to *r* at all. Therefore *r* is then only an imaginary focus; the rays, though they are directed thither, never meet there. Now if *rv*, the diſtance of this imaginary focus from the lens is leſs than *pv*, which is the principal focal diſtance, the rays, after they are refracted, will continue to converge in lines *af*, *vf*, but leſs than they did at their incidence: ſo that *af* and *vf*, the refracted rays, will converge leſs than *mar*, *xvr*, which are the directions of the incident rays. For if the rays, when they fell upon the lens, had converged to *p*, inſtead of *r*, that is, if the imaginary focus had been at the principal focal diſtance, the refracted rays would have been parallel, or the refraction of the lens would have juſt deſtroyed all the convergency which the rays had at their incidence. But *rv*, the diſtance of the real focus, is leſs, by the ſuppoſition, than *pv* the principal focal diſtance. Therefore the rays, as they converge to a nearer focus than *p*, converge more than if they had converged to *p*, by prop. 29. and conſequently they have a greater convergency than the refraction of the lens can deſtroy; ſo that they will continue to converge after refraction: but their convergency will be diminiſhed, for it will be proportional to the difference between what the incident rays had and what the refraction of the lens deſtroys.

Now ſince the refracted rays *af* and *vf* converge leſs than the incident ones *ma* and *xv*, the focus *f*, where the refracted rays meet, or the real focus, will be at a greater diſtance from the lens than the focus *r*, to which the incident rays tended, or than the imaginary focus, by propoſition 29.

70. *When the imaginary focus is nearer to a concave, or plano-concave lens, than its principal focus, as the difference between the diſtances of the imaginary focus and principal focus to the diſtance of the principal focus, ſo is the diſtance of the imaginary focus to the diſtance of the real focus..*

The diſtance of the imaginary focus is *rv*, Plat. X. fig. 7. the diſtance of the principal focus is *pv*; the difference between them is

$pv - rv = rp$. So that, if rv and pv are known, vf, the diſtance of the real focus from the lens, may be thus found: as rp is to pv, ſo is rv to a fourth proportional, which is the diſtance vf. This propoſition might be deduced from prop. 66. in the ſame manner that prop. 68. is deduced from prop. 64. But a readier way of proving it is to ſhew that it is in all reſpects the ſame with prop. 68, which has been proved already, except only, that every thing is here taken on the contrary ſide, that is, this propoſition is only propoſition 68. in negative terms; or fig. 7. is the negative of fig. 6. For firſt, R in fig. 6, is a radiant from whence the incident rays diverge, and r fig. 7, is a focus on the contrary ſide of the lens, to which the incident rays converge. Secondly AB is a plano-convex lens, and ab is a plano-concave one. Thirdly, the principal focal diſtance PV is taken on the ſame ſide of the lens with the incident rays, and the principal focal diſtance pv is taken on the ſame ſide with the refracted ones. Fourthly, the refracted rays AM and VX diverge, and the refracted rays af and vf converge. Therefore fifthly, the imaginary radiant F is on the ſide of the incident rays, as the real focus f is on the ſide of the refracted ones. Since therefore, whether diverging rays fall upon a plano-convex lens, or converging ones upon a plano-concave lens, every thing is the ſame, only taken on the contrary ſides, provided the radiant in one caſe, and the imaginary focus in the other, is nearer to the lens than its principal focus, the proportions will be the ſame in both caſes; that is, ſince RP is to PV, as RV to FV; rp is likewiſe to pv, as rv to fv.

Before we leave this ſubject, we may deduce three conſequences from ſome of the foregoing propoſitions, which will be very uſeful to us in ſome of our future enquiries. Theſe conſequences are the ſubject of the propoſitions, which follow in this chapter.

71. *When a radiant is farther from a convex or plano-convex lens, than its principal focus, as the radiant approaches to the lens on one ſide, the focus departs from it on the other ſide, and on the contrary, as the radiant departs the focus approaches.*

Whilſt the radiant R, Plat. X. fig. 4, is at a greater diſtance from the lens AB than PV, which is the principal focal diſtance of the lens, as R approaches to the lens on one ſide, F, which is the focus, will depart farther from it on the other ſide.

Provided R is at a greater diſtance from the lens than P its principal focus, the refracted rays will converge, by prop. 63. The nearer R is the more the incident rays diverge, by prop. 28. Now the convergency, which the refracted rays have, is proportional to the difference between the divergency, which the incident rays have, and that which the lens will deſtroy, by what was ſaid in proving prop. 64. And the divergency, which the lens deſtroys is a given quantity, by prop. 63, for it is always juſt as much as rays would have that came from the lens's principal focus. From hence then it follows, that the more the incident rays diverge the difference between the divergency which they have, and that which the lens will deſtroy, is ſo much the leſs. Therefore the more the incident rays diverge, or the nearer the radiant is the leſs the refracted rays will converge; and conſequently the more remote will be the focus, by prop. 29. So that the nearer the radiant is to the lens, the more remote the focus will be, or as the radiant approaches, the focus will depart. In like manner, the more remote the radiant is, the leſs the incident rays will diverge, and conſequently the refracted rays will converge the more, for which reaſon the focus will be the nearer, by prop. 29, or as the radiant departs, the focus will approach.

Or otherwiſe; as RP to PV, ſo is RV to FV. by prop. 64. But as the radiant R approaches the lens RP decreaſes, and conſequently RP will bear a leſs proportion to PV. But if the firſt term bears a leſs proportion to the ſecond, the third, or RV will bear a leſs proportion to the fourth or to FV. Now when RP decreaſes it is evident that RV decreaſes; that is, when the firſt term decreaſes the third will likewiſe decreaſe at the ſame time. And if theſe two quantities RP and RV, as they decreaſe together were likewiſe to decreaſe in the ſame proportion, then becauſe PV the third term is a given quantity FV the fourth term would be a given quantity too, and therefore however the radiant was nearer to the lens, or farther from it, RP might always be to PV as RV to FV, without any alteration being made in FV, or in the focal diſtance. But RP and RV, though they decreaſe together, cannot decreaſe in the ſame proportion. For RP is a leſs quantity than RV, and it is plane, that when R approaches to the lens, RP and RV will decreaſe equally, that is, juſt as much as RP is diminiſhed, juſt ſo much RV will be diminiſhed: therefore theſe two quantities are equally diminiſhed indeed, but not in the ſame proportion: for if from two quantities, one a leſs and the other a greater, you take equal quantities, you will diminiſh the leſs more in proportion than you do the greater. So that RP decreaſes faſter than RV. Conſequently the proportion of RP to PV would decreaſe faſter than that of RV to FV, if as PV always continues of the ſame length, ſo FV was likewiſe to continue always of the ſame length. But the proportion of RP to PV is never leſs than that of RV to FV, by prop. 64. Therefore as R approaches to the lens, to preſerve the proportion between RV and PV the ſame with that between RP and RV, not only RV muſt decreaſe but FV muſt likewiſe encreaſe; or as the radiant approaches the focus will depart. That RP decreaſes faſter than RV, will appear farther by conſidering what would be the caſe, if R the radiant was at P the principal focus. In this approach of the radiant to the lens, R and P coincide; therefore RP decreaſes till it becomes nothing. But RV will not be decreaſed to nothing: for when R and P coincide RV is then equal to PV. But if, when theſe two quantities RP and RV are both of them decreaſing, if RP becomes nothing before RV does, then RP muſt decreaſe faſter in proportion than RV.

It may here not be amiſs to take notice how far the focus will have departed from the lens, when the radiant by approaching towards it has gotten to the principal focal diſtance. When R and P coincide RP=0. Therefore RP bears no proportion to PV: and conſequently RV will bear no proportion to FV. But RV is then equal to PV. Therefore when the radiant is at the principal focal diſtance, the diſtance of the focus to which the rays converge, or FV, is ſo great that PV bears no proportion to it; and conſequently FV muſt be infinite. That is when the radiant is in the principal focus of the lens, the refracted rays converge to a point at an infinite diſtance, and conſequently are parallel, by propoſition 31. And ſo we found they ought to be, by propoſition 61.

If R departs from the lens, then RP encreaſes and ſo likewiſe does RV. But RP and RV encreaſe equally, therefore they cannot encreaſe in the ſame proportion: becauſe RP is leſs than RV, and if equal quantities are added to RP and RV, this equal addition to both will make a greater encreaſe in proportion in RP the leſſer of theſe two quantities. Therefore RP encreaſes faſter in proportion than RV. Conſequently, ſince RP is always to PV, as RV to FV, if the firſt quantity encreaſes faſter than the third, to keep the proportion between RV and FV, the ſame with that between RP and PV, beſides the encreaſe of RV it will be neceſſary that FV ſhould decreaſe; or as the radiant R departs from the lens ſo as to make RV encreaſe, the focus F will approach to it, ſo as to make FV decreaſe. That RP encreaſes faſter than RV will be evident from the following conſideration. RP is leſs than RV, when they begin to encreaſe; but when R is at an infinite diſtance from V or from the lens it will likewiſe be at an infinite diſtance from P: for if RV is infinite RP muſt be infinite too; becauſe if from RV an infinite quantity,

tity you take the finite quantity PV, the remainder will ſtill be infinite; becauſe the part taken away is too ſmall in reſpect of the whole to bear any proportion to it, or to make any alteration in it by being taken away. But when RP and RV are both of them infinite they are equal to each other, and conſequently RP muſt have encreaſed faſter in proportion than RV; for, as they were unequal at firſt ſetting out, they could not have been equal at laſt, unleſs the ſmaller of the two quantities or RP had encreaſed faſter in proportion than the larger or than RV.

Here again we may obſerve when the diſtance of R, or when RV, is infinite, what will be the diſtance of F the focus. As RP to PV, ſo is RV to FV. Now when R is at an infinite diſtance, RP and RV are both of them infinite, and conſequently are equal to one another, ſo that the firſt term is then equal to the third: therefore the ſecond term or PV will be equal to the fourth or FV. Now PV is the principal focal diſtance of the lens, and FV is the diſtance of the point to which the refracted rays converge. Therefore when the radiant is at an infinite diſtance, that is, by propoſition 30, when the incident rays are parallel, the refracted rays will converge to the principal focal diſtance, as we have already ſeen they will do in the definition of the principal focal diſtance, propoſition 60.

72. *When a radiant is nearer to a convex or plano-convex lens than its principal focus, as the real radiant approaches to the lens the imaginary radiant will approach to it, and as the real radiant departs the imaginary one will depart.*

If the radiant R, Plat. X. fig. 6. is nearer to the lens AB than P its principal focus; then, by propoſition 67. the refracted rays diverge, but leſs than the incident ones did; ſo that, if RA and RV are the incident rays, the refracted rays will be AM and VX. Now ſince RA and RV diverge more than AM and VX, the real radiant R, from whence the incident rays diverge, will be nearer to the lens than the imaginary one F, from whence the refracted rays appear to diverge, by propoſition 28. And as RP is to PV, ſo is RV to FV, by propoſition 68. If in this caſe the real radiant R moves towards the lens the imaginary one F will likewiſe move towards it; and if the real radiant moves farther from the lens, provided it does not move to a greater diſtance than P or the principal focus, the imaginary one will likewiſe move from it.

The nearer the radiant R is to the lens, ſo much more the rays, which fall upon the lens, diverge from one another, by propoſition 28. But whatever their divergency is, whether it be great or ſmall, the refraction of the lens will always deſtroy juſt the ſame part of it; for, by propoſition 61, it deſtroys juſt as much divergency as rays would have that come from P the principal focus. From hence then it follows that the greater the divergency is before refraction ſo much the greater it will be afterwards; becauſe the greater the whole is, the greater likewiſe will be the remainder when a given part is taken from it. Therefore the more the incident rays diverge, or the nearer the real radiant is, the more the refracted rays will diverge, and conſequently the imaginary focus will be the nearer. So that when the diſtance of RV decreaſes, or when the real radiant R approaches to the lens, the diſtance FV will decreaſe, or the imaginary radiant will likewiſe approach it. If R departs from the lens the incident rays diverge the leſs, by propoſition 28, and conſequently the refracted rays will diverge the leſs, by what has been already ſhewn; therefore the imaginary radiant will be the more remote from the lens; or will depart from it, when the real radiant departs.

Or otherwiſe; RP is to PV as RV to FV, by propoſition 68. Now as R approaches to V, RP will bear a greater proportion to PV; becauſe PV being the principal focal diſtance of the lens, is therefore, in all the motions of R, an invariable quantity. But when the proportion of RP to PV encreaſes, the proportion of RV to FV muſt likewiſe encreaſe. Becauſe at all times whatever proportion RP bears to PV, RV bears the ſame proportion to FV. Conſequently when R approaches the lens, the proportion of RV to FV muſt encreaſe. But when R approaches the lens, RV or its diſtance from the lens decreaſes: from whence it follows that the proportion of RV to FV cannot encreaſe, unleſs when RV decreaſes FV was to decreaſe. Therefore, ſince this proportion does encreaſe, whilſt RV decreaſes FV will decreaſe likewiſe: or when the real radiant R approaches the lens, the imaginary one F will likewiſe approach it. In like manner, when R moves from the glaſs PR decreaſes; therefore in this motion of the radiant RP will bear a leſs proportion to PV; and conſequently RV, though it encreaſes, will bear a leſs proportion to FV. But this would be impoſſible unleſs FV was to encreaſe. Therefore when RV encreaſes or when the real radiant moves from the lens, FV will likewiſe encreaſe or the imaginary radiant will move from the lens at the ſame time.

73. *If the real radiant is cloſe to a convex or plano-convex lens, the imaginary radiant will likewiſe be cloſe to it.*

When R, or the point from whence the incident rays diverge, Plat. X. fig. 6. touches V the vertex of the lens, then F, or the point from whence the refracted rays appear to diverge will likewiſe touch the lens at V.

The radiant R, when it is at V will be between the lens and the principal focus. Therefore, by propoſition 68, as RP to PV, ſo is RV to FV. But when R is cloſe to V, the diſtance of it from P is equal to the diſtance of V from P, or RP is equal to PV. Therefore RV will likewiſe be equal to FV. Now when R touches the lens at V the diſtance of R from V or RV is nothing, and conſequently, the diſtance of F from V or FV will be nothing, that is, F will likewiſe touch the lens at V.

CHAP. VI.

Of the refraction of converging rays by a convex lens and of diverging rays by a concave lens.

WE have in the laſt chapter conſidered what effect a convex or plano-convex lens will have upon ſuch rays as diverge at their incidence; and what effect a concave or plano-concave lens will have upon ſuch as converge. It ſtill remains for us to conſider what effect theſe lenſes will produce in the oppoſite caſes, that is, what will be the effect of a convex or plano-convex lens, when the incident rays converge, and what the effect of a concave or plano-concave one, when they diverge.

74. *If rays converge, when they fall upon a convex or plano-convex lens, they will converge more, after they have paſſed through it and have been refracted.*

If the rays MA and XV, Plat. X. fig. 6. converge to the point F, but before they arrive there, are to paſs through a plano-convex lens AB, theſe rays will not meet at F after they are refracted, but will converge more than they did at their incidence and will meet at R. For, by propoſitions 67, 68, if the rays RA and RV were to diverge from R, they would be refracted into the lines AM and VX by paſſing through the lens, and would diverge leſs than they did at their incidence, juſt as if they had come ſtrait forwards from the point F. Therefore if theſe rays AM and VX are turned back again in the lines MA and XV ſo as to converge to F, they would be refracted into their firſt directions AR and VR, ſo as to be made to converge more and to meet at the point R, by propoſition 41. In like manner, any other rays, which diverged before refraction, and diverge likewiſe after it, will diverge leſs, by propoſition 67. And all converging rays may be conſidered as ſuch rays turned back again; and, ſince all ſuch rays, when turned back again, will be refracted

ted into the lines, which they deſcribed at their incidence, by propoſition 41, therefore all converging rays, when they have paſſed through the lens, will be made to converge more.

75. *When converging rays paſs through a convex or plano-convex lens; as the ſum of the diſtances of the principal focus and imaginary focus is to the diſtance of the imaginary focus, ſo is the diſtance of the principal focus to the diſtance of the real focus.*

If MA and XV are the incident rays, they converge to F; and, becauſe the refraction, which they ſuffer in paſſing through the lens, will prevent them from meeting there, F is only an imaginary focus; the refracted rays will meet at R, and conſequently R is the real focus. Now if FV the diſtance of the imaginary focus, and PV the principal focal diſtance are known, RV the diſtance of the real focus may be thus found, as PV+FV, is to FV, or as the diſtance of the principal focus added to the diſtance of the imaginary focus is to the diſtance of the imaginary focus, ſo is PV to RV, or ſo is the diſtance of the principal focus to the diſtance of the real focus. That R will be the real focus, if F is the imaginary one, appears from propoſitions 68, 41. For, if the incident rays RA and RV had diverged from R, the refracted rays AM and VX would diverge from F, by propoſition 68. And if the rays AM and VX are turned back again in the lines MA and XV ſo as to converge to F, they will be refracted into the lines of their former incidence AR and VR, becauſe all refraction is reciprocal, by propoſition 41. Now, by propoſition 68, RP is to PV, as is RV to FV. Therefore PV one conſequent is to FV the other conſequent, as RP one antecedent is to RV the other antecedent. Euc. b. V. prop. 16. And ſince PV is to FV, as RP to RV, PV+FV is to FV, as RP+RV is to RV. Euc. b. V. prop. 17. But RP+RV is equal to PV. Therefore PV+FV is to FV, as PV is to RV.

76. *If rays diverge, when they fall upon a concave or plano-concave lens, they will diverge more, after they have paſſed through it and have been refracted.*

If there is any radiant at *f*, Plat. X. fig. 7. from which the rays *fa* and *fv* diverge; when they have been refracted by paſſing through the plano-concave lens *ab*, they will go on in the lines *am* and *vx* ſo as to diverge more, and will have the ſame direction in reſpect of each other, as if they had come ſtrait forwards from a radiant at *r*. If the rays *ma* and *xv*, which converge to the focus *r* before refraction, were to paſs through the lens, they would converge leſs after refraction and would meet at the focus *f*, by propoſitions 69, 70. And if theſe rays were to be returned back from *f*, in the lines *fa* and *fv* ſo as to diverge from *f*, they would be refracted back into their former directions *am* and *vx*, by propoſition 41. In like manner, all rays, which converge at their incidence, and converge likewiſe after they are refracted, will converge leſs after refraction than they did before it: and all rays, which diverge at their incidence, may be conſidered as ſuch rays turned back again, and ſuch rays, when turned back again, will be refracted into their firſt directions, and ſo be made to diverge more. Therefore all rays, which diverge at their incidence, will likewiſe be made to diverge more, when they have paſſed through the lens *ab*.

Since the refracted rays *am* and *vx* diverge more than the incident ones *fa* and *fv*, the imaginary radiant *r* will be nearer to the lens than the real radiant *f*, for the diſtance of the radiant is inverſely as the divergency of rays, by propoſition 28.

77. *When diverging rays paſs through a concave or plano-concave lens; as the ſum of the diſtances of the principal focus and real radiant is to the diſtance of the real radiant, ſo is the diſtance of the principal focus to the diſtance of the imaginary radiant.*

If *f*, Plat. X. fig. 7. is the radiant from whence the rays come, or the real radiant, then its diſtance from the lens is *fv*: the principal focal diſtance of the lens is *pv*. When theſe two are known the diſtance of *r* the imaginary radiant, or point from whence the refracted rays will appear to come, that is, the diſtance *rv* may be thus found. As $pv+fv$ is to *fv*, or as the principal focal diſtance added to the diſtance of the real radiant, is to the diſtance of the real radiant, ſo is *pv* to *rv*, or ſo is the principal focal diſtance to the diſtance of the imaginary radiant. That *r* will be the imaginary radiant, if *f* is the real one, follows from propoſitions 70, 41. For if the incident rays *ma* and *xv* had converged to *r* the refracted rays *af* and *vf* would have converged to *f*. Therefore, if the refracted rays *af* and *vf* were to be turned back again, and ſo were to diverge from *f* in the lines *fa* and *fv*, they would be refracted back into the lines of their firſt direction *am* and *vx*, by propoſition 41. Now by propoſition 70, *rp* is to *pv* as *rv* to *fv*. Therefore *pv* is to *fv* as *rp* to *rv*. Euc. b.V. prop. 16. And $pv+fv$ is to *fv* as $rp+rv$ is to *rv*. Euc. b.V. prop. 17. But $rp+rv$ is equal to *pv*. Therefore $pv+fv$ is to *fv* as *pv* is to *rv*.

This propoſition might have been proved otherwiſe, for it might be deduced from propoſition 75: becauſe in effect it is the ſame as propoſition 75, only every thing is taken on the contrary ſide; that is, this propoſition is only propoſition 75 in negative terms, or fig. 7. is the negative in all reſpects of fig. 6. For firſt, F, in fig. 6, is a focus to which the incident rays converge, and *f*, fig. 7. is a radiant on the contrary ſide of the lens, from which the incident rays diverge. Secondly AB is a plano-convex lens, and *ab* is a plano-concave one. Thirdly, the principal focal diſtance PV is taken on the ſide of the refracted rays, and the principal focal diſtance *pv* is taken on the ſide of the incident rays. Fourthly, the refracted rays AR and VR converge, and the refracted rays *am* and *vx* diverge. Therefore fifthly, the real focus R is on the ſide of the refracted rays, and the imaginary radiant *r* is on the ſide of the incident ones. Since therefore whether converging rays fall upon a plano-convex lens, or diverging rays upon a plano-concave lens, every thing is the ſame only taken on the contrary ſides, the proportions will be the ſame in both caſes; and, becauſe PV+FV is to FV as PV is to RV, *pv*+*fv* is to *fv*, as *pv* to *rv*. Or, ſince the propoſition is true, when its terms are affirmative, it will likewiſe be true, when they are negative.

There are two conſequences from what has been ſaid in relation to concave and plano-concave lenſes, which will be of uſe to us hereafter; and therefore we will take notice of them diſtinctly in the two following propoſitions.

78. *When diverging rays fall upon a concave or plano-concave lens, if the real radiant moves either towards the lens or from it the imaginary radiant will move the ſame way.*

If *f*, which is the real radiant, Plat. X. fig. 7, approaches to the lens *ab*, *r*, which is the imaginary radiant will likewiſe approach to it; or if *f* departs from the lens, *r* will depart from it. For the nearer *f* is to the lens ſo much more the incident rays diverge, by propoſition 28. The more the incident rays diverge the more the refracted rays will diverge. And the more the refracted rays diverge, the leſs will be the diſtance of *r*, or of the radiant from which they appear to come, by propoſition 28. Therefore as *f* approaches the lens, *r* will move the ſame way. In like manner the farther *f* is from the lens the leſs the incident rays will diverge, and conſequently the refracted rays will likewiſe diverge the leſs, therefore *r*, the imaginary radiant from which they appear to diverge will be the more remote, by propoſition 28. Therefore as *f* departs from the lens, *r* will move the ſame way.

Or otherwiſe *pv*+*fv* is to *fv*, as *pv* is to *rv*, by propoſition 77. Now when *f* approaches to the lens *fv* decreaſes, and conſequently *pv*+*fv* muſt decreaſe at the ſame time. But ſince *pv* or the principal focal diſtance

ſtance is always the ſame, both $pv+fv$ and fv will decreaſe only juſt as fv decreaſes. That is in the approach of f to the lens $pv+fv$ and fv will decreaſe equally. But if from two unequal quantities ſuch as $pv+fv$ and fv, equal parts are taken, the leſſer quantity or fv will decreaſe faſter in proportion to the whole than the greater quantity or $pv+fv$. Now if in the approach of f to the lens fv decreaſes faſter in proportion than $pv+fv$, it follows that fv will bear a leſs proportion to $pv+fv$, or that the proportion of fv to $pv+fv$ will decreaſe. But ſince it is always as $pv+fv$ to fv ſo pv to rv, when the proportion of the ſecond term to the firſt decreaſes, the proportion of the fourth term to the third, or of rv to pv muſt decreaſe likewiſe. Therefore in the approach of f to the lens, rv will bear a leſs proportion to pv. But pv is an invariable quantity, and conſequently rv cannot bear a leſs proportion to it unleſs rv decreaſes. Therefore in the approach of f to the lens rv or the diſtance of r from the lens will decreaſe, that is r will likewiſe approach the lens. In like manner, when f departs from the lens, or when fv encreaſes, $pv+fv$ will encreaſe at the ſame time. But then becauſe pv is a given quantity, $pv+fv$ and fv will encreaſe equally: and ſince fv is a leſs quantity than $pv+fv$, where an equal addition is made to both of them, this equal addition will make fv or the leſſer quantity encreaſe more in proportion to the whole, than $pv+fv$ or the greater quantity. But if fv encreaſes faſter than $pv+fv$, ſuch an encreaſe will make fv bear a greater proportion to $pv+fv$; that is, in the departure of f from the lens the proportion of fv to $pv+fv$ encreaſes. But ſince as $pv+fv$ to fv, ſo is pv to rv, if the proportion of the ſecond term to the firſt, or of fv to $pv+fv$ encreaſes, the proportion of the fourth to the third or of rv to pv muſt encreaſe likewiſe. Now pv or the third term is the principal focal diſtance of the lens, and conſequently is an invariable quantity: ſo that the proportion which rv bears to it cannot encreaſe unleſs rv encreaſes. Therefore in the departure of f or the real radiant from the lens, rv or the diſtance of the imaginary radiant from the lens will encreaſe, that is r will likewiſe depart from the lens at the ſame time.

79. *If the real radiant is cloſe to a concave or plano-concave lens, the imaginary radiant will likewiſe be cloſe to it.*

If f, Plat. X. fig. 7, the real radiant, or the point, from which the incident rays come, is cloſe to the plano-concave lens ſo as to touch it at v, then r the imaginary radiant, or point, from which the refracted rays appear to come, will likewiſe touch it at the ſame point. Or, when fv the diſtance of the real radiant from the lens is nothing, then rv the diſtance of the imaginary radiant will be nothing. For as $pv+fv$ to fv ſo is pv to rv, by prppoſition 77. Now when f touches the lens, fv is nothing: therefore $pv+fv$ is to fv, as $pv+0$ is to 0, and conſequently pv will be to rv as pv to 0, or when the ſecond term in this proportion or fv is nothing, the fourth term or rv will likewiſe be nothing, that is when the real radiant f touches the lens, the imaginary radiant r will touch it too.

CHAP. VII.

Of the principal focus of meniſcuſes, and entire ſpheres of either glaſs or water.

80. *In a meniſcus, as the difference between the ſemidiameters of the two ſurfaces, to the ſemidiameter of either, ſo is double the ſemidiameter of the other to the principal focal diſtance.*

A Meniſcus has one of its ſides convex and the other concave. So that if the plane ſides of the plano-convex lens AB, Plat. X. fig. 8. and of the plano-concave one CD were cloſe together they would make the meniſcus EF or GH; for EF and GH differ in ſhape from one another only upon account of the different thickneſs of the glaſs. From hence it is evident that the properties of a meniſcus are the ſame either with thoſe of a plane glaſs, or with thoſe of a convex lens, or with thoſe of a concave one. For firſt; if the convexity on one ſide is exactly equal to the concavity on the other, then theſe two ſides will produce in the refracted rays, contrary effects and in an equal degree; ſo that the direction of the rays, when they have paſſed through the meniſcus, will be the ſame that it was, before they entered it. And this, as was ſhewn in propoſition 47, is the property of a plane glaſs. Or, when the convexity on one ſide of the meniſcus is equal to the concavity on the other, the two ſurfaces of the meniſcus are parallel to each other, as the two ſurfaces of a plane glaſs are, ſo that the meniſcus is nothing but a plane glaſs that is bent. Of this ſort are watch-glaſſes and the ſide of a glaſs decanter, which in refracting the rays of light produce no more alteration in the direction of them, than any other plane glaſs would have done. Secondly, if the convexity is greater than the concavity, then likewiſe the two ſurfaces of the meniſcus produce contrary effects, but not in an equal degree; for the convex ſurface refracts the rays more than the concave one; and therefore the meniſcus will have all the properties of a convex lens. And thirdly, if the concavity is greater than the convexity, then the refraction will be greater at the concave ſurface than at the convex one, and conſequently the meniſcus will have the properties of a concave lens. From hence it follows that when the convexity is greater than the concavity, if the incident rays were parallel, the refracted ones will converge, by propoſition 53, and the focus will be a real or affirmative one. But if the concavity is greater than the convexity, then the refracted rays will diverge, by propoſition 54, and the focus will be a virtual or negative one. The diſtance of this focus whether affirmative or negative is the principal focal diſtance of the meniſcus; and the rule for finding it is this, which is laid down in the propoſition; as the difference between the ſemidiameters of the convexity and concavity is to the ſemidiameter of either, ſo is double the ſemidiameter of the other to the principal focal diſtance.

This propoſition does not want to be demonſtrated, becauſe it is the ſame in effect with propoſition 55. Here, in the meniſcus, the firſt term in the proportion is the difference between the ſemidiameters of the two ſurfaces; and in the convex lens it was the ſum of them. But the reader will eaſily underſtand that this makes no material alteration in the propoſition, if he recollects that both ſurfaces of a convex lens are of the ſame ſort, and that the ſemidiameters of theſe ſurfaces are both of them affirmative; whereas the two ſurfaces of a meniſcus are not of the ſame ſort, for one of them is convex and the other concave, and conſequently, if the ſemidiameter of the convexity is conſidered as affirmative, the ſemidiameter of the concavity muſt be conſidered as negative. Thus in a convex lens if the ſemidiameter of one convexity is called r, and that of the other s, then both r and s are affirmative and one of them added to the other makes r+s, which is the ſum of theſe two ſemidiameters. But in a meniſcus, if the ſemidiameter of the convexity is called r, and the ſemidiameter of the concavity is called s, then r is affirmative and s is negative, that is, s muſt have the negative ſign before it, and the notation of it or manner of expreſſing it is by this mark —s. In this caſe therefore, when r and —s are added together they make r—s, which is the difference between r and s. So that the firſt term, both in propoſition 55, and in this, is the ſame in effect; for it is the two ſemidiameters added together: but the two ſemidiameters added together is the ſum of them, when they are both affirmative, and is the difference between them, when only one of them is affirmative, and the other negative. Upon the whole therefore this propoſition may be conſidered as one caſe of propoſition 55, or a meniſcus may be conſidered as a ſort of convex lens having the convexity affirmative at one ſurface and negative at the other.

Now when the incident rays are parallel, the refracted rays will converge and the focus will be affirmative, if the convexity of the meniſcus is greater than its concavity; but they will diverge and the focus will be negative, if the concavity is greater than the convexity. If the ſemidiameter of the convexity is called r, as before, and the ſemidiameter of the concavity is called —s; then the more convex one ſide of the meniſcus is, ſo much leſs is the ſphere of which that ſide is a ſegment; for any given part in a ſpheres ſurface is ſo much the more convex as the ſphere is leſs: and conſequently the more convex one ſide is, ſo much the ſhorter will be the ſemidiameter of the ſphere of which that ſide is a ſegment, or the quantity r will be ſo much the leſs. And for the ſame reaſon the leſs convex this ſide of the meniſcus is, the greater will be the ſemidiameter of the convexity or the quantity r. In like manner the more concave one ſide of the meniſcus is, the leſs is the ſemidiameter of the concavity or —s, and the leſs the concavity the greater is —s. Now when r is greater than —s, r—s will be affirmative. Thus, for inſtance, if r or the ſemidiameter of the convexity is 10 inches and —s, or the ſemidiameter of the concavity is 6 inches. $r - s = 10 - 6 = 4$. In this caſe r—s or 4 is an affirmative quantity: for in all ſubſtraction if a leſs quantity is taken from a greater the remainder is affirmative. But when the ſemidiameter of the convexity is greater than that of the concavity, or when r—s is affirmative, the convex ſurface is a ſegment of a greater ſphere than the concave one, and conſequently the convexity on one ſide of the meniſcus will be leſs than the concavity on the other. And when the convexity is leſs than the concavity, the meniſcus will have the effect of a concave lens, as appears from what has been already proved. Therefore when r—s is affirmative, the rays, if they were parallel at their incidence, will diverge after refraction and the focus will be negative. But when r is leſs than —s, r—s is a negative quantity: for if an equal quantity is ſubſtracted from an equal one the remainder will be nothing, therefore if a greater quantity is taken from a leſs the remainder will be leſs than nothing. Thus if r is 10 inches, and —s is 12 inches, $r - s = 10 - 12 = -2$. Now if the ſemidiameter of the convexity or r is leſs than that of the concavity or than —s, the convex ſurface of the meniſcus is a ſegment of a leſs ſphere than the concave one; and conſequently the convexity on one ſide is greater than the concavity on the other; ſo that the meniſcus will have the effect of a convex lens. Therefore when r is leſs than —s, or when r—s is negative, the rays, if they were parallel at their incidence will converge after refraction, and the focus will be affirmative.

The reader, if he underſtands the firſt rules of algebra, will eaſily find that this follows from the rule laid down in the propoſition for finding the principal focus of a meniſcus. For, calling the principal focus x, as $r-s$ is to r, ſo is $-2s$ to x, by the propoſition; or; ſince the ſecond term may be either ſemidiameter, either r or $-s$ and then the third muſt be double the other; as $r-s$ is to $-s$, ſo is $2r$ to x. The quantity x is to be found by the rule of three, by multiplying the firſt term into the ſecond, and then dividing the product by the firſt, for the quotient of this diviſion is x. Now here we muſt obſerve that in a given meniſcus the product of the firſt term multiplyed into the ſecond is the ſame in either notation for either $r\times-2s$ or $-s\times2r$ produces $-2rs$: and farther we muſt obſerve that the product of the ſecond term multiplyed into the third is always negative; becauſe one term is always affirmative and the other negative. But whether x is negative or affirmative depends upon the nature of the firſt term $r-s$, by which this negative product $-2rs$ is to be divided in order to find out the quantity x. Now if $r-s$ is affirmative, the quotient x ariſing from $-2rs$ divided by $r-s$ will be negative, becauſe all quotients are ſo, where a negative quantity is divided by an affirmative one. But if $r-s$ is negative, the quotient x will be affirmative, becauſe all quotients are ſo, where a negative quantity is divided by a negative one. From hence therefore it appears, as before, that if r is greater than $-s$, or the ſemidiameter of the convexity is greater than that of the concavity, that is, if $r-s$ is affirmative, the rays, that were parallel when they fell upon the meniſcus, will diverge after refraction, and the principal focus will be a negative or virtual one. But if r is leſs than $-s$, or the ſemidiameter of the convexity is leſs than that of the concavity, that is, if $r-s$ is negative, the rays, that were parallel when they fell upon the meniſcus, will converge after refraction, and the focus will be an affirmative or real one.

When we have thus determined the principal focal diſtance of a meniſcus, or ſhewn what effect ſuch a glaſs will have upon parallel rays, it would be tireſome as well as needleſs to ſhew what effect it would have upon diverging or converging ones. For ſince a meniſcus, unleſs the ſurfaces of it are parallel to one another, has the ſame effect, either that a convex lens or a concave one would have, all the caſes of diverging or converging rays, that are refracted by it, will be the ſame with thoſe already explaned in the inſtances of convex and concave lenſes.

81. *When rays, that were parallel, have paſſed out of the air and gone through a ſphere of glaſs, they will converge to a focus, the diſtance of which from the ſphere will be equal to half its ſemidiameter.*

If AGN, Plat. X. fig. 9. is a glaſs ſphere, and QA, BC are two parallel rays, or two rays which come from the ſun; the ray BC continued paſſes through E the center of the ſphere; and conſequently BC is perpendicular to the ſurface, upon which account it ſuffers no refraction as it enters the ſphere at C; and for the ſame reaſon likewiſe it ſuffers no refraction when it goes out of it at D: ſo that the ray BC paſſes ſtrait forwards through the ſphere in the direction BT; by propoſition 41. But the ray QA which falls obliquely upon the ſurface of the ſphere at A, as it paſſes out of air into glaſs, will be refracted, and will be ſo made to converge to its axis, that if it was to proceed in the direction AGT, which it acquires by this refraction, it would meet its axis, or the focus would be, at T, by propoſition 48. And upon the ſuppoſition here made, that the ray is only once refracted CT, or the diſtance of the focus from the ſurface where the rays enter, is equal to three ſemidiameters of the ſphere, by propoſition 50.

Now if CT is three ſemidiameters of the ſphere, CD or the diameter contains two of them, and conſequently DT, or the focus produced by the firſt refraction, is at the diſtance of a ſemidiameter from the ſecond ſurface of the ſphere. But this focus T is only an imaginary one, for the rays, though they converge to T after their firſt refraction, will never meet there; becauſe when AG or any other oblique ray comes out of the glaſs into the air it will be refracted a ſecond time, and this ſecond refraction will make it converge more: ſo that the ray AG, inſtead of going ſtrait forwards in the direction GT, will be refracted at G and will deſcribe the line GF ſo as to meet its axis at F.

The rule for finding DF or the diſtance of the real focus after the rays are thus refracted a ſecond time is this; divide DT the diſtance of the imaginary focus from the ſphere into two equal parts, and DF or half DT will be the diſtance of the real focus.

Continue the incident ray QA to H. Since QAH is the direction of the incident ray, and AG is its direction after it is once refracted at entering the ſphere, the angle HAG contained between the incident and refracted ray is the angle of refraction. Now when the refracted ray AG goes out of the ſphere at G, it paſſes out of glaſs into air, as it paſſed out of air into glaſs, when it entered it at A; and becauſe the two mediums are the ſame in both caſes, the ray as it paſſes out of the ſphere will be juſt as much refracted from a perpendicular as it was refracted towards one upon its entrance. If this is not ſelf-evident it may be ſhewn to be true from the following conſiderations. Suppoſe the ray AG, inſtead of paſſing out of the ſphere at G, to be turned back again in the direction GA and to paſs out at A. Upon this ſuppoſition it will be juſt as much refracted in paſſing out at A as it was at its entrance, by propoſition 41. But upon this ſuppoſition the ray paſſes out of the glaſs ſphere into the air, and ſo it would have done, if it had not been turned back again, but had paſſed out at G. Therefore the ray would be equally refracted, whether it was turned back again and paſſes out at A, or is not turned back again and paſſes out at G: for in both caſes it paſſes out of glaſs into air and through the ſurface of the ſame ſphere. But if it was turned back again, the refraction, when it paſſes out at A, would be equal to the refraction, when it entered the ſphere at A. Therefore the refraction, when it paſſes out at G, is likewiſe equal to the refraction which it ſuffered upon entering the ſphere at A. That is; the ray AG when it paſſes out of the ſphere does not go ſtrait forwards in the direction GT, but is refracted into ſome line GF ſo as to make the angle of refraction TGF equal to the angle of refraction HAG. But ſince QH and BT, the directions of the rays at their incidence upon the ſphere, are parallel, by the ſuppoſition, and AT croſſes theſe two parallel lines ATF, or GTF which is the ſame angle, is equal to the alternate angle HAG. Euc. b. I. prop. 29. Therefore ſince TGF and GTF are each of them equal to HAG they are equal to one another; that is, the angles at the baſe of the triangle GFT are equal; and conſequently the ſides FG and FT, which ſubtend theſe equal angles, are likewiſe equal. Euc. b. I. prop. 6. But when G and D are very near each other DF is nearly equal to GF. Therefore DF is likewiſe equal to FT, or DF is half DT, or the focus F divides DT into two equal parts. Now if the ſphere is glaſs DT, as has been proved, is a ſemidiameter of the ſphere, and conſequently DF, or the focal diſtance is half a ſemidiameter.

82. *When rays that were parallel at firſt, have paſſed out of the air, and have gone through a ſphere of water, they will converge to a focus, the diſtance of which from the ſphere will be equal to its ſemidiameter.*

If AGN, Plat. X. fig. 9. is a hollow ſphere of glaſs, and this ſphere is filled with water, then ſince AC or GD or any other part in the ſide of the glaſs is a meniſcus whoſe convexity on one ſide is equal to its concavity on the other, the rays of light as they paſs through any part of the glaſs will not have their direction at all altered, by propoſition 80. And conſequently the water contained within the glaſs may be conſidered as a ſphere of water encompaſſed with air. Now parallel rays as QA, BC, paſſing out of air into water through the convex ſurface AC, ſince they paſs out of a rarer medium into a denſer, will be made to converge, by propoſition 48: and the diſtance CT or the diſtance of the focus to which the rays converge by this firſt refraction is thus determined, by propoſition 50, as the ſine of refraction is to the ſine of incidence, ſo is the ſemidiameter of the ſphere to DT the focal diſtance. But the ſine of refraction is to the ſine of incidence in the paſſage of a ray out of air into water, as 1 to 4. Therefore the ſemidiameter is to the focal diſtance, after the rays have been once refracted, as 1 to 4; that is, CT the diſtance of the focus is equal to 4 times CE or to 4 ſemidiameters or 2 diameters of the ſphere. Now CD is one diameter, and conſequently DT will be another diameter. But when the ray comes out of the water at G, it will be refracted a ſecond time and will be made to converge more; ſo that it will croſs its axis DT nearer to the ſphere than T, at ſome point F, which divides DT into two equal parts, which may be ſhewn by what was ſaid in propoſition 81. Therefore DF will be half DT; and, ſince DT is a diameter, DF, or the focal diſtance after both refractions, which is half DT, muſt be a ſemidiameter of the ſphere.

The body of ſome glaſs decanters is pretty nearly ſpherical, and the water contained in it is a ſphere of water. Therefore from what has been ſaid it appears that parallel rays, or the rays of the ſun, by paſſing through ſuch a decanter full of water will be collected into a focus behind it at the diſtance of a diameter of the ſphere, in the ſame manner that a convex lens made of glaſs would collect them: and any ſuch bodies as eaſily take fire will be burned by thoſe rays, if the bodies are placed at that focus.

A lens of water may likewiſe be contrived which would collect the rays of the ſun and kindle fire in the ſame manner that a burning glaſs does. For if each ſurface of the lens AB, Plat. IX. fig. 6. was a watch glaſs with its convexity outwards and the edges, where the two glaſſes are joyned, were cemented together ſo as to make a hollow lens; and if this hollow lens was to be filled with water, the contents would be a lens of water. The two glaſſes ACB, ADB, are meniſcuſes of ſuch a ſort as would make no alteration in the direction of the rays that paſs through them, as was proved in propoſition 80. But the water, that is between the lenſes, being denſer than the air would make the rays of the ſun converge for the ſame reaſons that we have already ſeen a lens of glaſs would produce this effect. The only difference would be, that ſince the denſity of water is leſs than that of glaſs, rays by paſſing through a lens of water will be leſs refracted, and will therefore converge leſs, and conſequently, by propoſition 29, will have their focus more remote than if they had paſſed through a lens of glaſs. It would not be worth our while to enquire how much more remote the focus would be; all that we wanted to prove was that the rays of the ſun may be made to burn by paſſing through cold water.

CHAP. VIII.

Of the pictures in a dark room.

83. *Rays of light, which flow from ſeveral points in any object, and then paſs through a convex or plano-convex lens, will afterwards converge to ſo many other points, provided the object is at a greater diſtance from the lens than its principal focus.*

IF ABC, Plat. XI. fig. 1. is a luminous object, or object that emits light; ſuch as the ſun is, or the moon, or the flame of a candle; all the points in this object, as A, B, C, and all the intermediate points, are radiants, they emit light or ſend it out in all directions, ſo that from each of theſe points beams of light diverge. Thus from the point A innumerable rays diverge in all directions; amongſt which are the rays AG, AH, and AK. The ſame may be ſaid of all the other points in the whole object from A, which is the higheſt, to C, which is the loweſt. Or, if ABC is not a luminous object, that is, if it does not emit light, as ſuppoſe it to be a croſs, or a tree, or a houſe, or a man, yet when the ſun ſhines upon it, or in open day-light though the ſun does not ſhine out, or when it is ſtrongly enlightened by candles or by any other means, it will reflect the light, that falls upon it, from every point; and therefore in this caſe likewiſe, as the reflected light diverges from every point in the object, each point from A the higheſt to C the loweſt will be a radiant. Now if ML is a dark chamber, GK a hole in the window-ſhutter, and GHK a convex lens placed in the hole, the rays, which diverge from A, or B, or C, or any other radiant in the object will be made to converge by paſſing through the lens, provided the object is at a greater diſtance from the lens than the lens's principal focus, by propoſition 64. Amongſt the rays BG, BH, and BK, one ray as BH is the axis, and this will paſs through the lens without being refracted, as was proved in propoſition 48. But the collateral rays AG and AK, which are made to converge, will go on after refraction in the directions GE and KE ſo as to croſs their axis at E, that is, all the rays, which come from the point B in the object, will be united behind the lens or within the room in the point or focus E. In like manner amongſt the rays, AG, AH, and AK, which diverge from one and the ſame point A, though the ray AH, which is the axis, is refracted in paſſing through the lens, yet the direction of it is not altered by the refraction, for as the refracted ray is parallel to the incident one, the ray may be conſidered as if it went ſtrait through the lens,

PLATE X.

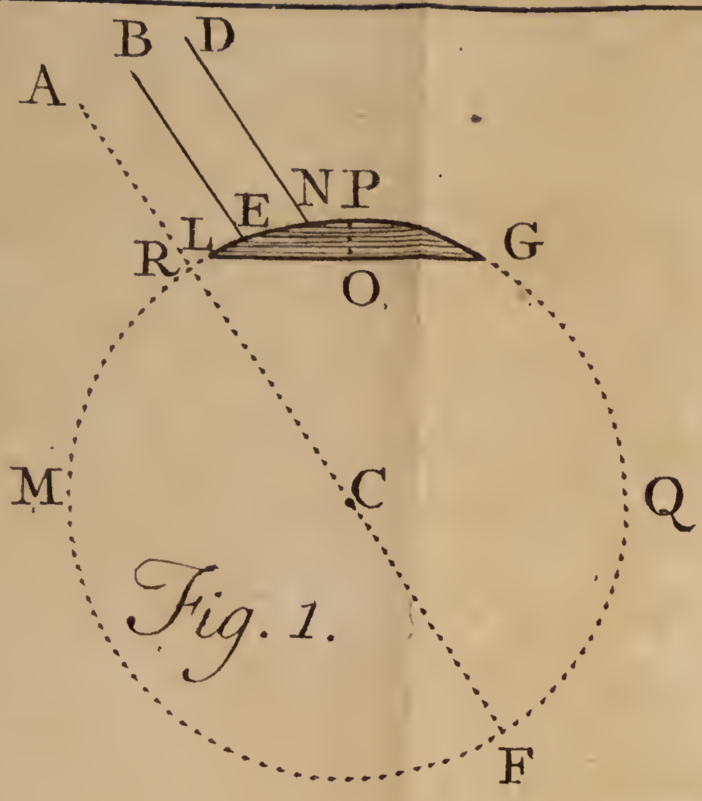

Fig. 1.

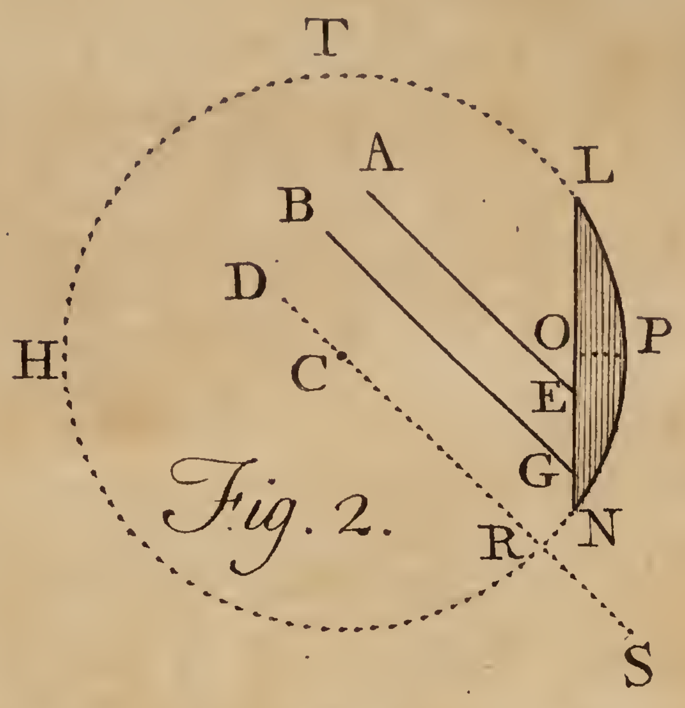

Fig. 2.

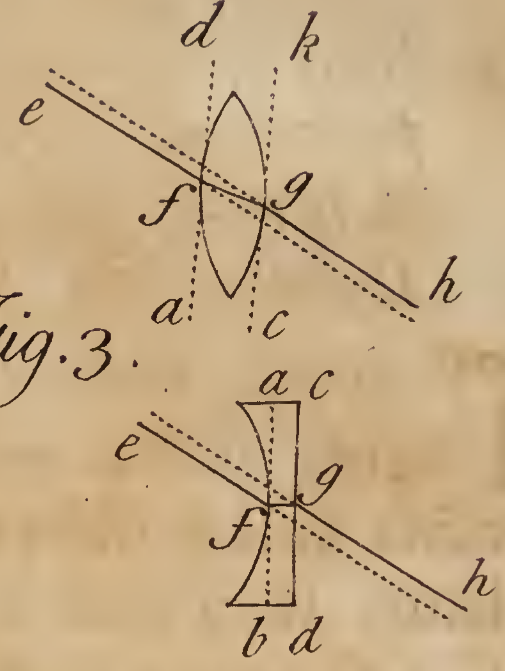

Fig. 3.

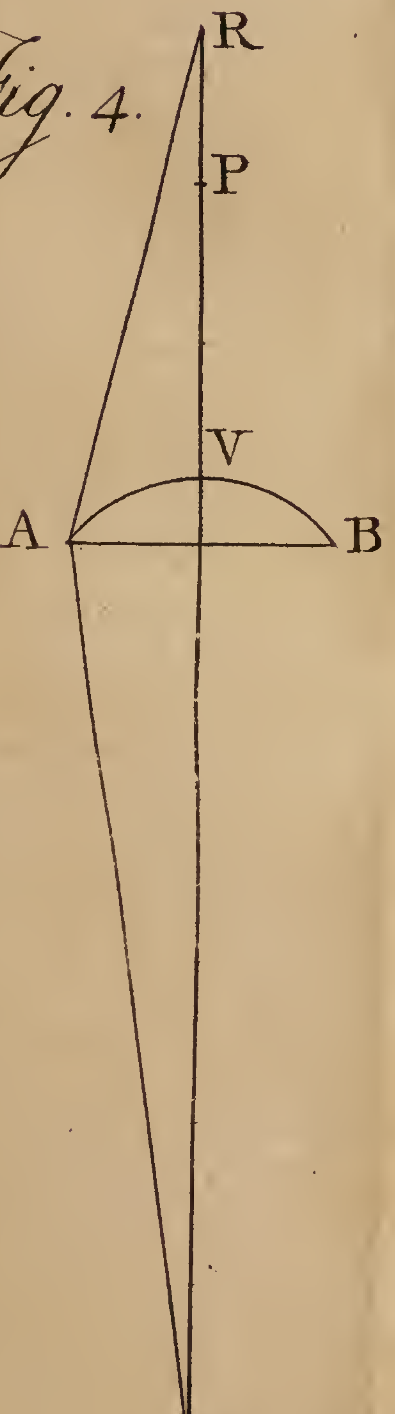

Fig. 4.

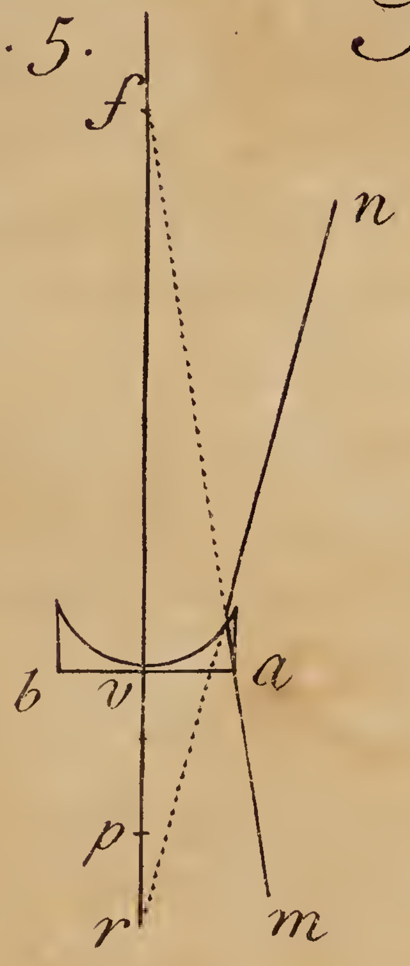

Fig. 5.

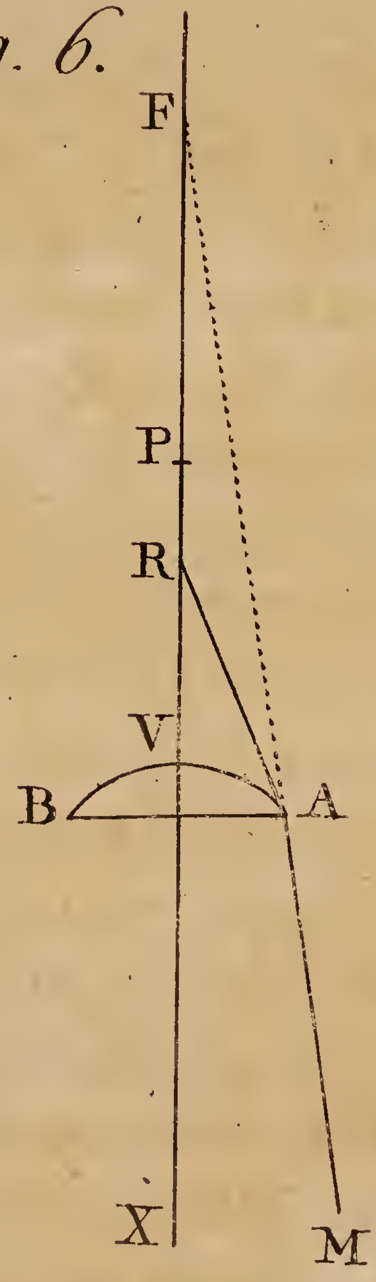

Fig. 6.

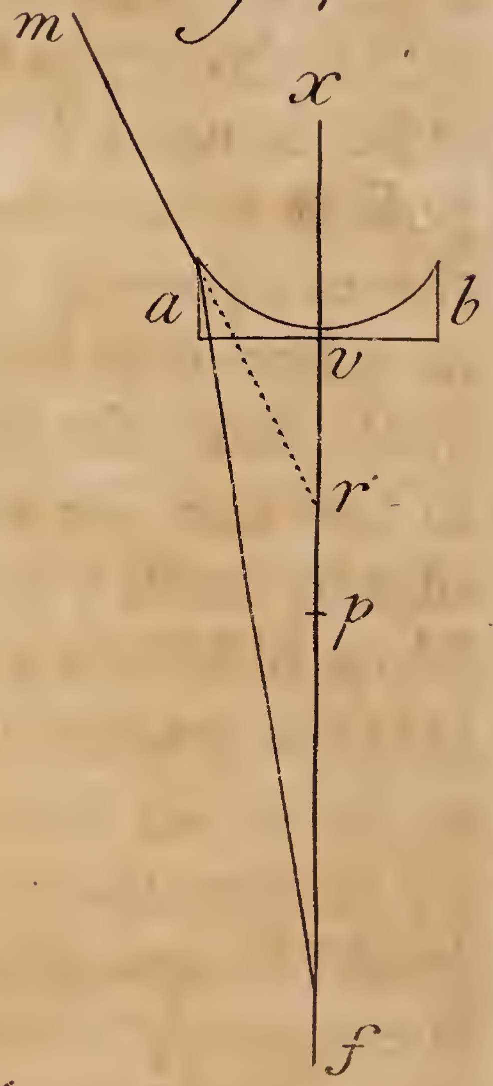

Fig. 7.

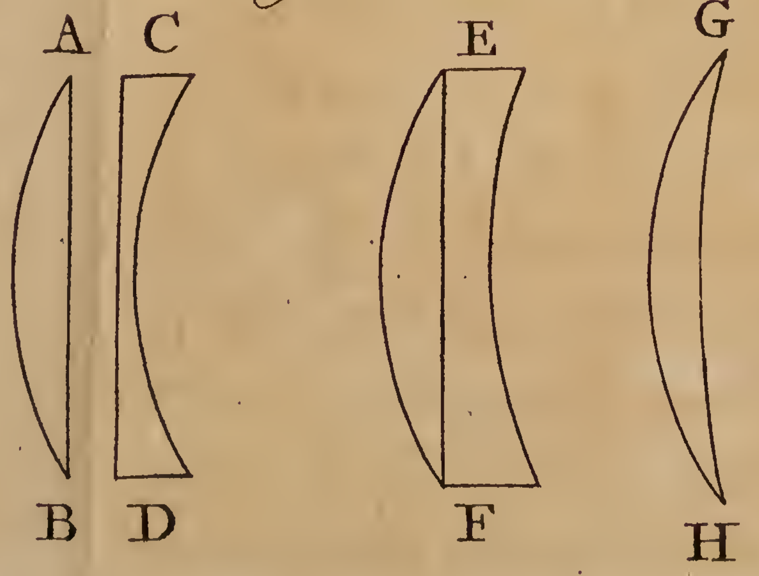

Fig. 8.

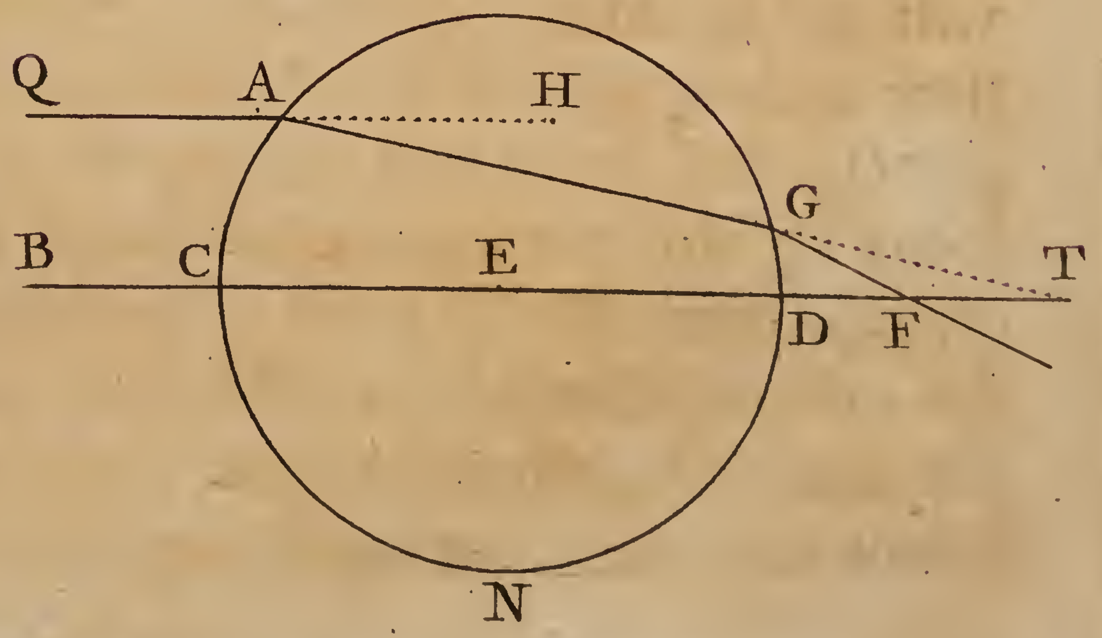

Fig. 9.

as was ſhewn in propoſition 59. But any other rays proceeding from the ſame point, ſuch as AG, and AK, will be made to converge to their axis and will be united in a focus or point at F. And thus likewiſe all the rays, which proceed from C, will be united in a focus or point at D. The ſame may be ſaid of the rays which diverge from each point in the whole object: the rays from each point will by the refraction of the lens be made to converge to a point on the other ſide of it: and therefore upon the whole the rays of light, which flow from ſeveral points in the object and then paſs through the lens, will afterwards converge to ſo many other points.

It is neceſſary the object ſhould be more remote from the lens than the lens's principal focus. For if it was in the principal focus, then the rays diverging from the ſeveral radiants A, B, C, &c. would not converge to ſo many other points after refraction, for they would not converge at all, but would be parallel, by propoſition 61. Or if the object was nearer than the principal focus, then likewiſe the refracted rays would not converge at all, but would diverge, by propoſition 67.

The reader will ſee, without having it explaned to him, that a plano-convex lens would produce the ſame effect. And as a double convex one is moſt commonly uſed for the purpoſes to be mentioned in the following propoſitions, it will be ſufficient here to have taken notice once for all, that what is hereafter proved to be true of one ſort of lens, will be equally true of the other ſort, allowing for the different diſtances of the principal focus of a plano-convex lens and a double convex one.

84. *Wherever the rays, which come from the ſeveral points of any object, meet again in ſo many points, after they have been made to converge by the refraction of a convex lens, there they will make a picture of the object upon any white body on which they fall.*

If the rays which diverge from the ſeveral points A, B, C, &c. in the object AC, Plat. XI. fig. 1. paſs through the convex lens GHK fixed in the hole of the window ſhutter, and, when they come into the dark room ML, converge to juſt as many points F, E, D, &c. they will make a picture of the object upon a piece of white paper held at the place DF.

There will be juſt as many focuſes upon the paper as there are radiant points in the object, from which the rays come; and theſe focuſes will be diſpoſed in the ſame manner in reſpect of one another that the radiants are, by propoſition 83. Thoſe focuſes will be the moſt bright, in which the moſt rays of light are united, and thoſe will be the leaſt bright or the deepeſt ſhaded, in which the feweſt rays are united. And the moſt rays muſt be united in ſuch focuſes, as correſpond to thoſe radiants,

diants, from which the moſt light come; and the feweſt rays muſt be united in ſuch focuſes as correſpond to thoſe radiants, from which the leaſt light come. Therefore the light and ſhade upon the paper or at the focuſes will be every where anſwerable to the light and ſhade upon the ſurface of the object or at the radiants. And laſtly, ſince the colours of the rays are not changed, each focus upon the paper will have the ſame colour with the correſponding radiant. Now when the rays from theſe focuſes are reflected by the paper and enter the eye of a ſpectator, who looks at the paper, he will there ſee the picture or likeneſs of the object. For the figure made up of theſe focuſes will be like the figure of the object; becauſe the focuſes are diſpoſed in the ſame manner in reſpect of one another, that the radiants in the object are, The light and ſhade upon the paper is every where anſwerable to the light and ſhade upon the ſurface of the object. And the colouring of each particular part through the whole figure upon the paper is the ſame with the colouring of the correſpondent part in the object.

It is neceſſary that the room ſhould be darkened ſo that no light may come into it, but what comes through the hole GK. For if beſides the rays which flow from this object any other ſtrong light, ſuch as open day-light, was to fall upon the paper; this would dilate and weaken the ſhadings and colourings of the picture ſo as to make it diſappear: the few rays that produce the picture become imperceptible, when they are mingled with the much more numerous rays of open day-light. Indeed though the room is not ſo abſolutely darkened as to exclude all light but what enters at the hole, though there ſhould be ſome other rays come into it and mingle themſelves with the picture; yet if the room is almoſt dark, that is, if there are but few of theſe rays, the picture will not diſappear, but will only be made fainter. The picture is the moſt vivid when the room is quite dark; it becomes fainter by letting in a ſmall quantity of light; and will be made wholly to diſappear by letting in too much.

If in a room that is quite darkened and has no hole in the window-ſhutter to let any light at all into it, you place a lighted candle, then by holding a convex lens at ſome diſtance from the candle, and a piece of white paper at a proper diſtance beyond the lens, you may collect the rays that flow from the ſeveral points in the candle into as many focuſes upon the paper; and theſe focuſes will make the picture of the candle upon the paper. This picture may eaſily be ſeen notwithſtanding it is produced only by rays from the candle, and the ſame candle enlightens every part of the room. For the rays, which have paſſed through the lens and are collected by it, will throw a ſtronger light upon the paper than the direct rays, or thoſe which have not been thus collected, will produce any where elſe: therefore though the paper is at the ſame time enlightened by the direct rays, which come from the candle, as well as any other part of the room, yet theſe will be much fainter than the picture, and, though they are mingled with it, will not hinder it from appearing.

In like manner, when we uſe a common burning-glaſs, which is a convex lens, it collects the ſuns rays into a focus, which, when it is received upon a white paper, is a bright round ſpot. We call it a focus, though properly ſpeaking it conſiſts of innumerable focuſes; for the rays which come from the ſeveral points in the ſun are made by the glaſs to converge to as many points or focuſes upon the paper: ſo that the bright ſpot, which is produced at the focus of a burning glaſs by collecting the ſuns rays, is nothing elſe but the picture of the ſun. But this picture or bright ſpot is eaſily ſeen, though it is produced in open day-light and not in a dark room: becauſe the light of the rays, when they are thus collected, is ſo much greater than the light of the direct rays of the ſun, or than common day-light, that common day-light, though it is mingled with the picture, will make little or no alteration in it.

The reader has by this time ſeen the reaſon, why, in propoſition 25, I gave the name of a pencil of rays not to the incident beam, AG, AH, AK, alone, but to this incident beam and the refracted one, GF, HF, KF, taken together: for it is not the incident beam alone, which produces the effect of a painters pencil; it is the incident beam and refracted one together which paint the picture of the point A in the object at the point F upon the paper.

For the better underſtanding many things which follow, it will be neceſſary here to conſider ſeveral properties of ſuch pictures as are produced in the manner juſt now deſcribed.

85. *The picture in a dark room is inverted in reſpect of the object.*

In whatever poſition the object AC, Plat. XI. fig. 1. is placed, the picture DF will be in the contrary poſition. The focus, in which the rays, that come from A, ſuch as AG, AH, and AK, are united, is in the axis AH; for this axis goes ſtrait through the lens GHK, and the reſt of the rays are made to converge towards it. So likewiſe, as appears from what was ſaid concerning the axis of an oblique beam the focus, where the rays that come from C, ſuch as CG, CH, and CK, are united, is in the axis CH for the ſame reaſon. Now the higheſt point A is painted upon that part of the paper where the rays, that come from A, are united; and the loweſt point C is painted upon that part of the paper where the rays, that come from C, are united. Therefore the picture of the higheſt point is ſomewhere in the axis AH continued beyond the lens into the room; and the picture of the loweſt point is ſomewhere in the axis CH continued in the ſame manner. But the axis AH is the middle ray of the cone of rays GAK, by propoſition 22. And this cone has the lens GHK for its baſe, by propoſition 21: and conſequently the axis AH paſſes through the middle of the lens. For the ſame reaſon the axis CH muſt paſs through the middle of the lens. And if both the axis AH and the axis CH paſs through the middle of the lens, they muſt croſs one another in paſſing: ſo that though AH is the higher and CH the lower of the two on the ſide of the lens next the object; yet on the other ſide of the lens, when the axes are within the room, they will have croſſed each other; and HF, which is AH continued, will be the lower of the two, and HD, which is CH continued, will be the higher. But we have proved already that the point A will be painted ſomewhere in the axis AH continued beyond the lens, or ſomewhere in the line HF, and that the point C will be painted ſomewhere in the axis CH continued beyond the lens, or ſomewhere in the line HD. Therefore A or the higheſt point of the object will be painted lower upon the paper than C, and C or the loweſt point of the object will be painted higher upon the paper than A; that is, the picture FED will be in a contrary poſition to the object, or, in reſpect of the object, will be inverted.

86. *The picture in a dark room is not diſtinct, unleſs the paper, upon which it is painted, is placed where the ſeveral beams of rays are each of them united in their reſpective focuſes.*

The picture is then only diſtinct when all the rays that come from one and the ſame point in the object are collected into one and the ſame point upon the paper. For, if the rays, which come from A, for inſtance, are not united in a focus at F, but are ſcattered ſo as to be ſpread over a ſpot upon the paper inſtead of falling all together upon a ſingle point; and if the rays likewiſe, which come from the point next below A in the object, are ſcattered in the ſame manner; then the rays which come from different points in the object, will not be ſeparated from one another upon the paper, but as the rays from the point A are ſcattered at F and the rays that come from the point next below A are likewiſe ſcattered juſt above F, theſe two beams, though they come from different points in the object, will be mingled upon the paper, and conſequently in the picture theſe two points will not appear ſo diſtinct from one another as they do in the object. This will be the caſe of all the other parts of the picture, unleſs the paper is placed where the ſeveral beams are united in their reſpective focuſes.

87. *As the object approaches to the lens, the diſtinct picture departs from it; and as the object departs, the diſtinct picture approaches.*

The picture, by propoſition 86, is diſtinct where the ſeveral beams of rays, that come from different points or radiants in the object, are united in their reſpective focuſes, where the rays, for inſtance, which come from one and the ſame radiant A are all of them united in one and the ſame focus F, and thoſe, which come from B, are united in one and the ſame focus E. But as the object approaches to the lens the ſeveral radiants A, B, C, &c. approach it, and conſequently the ſeveral focuſes F, E, D, &c. depart from it, by propoſition 71. Therefore as the object approaches to the lens the diſtinct picture, which is always in the ſame place with theſe focuſes, departs from it. In like manner as the object, or the ſeveral radiants A, B, C, &c. departs from the lens, the ſeveral focuſes F, E, D, &c. and conſequently the diſtinct picture will approach to it, by propoſition 71.

From hence then it follows, that if the picture is diſtinct upon the paper, when the diſtance of the object is HB, and the diſtance of the paper HE; then ſuppoſing the object to be brought nearer to the lens, the picture upon the paper will become confuſed; for the diſtance of the diſtinct picture is encreaſed by bringing the object nearer: and in order to make the picture appear diſtinct again upon the paper, it is neceſſary that the diſtance HE ſhould be encreaſed as much, or that the paper ſhould be removed farther from the lens. But if the object is removed farther from the lens, then the diſtinct picture gets nearer to it, and if the paper ſtill continues at the ſame diſtance HE the repreſentation upon the paper will be confuſed: therefore in order to reſtore the diſtinctneſs of this picture or repreſentation, the diſtance HE muſt be diminiſhed, or the paper muſt be brought nearer to the lens.

If rays that come from ſeveral objects enter the lens at the ſame time and paſs through it into the dark room, then the diſtinct picture at the focus, will conſiſt of the repreſentations of all thoſe objects. But to make all the parts of the picture diſtinct, the ſeveral objects, that are repreſented in it, muſt be at equal diſtances from the lens: for if ſome of them are farther off than others, then by holding the paper ſo near to the lens as to have the remoter objects repreſented diſtinctly upon it; theſe parts of the picture, where the nearer objects are repreſented, will appear confuſed;

fuſed; or on the contrary by holding the paper ſo far from the lens as to have the nearer objects repreſented diſtinctly, thoſe parts, where the remoter ones are repreſented, will appear confuſed. Though if the diſtances of the ſeveral objects are not very different, the confuſion in either caſe will be but inconſiderable.

When the object AC is removed to an infinite diſtance from the lens, or when the ſeveral radiants A, B, C, &c. are ſo remote that the rays, which come from them, may, by propoſition 30, be conſidered as parallel; then theſe rays will be collected into their ſeveral focuſes, in the principal focus of the lens, by propoſition 60: and therefore the diſtinct picture of the object will likewiſe be at the principal focus of the lens, by the propoſition now before us. This is the caſe of the ſun: it is ſo remote an object, that the rays, which come from the ſame point, are parallel to one another, and conſequently the diſtinct picture, which is the burning ſpot, when the ſolar rays are collected by a burning glaſs, is at the principal focus of the glaſs.

Here we may obſerve that though, whilſt the object keeps departing from the lens, the picture keeps approaching to it; yet the picture can never be nearer to the lens than its principal focus; ſince this is the place of the picture, when the object has departed to an infinite diſtance.

88. *When the object is parallel to the picture, the diameter of the object is to the diameter of the diſtinct picture, as the diſtance of the object from the lens is to the diſtance of the picture from the lens.*

If the paper is held parallel to AC, ſo that the object and diſtinct picture may be parallel to one another; then AC, which is the height of the object, or a diameter drawn from the top of it to the bottom, bears the ſame proportion to FD, which is the height of the picture, that CH or the diſtance of the object from the lens bears to DH or to the diſtance of the picture from the lens; that is, AC is juſt as much longer or ſhorter than EF, as CH is longer or ſhorter than DH.

Now A is the higheſt radiant in the object, and C is the loweſt radiant, and the height of the object is the diſtance between theſe two or AC. The radiant A is painted ſomewhere in the right line AH continued behind the lens, and the radiant C is painted ſomewhere in the right line CH continued in the ſame manner, by propoſitions 83, 84. So that the height of the object is AC or the diſtance between theſe two lines at the object, and the height of the picture will be FD or the diſtance between the ſame lines, when they are continued to the place where that picture is repreſented.

Since therefore CD and AF are right lines, and AC is parallel to FD; it follows that the angle ACD is equal to the alternate one CDF; and that the angle CAF is likewiſe equal to the alternate one AFD. Euc. b. I. prop. 29. Therefore in the triangles AHC and FHD the angles at A and F are equal, and ſo are the angles at C and D. And CHA and DHF are equal, becauſe they are vertical. Euc. b. I. prop. 15. So that all the angles in the triangles AHC are reſpectively equal to all the angles in the triangle FHD, or theſe two triangles are equiangular and conſequently the ſides about the equal angles are proportional. Euc. b. VI. prop. 4. AC is to CH, as FD is to DH, or the height of the object is to the height of the picture, as the diſtance of the object from the lens is to the diſtance of the picture from the lens.

In the ſame manner we might prove that the breadth of the object is to the breadth of the picture, or that any diameter or line drawn acroſs the former is to any correſponding diameter or line drawn acroſs the latter, as the diſtance of the object from the lens on one ſide is to the diſtance of the picture from it on the other.

From hence it follows that the diameter of the picture may either be ſhorter than that of the object, or longer than that of the object, or equal to it. If DF is nearer to the lens than AC, that is, if the diſtance of the picture is leſs than the diſtance of the object, the diameter of the picture will be leſs than the diameter of the object, or DF will be ſhorter than AC, in the ſame proportion. But if the picture was farther from the lens than the object, any diameter of the picture would be greater than the correſponding diameter of the object: the height, for inſtance, of the picture would be greater than the height of the object. Or if both the picture and object were at an equal diſtance from the lens, the picture would in all reſpects be equal to the object.

89. *The picture, when it is confuſed, is ſomething bigger, than when it is diſtinct.*

When the paper is held either too near to the lens, or too far from it the picture will be confuſed. by propoſition 86. And in either caſe it will be a little longer and a little broader than if it was made diſtinct by holding the paper at a proper diſtance.

When the paper is held at the focus, or when the picture is diſtinct, all the rays, that come from the ſame point in the object, are collected into one ſingle point upon the paper. Thus the rays, for inſtance, that come from A, Plat. XI. fig. 1. or from the higheſt point in the object, are collected into the point F, or into the loweſt point in the picture, and all the rays, that come from C, or from the loweſt point in the object, are collected into the point D, or into the higheſt point of the picture, by propoſition 84, 86. It is plane therefore that the length of the picture is the diſtance between F and D, or the diſtance between that place upon the paper where the rays fall which come from the higheſt point of the object, and that place where thoſe fall which come from the loweſt point.

Now when the picture is diſtinct F and D, as has juſt been obſerved, are only ſingle points. But if the paper was to be brought nearer to the lens, ſo as to receive the rays of each pencil before they are united in their ſeveral focuſes, the rays that come from A would not be collected into a ſingle point at F, but would be ſpread over a little circular ſpace upon the paper. In like manner the rays that come from C would not be collected into a ſingle point at D but would be ſpread over a little circular ſpace. And the ſame would happen to the rays of every other pencil. This will make the picture appear confuſed, by propoſition 86. It will likewiſe make the picture longer. For the length of the picture is the diſtance between that place upon the paper where the rays fall which come from the higheſt point of the object, and that place where thoſe fall which come from the loweſt point. This, when the picture is diſtinct, is the diſtance between the two points F and D. But when the picture is confuſed the rays at F are ſpread over a ſmall circle, and therefore will be extended in all directions a little way round the point F, and conſequently will reach below the point F: and for the ſame reaſon the rays at the other end of the picture will reach above the point D. But the picture is extended upon the paper as far as the rays at F reach one way and the rays at D the other way. Therefore the picture, when it is confuſed, is ſomething longer than when it is diſtinct. In the ſame manner we might prove that it is ſomething broader too. And conſequently the confuſed picture is every way ſomething bigger than the diſtinct one.

There is another way of making the picture confuſed beſides bringing the paper too near to the lens, and that is by removing it to too great a diſtance. For when the rays of any pencil have met at their focus, as at F, for inſtance, if there is nothing there to ſtop them, they will proceed ſtrait forwards, and, after they are paſſed the focus, will begin to diverge from thence as from a radiant, by propoſition 15. So that when the paper is more remote than this focus, theſe rays, inſtead of being collected into a ſingle point at F will be in a diverging ſtate, and conſequently will be ſpread over a ſmall circle at F. The ſame likewiſe will happen to the rays at D. And this, for the ſame reaſons as before, will make the picture a little bigger every way, than it would be if it was diſtinct.

90. *The object and the diſtinct picture are ſimilar ſurfaces.*

It will here be neceſſary to explane what is meant by the ſurface of the object. When an object is conſidered as reflecting light from that ſide of it which is towards the lens, whatever is the ſhape of the object on that ſide, we may always look upon it as a plane ſurface differently ſhaded. Thus if AC, Plat. XI. fig. 1. is the body of a man; though the ſide of the man, which is towards the lens, is not a plane ſurface, yet the light is reflected from it, juſt as if it was the figure of a man drawn upon the plane ſurface of a piece of canvas and differently ſhaded in different parts of it. Upon this account we conſider the ſide of the man, or other object, which is next the lens, as ſuch a plane ſurface differently ſhaded. And in this view we affirm that the ſurface of the object will be ſimilar to the ſurface of the diſtinct picture. Theſe two ſurfaces are ſimilar, if the height of the object bears the ſame proportion to the height of the picture, that the breadth of the object in any part of it bears to the breadth of the picture in the correſponding part. The proportion that the height of the object bears to the height of the picture, is always the ſame with the proportion, that the breadth of the object in every part of it bears to the breadth of the picture in the correſponding part, for each of theſe proportions is the ſame with that which the diſtance of the object from the lens bears to the diſtance of the picture from the lens, by propoſition 88. Therefore the object and picture are always ſimilar to one another.

Indeed it is in this ſimilarity that a principal part of the likeneſs between the object and picture conſiſts. For though they are ſhaded and coloured in the ſame manner; yet one of them would not be like the other unleſs their outlines were ſimilar, that is, unleſs the ſurface included within the outlines of the object was ſimilar to the ſurface included within the outlines of the picture.

91. *When the object is given, the diameter of the diſtinct picture is inverſely as the diſtance of the object from the lens.*

The diameter of the picture encreaſes as its diſtance from the lens encreaſes, and decreaſes as its diſtance decreaſes, by propoſition 88. For when it is at the ſame diſtance with the object, it is equal to the object; when it is more remote than the object its diameter is greater than that of the object in proportion as its diſtance is greater; and when it is nearer than the object, its diameter is leſs in proportion as its diſtance is leſs. Now the diſtance of the picture encreaſes as the diſtance of the object decreaſes, and decreaſes as the diſtance of the object encreaſes, by propoſipoſition 47. Since therefore the diameter of the picture encreaſes as its diſtance encreaſes, and ſince its diſtance encreaſes as the diſtance of the object decreaſes; it follows that the diameter of the picture encreaſes as the diſtance of the object decreaſes, or that the diameter of the picture is greater as the diſtance of the object is leſs, or that the diameter of the picture is inverſely as the diſtance of the object.

92. *The area of the picture, when the object is given, is inverſely as the ſquare of the objects diſtance from the lens.*

Whatever is the diſtance of the object, and whatever, in conſequence of that, is the ſize of the picture, it has been proved, in propoſition 90, that the ſurface of the picture is always ſimilar to the ſurface of the object. And, becauſe the ſurface of the object, by the ſuppoſition, is always the ſame, the ſurface of the picture, which in all ſizes of it is ſimilar to the ſurface of the object, muſt in all ſizes of it be ſimilar to itſelf. But the areas of ſimilar ſurfaces are as the ſquares of their homologous diameters, that is, as the ſquares of their heights, or as the ſquares of their breadths. Euc. b.VI. prop. 20. corol. 1. Therefore the area of the picture is always as the ſquare of its diameter. But the diameter of the picture is inverſely as the diſtance of the object, by propoſition 91. Therefore the ſquare of the diameter of the picture, or the area of it, is inverſely as the ſquare of the diſtance of the object.

We may apply this propoſition and the foregoing one at the ſame time to an inſtance or two by the help of numbers. When the object AC, Plat. XI. fig. 1. is at the diſtance BH from the lens, the height of the picture is DF. But if the diſtance of the object was twice BH, then the height of the picture would be only $\frac{1}{2}$ DF. And when the height of it is thus reduced to $\frac{1}{2}$, the breadth will at the ſame time and for the ſame reaſon be likewiſe reduced to $\frac{1}{2}$. For, by propoſition 91, either the height or the breadth of the picture is inverſely as the diſtance of the object: ſo that by doubling the diſtance of the object both the height and the breadth of the picture are halved. But if by doubling the diſtance of the object, the picture is made only $\frac{1}{2}$ as long and $\frac{1}{2}$ as broad as it was, the ſurface of it will be reduced to $\frac{1}{4}$. That is, if we call the diſtance BH 1, then twice BH will be 2: the ſquare of 2 is 4; and by encreaſing the diſtance of the object from BH to twice BH, the area of the picture is diminiſhed to $\frac{1}{4}$ of what it was. But $\frac{1}{4}$ is the reciprocal of 4 or of the ſquare of the diſtance. Therefore the area of the picture is as the reciprocal of the ſquare of the diſtance, or inverſely as the ſquare of the diſtance. In like manner if the diſtance of the object was three times HB, the height of the picture would be $\frac{1}{3}$ of DF, and the breadth of it would likewiſe be reduced to $\frac{1}{3}$ of what it was before. Therefore the area of the picture would be reduced to $\frac{1}{9}$: for if at the diſtance 3HB the ſurface of it is but $\frac{1}{3}$ in length and $\frac{1}{3}$ in breadth what it is when the diſtance was HB, the area of that ſurface will be but $\frac{1}{9}$ of what it was. And thus in all other inſtances, the area of the picture, or the ſurface of that part of the paper, where the rays of light fall which form the picture, is always as the reciprocal of the ſquare of the objects diſtance: ſo that if we call that ſurface 1 at any certain diſtance of the object, at twice the diſtance of the object it will be only $\frac{1}{4}$; at three times the diſtance of the object it will be $\frac{1}{9}$; at four times the diſtance it will be $\frac{1}{16}$; &c.

93. *When the diſtance of the object is given, the diameter of the diſtinct picture is directly as the diameter of the object.*

If the diſtance of the object is given, that is, if the diſtance of the object does not vary, the picture will always be at the ſame diſtance from the lens. If the object is always at the diſtance BH from the lens, Plat. XI. fig. 1. the image will always be at the diſtance EH, by propoſition 64, 84. In this caſe the diameter of the picture DF will be greater or leſs as the diameter of the object AC is greater or leſs. For AC the diameter of the object always bears the ſame proportion to DF the diameter of the picture that BH the diſtance of the object bears to EH the diſtance of the picture, by propoſition 88. But when BH and EH are given, that is, when the object and picture continue at the ſame diſtance from the lens, the proportion which BH bears to EH is likewiſe given; for the ſame lines always bear the ſame proportion to each other. Therefore when BH and EH are given, the proportion, which AC bears to DF, muſt be given too. Now where the proportion of AC to DF is invariable, it follows that if AC encreaſes DF muſt encreaſe in the ſame proportion, or if AC decreaſes DF muſt decreaſe in the ſame proportion; that is, where the diſtance of the object, and conſequently the diſtance of the picture is given, as the height or any other diameter of the object is greater or leſs the height or any other correſponding diameter of the picture will be greater or leſs in the ſame proportion. But quantities which vary in the ſame proportion are directly as each other. Therefore where the diſtance of the object from the lens is given, the diameter of the picture is directly as the diameter of the object. So that if AC was twice as long as it is in the figure before us, the length of DF would be doubled, if AC was thrice as long, then DF would be tripled, or in whatever other proportion AC encreaſes, DF will encreaſe at the ſame time and in the ſame proportion. Or if AC was half as long as it is in the figure, DF would be but half as long, if the length of AC was only one third of what it is, the length of DF would likewiſe be only one third, or in whatever other proportion AC decreaſes, DF will decreaſe at the ſame time and in the ſame proportion.

94. *Whatever is the diſtance of the object from the lens, the diameter of the diſtinct picture will always be the ſame, provided the diameter of the object is proportional to its diſtance.*

If the diſtance of the object AC, Plat. XI. fig. 1. was twice HB, this would make the picture DF half as long as it is, when the diſtance of the object is only once HB, by propoſition 91. But if the length of AC was to be doubled, this would make the picture twice as long, by propoſition 93. Now if, when the diſtance of the object is doubled, the length of it was to be doubled at the ſame time, the length of the picture would be halved upon one account, and at the ſame time would be doubled upon the other account, that is, the length of the picture would be diminiſhed and encreaſed equally, and conſequently upon both accounts taken together it would continue the ſame. In like manner, if the diſtance of AC the object was only half BH, this would make the picture DF twice as long as it is, by propoſition 91. And if the length of the object was half AC this would make the picture half as long as it is. If therefore, when the diſtance of the object is halved the length of it is halved at the ſame time, that is, if the length of the object decreaſes in the ſame proportion with its diſtance, this will make the picture twice as long upon one account and half as long upon the other account, and conſequently upon both accounts taken together its length will be encreaſed and diminiſhed equally, and therefore will continue the ſame. This method of reaſoning may eaſily be applyed to any other inſtance where the diameter of the object is proportional to its diſtance from the lens, that is, where the diameter of the object encreaſes or decreaſes in the ſame proportion with its diſtance.

Or otherwiſe. Call the height of an object X, and call the diſtance of it from the lens Z. Then the length of the picture is as X directly, by propoſition 93, and as Z inverſely, by propoſition 91. Therefore univerſally the length of the picture will be as the fraction $\frac{X}{Z}$; for ſuch a fraction, like all other fractions, is as its numerator directly and as its denominator inverſely, that is, it is directly as X and inverſely as Z. But if X is always directly as Z then $\frac{X}{Z}$ or the length of the picture will be a given quantity, that is $\frac{X}{Z}$ will always be the ſame, whatever be the values of X and Z, or let the diſtance and length of the object be what they will. For the fraction $\frac{X}{Z}$ is as X directly, or encreaſes in the ſame proportion that X encreaſes, and it is inverſely as Z, or decreaſes in the ſame proportion that Z encreaſes. But X is directly as Z, that is, in whatever proportion X encreaſes Z will encreaſe in the ſame proportion. Therefore in whatever proportion the fraction $\frac{X}{Z}$ encreaſes when its numerator X encreaſes, it will be diminiſhed at the ſame time in the ſame proportion by the encreaſe of its denominator Z. In like manner, if $\frac{X}{Z}$ is diminiſhed becauſe X its numerator decreaſes; it will at the ſame time be proportionally encreaſed by the decreaſe of its denominator Z. Therefore in all the variations of X and Z, the value of the fraction $\frac{X}{Z}$ continues the ſame; or in all the changes of the objects height and diſtance; the length of the picture will continue the ſame; provided X is directly as Z, or provided the height of the object is directly as its diſtance.

Suppoſe there were two objects at different diſtances from the lens, and ſuppoſe their heights to be in the ſame proportion to one another that their diſtances are, for inſtance, call the height of one object X and the height of the other x, call the diſtance of the former object Z and the diſtance of the latter z; and let X be to x as Z to z: then the diſtinct picture of each object will be of the ſame length. The length of one picture will be as $\frac{X}{Z}$ and the length of the other will be as $\frac{x}{z}$, but ſince X is to Z as x to z, the firſt term divided by the third is equal to the ſecond divided by the fourth, that is, $\frac{X}{Z}=\frac{x}{z}$; that is, ſince $\frac{X}{Z}$ is a given quantity whatever are the values of X and Z, when they are both diminiſhed in the ſame proportion and reduced to x and z, the fraction $\frac{x}{z}$ will be equal to $\frac{X}{Z}$. Therefore each picture will be of the ſame length.

95. *When the diameter and diſtance of the object are given, the diameter of the picture will be directly as its diſtance from the lens.*

If the diameter and diſtance of the object are given, it is plane, from propoſitions 91, 93, that the diameter of the picture cannot be varyed, whilſt the ſame lens is made uſe of. But though the object continues the ſame and is placed at the ſame diſtance, yet by changing the lens GHK, Plat. XI. fig. 1, for one that is leſs convex or that has a longer principal focus, the rays will be made to converge leſs by paſſing through it, and the focuſes of the ſeveral pencills, or the picture made by thoſe pencills will be at a greater diſtance from the lens. In like manner, if a more convex lens is made uſe of, the refracted rays will converge the more, and the picture will be at a leſs diſtance. This may be proved more certainly by applying propoſition 64. When the diſtance of the picture is thus varyed only by changing the convexity of the lens, without any alteration being made either in the diameter or in the diſtance of the object, the diameter of the picture will be greater as its diſtance is greater, or leſs as its diſtance is leſs. For ſince the diſtance of the object is always to the diſtance of the picture, as the diameter of the object is to the diameter of the picture; and ſince the firſt and third terms, or the diſtance and diameter of the object, are ſuppoſed invariable, in whatever proportion the ſecond term, or diſtance of the picture, is encreaſed or diminiſhed, the fourth term, or diameter of the picture, muſt encreaſe or decreaſe in the ſame proportion; that is, the diameter of the picture will be directly as its diſtance from the lens.

96. *When the diameter and diſtance of the object are given, the area of the picture is as the ſquare of its diſtance from the lens.*

For the object and picture are always ſimilar ſurfaces, by propoſition 90; and conſequently the picture, whatever is the ſize of it, will always be ſimilar to itſelf, becauſe it is always ſimilar to the object. But the areas of ſimilar ſurfaces are as the ſquares of their correſponding diameters. Euc. b. VI. prop. 20. corol. 1. Therefore the area of the picture is as the ſquare of its diameter. Now the diameter of the picture, when the diameter and diſtance of the object are given, is as its diſtance from the lens, by propoſition 95. Therefore the ſquare of its diameter, or the area of it, is as the ſquare of its diſtance from the lens.

97. *Though the diſtance of the object from the lens ſhould be varyed, yet the picture may be preſerved diſtinct upon the paper without varying the diſtance of the paper from the lens.*

This indeed would be impoſſible, as appears from propoſition 87, if the ſame lens was to be made uſe of in all the different diſtances of the object. But we have obſerved, in propoſition 95, that a leſs convex lens will make the picture more remote from the lens, and a more convex one will make it nearer. Suppoſe therefore that the picture of any object is repreſented diſtinctly upon the paper in the dark room, and then let the object be removed to a greater diſtance from the lens; if in this greater diſtance of the object the ſame lens is made uſe of, the picture will be confuſed, unleſs the paper is brought nearer, by propoſition 87. But if in this caſe the lens is changed for one of a leſs convexity, this will throw the picture farther back. And if it is thrown as much back by diminiſhing the convexity of the lens, as it is brought forwards by the removal of the object; then by both cauſes concurring to produce equal and contrary effects it will be kept in the ſame place, and will continue diſtinct, though the paper is not ſtirred. In like manner if the object comes nearer to the lens, then, by propoſition 87, the paper muſt be removed backwards from the lens in order to preſerve the picture diſtinct, ſuppoſing the ſame lens to be made uſe of. But a more convex lens would bring the picture forwards. Therefore, when the object approaches the lens, if a more convex lens is made uſe of, the picture may by this means be brought forwards juſt as much as it is thrown backwards by the approach of the object. And thus again, by both theſe cauſes concurring to produce equal and contrary effects, the diſtinctneſs of the picture may be preſerved without ſtirring the paper.

98. *Though the diſtance of the object from the lens ſhould be varyed, yet the picture may be preſerved diſtinct upon the paper, without either moving the paper or changing the lens.*

When BH, Plat. XI. fig. 1. which is the diſtance of the object from the lens, is encreaſed, the diſtinct picture will approach to the lens, by propoſition 87; that is, when BH encreaſes EH decreaſes. Now if EH is the diſtance of the paper from the lens, and one way of keeping the picture diſtinct upon the paper is by moving the paper nearer to the lens as the picture moves nearer to it, yet this is not the only way: for as by moving the paper nearer to the lens the diſtance EH is diminiſhed, ſo likewiſe without ſtirring the paper the diſtance EH may be deminiſhed, if the lens is moveable and can be brought nearer to the paper. By the ſame means the picture may be kept diſtinct, when the object approaches to the lens: for as the diſtinct picture will in this caſe depart, the diſtance EH between the lens and the paper muſt be encreaſed till it is equal to the diſtance of the diſtinct picture; and this may be done not only by moving the paper farther from the lens, but by moving the lens farther from the paper without ſtirring the paper at all.

99. *When the object is very near the lens, though its diſtance is greater than the principal focal diſtance, yet, in order to make the picture diſtinct, the area of the lens muſt be very ſmall.*

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The picture is not diſtinct, unleſs all the rays, that come from one and the ſame radiant in the object, are united in one and the ſame point upon the paper, by propoſition 86. Now if *a*, Plat. XI. fig. 2. is a radiant, and the lens is very near it, as ſuppoſe at *np*, and ſuppoſe *np* to be the diameter of the lens: then the cone of rays, the baſe of which is the lens, will be *nap*. Of theſe rays the extreme ones as *an* and *ap* diverge much more than the middle ones *ad* and *ae*: ſo that if the paper, upon which the focus of theſe rays is to be received after they have paſſed through the lens, is placed where the middle or leſs diverging rays are collected, the extreme rays, becauſe they diverged more, will not be collected ſo near to the lens, and conſequently, if they fall upon the paper, they will make the picture confuſed, as they are not brought to a focus. But if theſe extreme rays are excluded, or prevented from paſſing through the lens, then the picture will be diſtinct. The way to exclude them is to diminiſh the area of the lens, or of the hole where the lens is placed: for if the diameter of it was *de* inſtead of *np*, only the middle rays could paſs through it, and the extreme ones would be ſtopped by the window-ſhutter. This would not be neceſſary, if the radiant was at a greater diſtance from the lens. Suppoſe the lens was at *ſr*, then the extreme rays as *an* and *ap* paſs along in the directions *nb* and *nc*, ſo that ſome of them will be above the lens and ſome below it: only the middle rays of the cone, *as* and *ar* which are *ad* and *ae* continued, and ſuch rays as are contained between theſe, will paſs through the lens, though the area of it is not diminiſhed, or though the diameter is *ſr* equal to *np*.

100. *When the diſtances of the object and picture and likewiſe the diameter of the object are given, the diameter of the picture will not be altered by altering the area of the lens.*

If GHK, Plat. XI. fig. 1. is the lens, and the hole in the window-ſhutter is big enough to leave the whole area of the lens open ſo that the rays may paſs through it, then, ſuppoſing DF to be the length of the picture, no alteration would be made in the length of this picture, though the hole in the ſhutter was to be diminiſhed, and by that means the whole area of the lens was to be covered, except a very ſmall part in the middle of it at H. The height of the picture DF is the diſtance between the two extreme focuſes D and F. One of theſe focuſes is always in the axis AHF of that cone which comes from A one extreme point of the object; the other is always in the axis CHD of the cone which comes from C the other extreme point of the object, and theſe two axes go ſtrait through the middle of the lens, as appears from what has been ſaid in demonſtrating propoſition 85. Since therefore DF the height of the picture is the diſtance between theſe two lines AHF and CHD, when they come to the paper, the height of the picture will be the ſame whether the whole area of the lens is open or only a ſmall part of it in the middle at H: becauſe theſe two extreme axes go ſtrait through the middle of the lens and there croſs each other; and conſequently the diſtance between them, when they come to the paper, or DF, will be the ſame, whether the area of the lens is great or ſmall.

101. *The brightneſs of the picture, when its diſtance from the lens is given, is directly as the area of the lens.*

The diſtinctneſs of the picture is not the ſame thing as its brightneſs: nor is the confuſion of its parts the ſame thing as its obſcurity. The picture may be diſtinct in all its parts, the rays, which come from one and the ſame point in the object, may be exactly collected into one and the ſame point upon the paper, and yet if but few rays ſhould paſs through the lens ſo that the ſpace or area upon the paper, where the picture is painted, ſhould be but faintly enlightened, this picture, though it is diſtinct, will be a faint or obſcure one. Or, though the picture is confuſed either by the papers being placed at an improper diſtance from the lens, or by any other means, yet if many rays ſhould paſs through the lens, ſo that the ſpace, where this picture is painted, ſhould be ſtrongly enlightened, this picture, notwithſtanding its confuſion, will be a bright one.

Now the brightneſs or faintneſs of the picture in every part, as we have here explaned what is meant by brightneſs and faintneſs, muſt evidently depend on the number of rays that come to that part; the picture will be bright or faint in proportion as it is formed by more or by fewer rays. And the quantity of light or number of rays that paſs from a given object into the room is greater or leſs as the hole, through which they paſs, or as the area of the lens is greater or leſs. Therefore the degree of brightneſs, that the picture has, will be greater or leſs as the area of the lens is greater or leſs; that is, it will be directly as this area.

From hence it is plane that though the ſize of the picture is not altered by diminiſhing the area of the lens, as was proved in propoſition 100, yet the brightneſs of it will be diminiſhed. Thus whatever degree of brightneſs the picture DF, Plat. XI. fig. 1. may have when the whole area of the lens GHK is open; if the greateſt part of this area was to be covered, and none but a ſmall round hole at H was to be left open for the light to paſs through, this would make the picture much fainter. For when the whole area is open, the entire cone of rays AG, AK, and all the other rays included between thoſe extreme ones, paſſes through the lens from the points A in the object: but when the area is diminiſhed to a ſmall hole at H, the greateſt part of this cone is excluded, no rays but the middle ones AH and a few that are very near to AH can paſs through the lens; and conſequently the focus F where the point A is repreſented will be but faintly enlightened. The ſame may be ſaid of every other cone of rays and of every other point in the picture. Therefore the whole picture will be made fainter or leſs bright by diminiſhing the area of the lens.

Here too we may obſerve farther, that though, when an object is very near to the lens, the picture is made more diſtinct by diminiſhing the area of the lens, as was ſhewn in propoſition 99, yet by this means it will at the ſame time be made more faint: for by diminiſhing the area of the lens many rays are excluded, and the part of the paper, where the picture is painted, will conſequently be leſs enlightened.

102. *The brightneſs of the picture, when the area of the lens and the diſtance of the object are given, is inverſely as the ſquare of its diſtance from the lens.*

When the area of the lens is given, we have ſeen in propoſition 101, that the number of rays, which paſs through it and form the picture, will be given too. Now the ſame quantity of rays, if they are ſpread over a large ſpace upon the paper will not enlighten it ſo ſtrongly, as they would a ſmaller ſpace, where they are kept cloſer together and all of them fall within a narrower compaſs. That is to ſay, the brightneſs at the paper will be greater, as the ſpace, upon which the rays fall, is leſs, and will be leſs as that ſpace is greater; or the brightneſs will be inverſely as that ſpace. But the ſpace upon which the rays fall, that produce the picture, is the area of the picture. Therefore the brightneſs of the picture will be inverſely as its area. Now the area of the picture, when the diſtance of the object is given, is as the ſquare of its diſtance from the lens, by propoſition 96, and conſequently its brightneſs is inverſely as the ſquare of its diſtance.

103. *When the area of the lens is given, the brightneſs of the picture will not be altered, let the diſtance of the object be what it will.*

It appears from propoſition 102, that the brightneſs of the picture is inverſely as the ſquare of its diſtance from the lens; when this diſtance is varyed by changing the convexity of the lens without altering the diſtance

ſtance of the object. And ſo likewiſe it would be when the diſtance of it is varyed by changing the diſtance of the object, if ſuch a change of the objects diſtance made no other alteration but in the diſtance of the picture. But there is another alteration made in this caſe, which will preſerve the degree of brightneſs in the picture always the ſame.

Now the area of the picture is inverſely as its brightneſs, as appears from what has been ſaid in proving propoſition 102. And the area of the picture is inverſely as the ſquare of the objects diſtance, by propoſition 92. From hence it follows that the brightneſs of the picture and the ſquare of the objects diſtance are varyed in the ſame proportion, for they are both of them inverſely as the area of the picture. But two quantities which are varyed always in the ſame proportion are directly as each other. Therefore, if all other circumſtances were equal, the brightneſs of the picture would be directly as the ſquare of the objects diſtance from the lens. Light is propagated in right lines from every point of the object, by propoſition 34. Therefore the quantity of light that falls upon a given ſpace, or the quantity that paſſes through a lens the area of which is given, will be inverſely as the ſquare of the objects diſtance, by propoſition 22 of mechanics. And conſequently the brightneſs of the picture, if all other circumſtances were equal, would be inverſely as the ſquare of the objects diſtance. We ſee therefore that upon altering the objects diſtance, the brightneſs of the picture is altered upon two accounts; firſt, it will be directly as the ſquare of the objects diſtance upon account of the alteration that is made in its area; and ſecondly, it will be inverſely as the ſquare of the objects diſtance, upon account of the alteration that is made in the quantity of light, by which it is formed: or, calling the objects diſtance D and the ſquare of its diſtance D^q, the brightneſs of the picture will be as D^q directly and as D^q inverſely, that is, it will be as $\frac{D^q}{D^q}$. But ſuch a fraction will be equal to 1, ſince the numerator and denominator are the ſame. And 1 is an invariable or given quantity. Therefore the brightneſs of the picture, will be the ſame or invariable let the diſtance of the object be what it will.

We have not here taken notice that the rays of light are to paſs through the atmoſphere from the object to the lens, and that ſome of them will neceſſarily be ſtopped in their paſſage by the air and by ſuch bodies as float in it. From whence it is evident that the more of the atmoſphere the rays have to paſs through the more of them will be ſtopped, and the leſs of it they have to paſs through the fewer of them will be ſtopped.

And conſequently more rays will be ſtopped in paſſing from a remote object than from a near one. This will make the picture of a remote object ſomething fainter than it would be if the object was nearer. So that this propoſition is true only upon ſuppoſition that the rays paſs freely from the object to the lens, and that none of them are intercepted in their paſſage.

104. *When all other circumſtances are equal, the picture will be brighter in proportion as the object is brighter.*

The reader could ſcarce want to be reminded of this: for it is plane, that in equal circumſtances, the more rays will paſs through the lens from the object in proportion as the object is brighter: and conſequently, ſince the brightneſs of the picture depends upon the number of rays that form it, the picture will be more or leſs bright in proportion as the object itſelf is more or leſs bright. Upon this account it is, that, when a houſe, or a man, or any other object abroad is repreſented upon the paper in a dark room, this picture will be leſs bright, when the ſun does not ſhine upon the object, than when it does.

105. *If the hole in the window-ſhutter is very ſmall, the inverted pictures of objects, that are abroad, will be repreſented upon a piece of paper in a dark room; though there is no lens in the hole.*

The picture in a dark room, when there is a convex lens in the hole of the window-ſhutter, is drawn upon a piece of white paper placed at the focus, becauſe the rays, that come from one and the ſame point in the object, will be collected there ſo as to fall upon one and the ſame point of the paper, by propoſition 84. Now if the hole is very ſmall this may in a great meaſure be brought about, though there is no lens made uſe of: for the ſmallneſs of the hole will in ſome ſort anſwer the purpoſe of a lens; becauſe it will admit only a ſlender cone of rays from each point in the object: and when any one of theſe cones comes to the paper, all the rays of it will be ſo near to one another as to fall nearly upon one and the ſame point. Let *abc*, Plat. XI. fig. 3. be an object, *ml* a dark room, and *ik* the diameter of a very ſmall circular hole in the window-ſhutter. Of all the rays that come from the point *a* only the ſlender cone contained between *aif* and *akf*, can be admitted into the room, on account of the ſtraitneſs of the hole through which the light is to paſs. The diameter of the baſe of this cone at the paper will be *f*, *f*, for all the rays of it are there included within this compaſs. And when the hole is very ſmall this circular baſe will neceſſarily be ſo too: ſo that all the rays, which come from one and the ſame point *a* in the object, though they do not fall exactly upon a ſingle point of the paper, fall upon only a very ſmall round ſpot: and at this ſpot the picture of the point *a* will be repreſented in the ſame manner that ſuch a point is repreſented in the confuſed picture deſcribed in propoſition 86, where a lens is made uſe of. For the ſame reaſons, the ſlender cone of rays that is admitted through the hole from the point *b* and falls upon the paper within the compaſs of a ſmall round ſpot, the diameter of which is *e*, *e*, will paint the picture of that point. And the ſlender cone, that is admitted from the point *c* and falls within the compaſs of a ſpot having *d*, *d*, for its diameter will paint the picture of that point. Thus the picture of each point in the object, and conſequently the whole object *abc* will be repreſented upon the paper. But then this repreſentation will be confuſed: becauſe the rays from each point of the object are not collected into a ſingle point upon the paper, but cover a circular ſpot upon it: by which means the rays of one point will be mingled a little with thoſe from another; the rays *f*, *f*, which come from the point *a*, with the rays *e*, *e*, that come from another point, and the rays *d*, *d*, with the ſame rays *e*, *e*.

This picture will be inverted in reſpect of the object. For the rays from *a*, which is the higheſt point, as they paſs through the hole, will croſs thoſe from *c*, which is the loweſt point: and therefore at the paper, after the rays have thus croſſed, thoſe which came from the higheſt point will fall lower than thoſe which came from the loweſt point. The rays muſt croſs in this manner; becauſe they deſcribe right lines by propoſition 34: and no two right lines can be drawn from *a* and *c* into the room *ml* through ſo ſmall a hole as *ik* without croſſing each other at the hole. That the rays do thus croſs each other, or that thoſe from the higheſt point *a* fall lower than thoſe from the loweſt point *c*, would be evident to any perſon ſtanding within the room. For if his eye was at *e*, *e*, directly before the hole he could then ſee, through ſo ſmall a hole, only the middle of the object about *b*: in order to ſee the higheſt point of it or *a*, that is, in order to receive rays from the higheſt point into his eye, he muſt ſink his eye to *f*, *f*, lower than *e*, *e*, and in order to ſee the loweſt point or *c*, he muſt raiſe his eye to *d*, *d*, higher than *e*, *e*.

106. *When there is no lens in the hole, through which the rays are admitted into a dark room, the diameter of the picture, if the diſtance of the object and picture are given, will be proportional to the diameter of the hole.*

It will be otherwiſe, as we have ſeen in propoſition 100, if a lens is made uſe of: for then neither the height nor breadth of the picture will be varyed by any alteration that can be made in the area of the hole. For when a lens is made uſe of the height of the picture will be the diſtance between the two extreme focuſes DF, Plat. XI. fig. 1. and theſe focuſes are always in the lines CHD and AHF, or in the axes of two cones of rays, one of which comes from C the loweſt and the other from A the higheſt point in the object. And as theſe two axes paſs through the middle of the lens, the diſtance between them, when they come to the paper, will be the ſame whether the area of the lens is great or ſmall. But if there is no lens in the hole, the picture is terminated at the bottom, not by the middle ray or axis of the cone that comes from *a*, Plat. XI. fig. 3, but by *af*, which is the loweſt ray of that cone; and it is terminated at the top, not by the axis of the cone that comes from *c*, but by *cd*, which is the higheſt ray of that cone. Now if the diameter of the hole *ik* was made leſs, it is plane that the loweſt ray *af* would be excluded at *k* and ſome other ray, which paſſes nearer to *i* than this, would be then the loweſt and would terminate the picture at the bottom: for the ſame reaſon the higheſt ray *cd* would be excluded at *i* and ſome other ray, which paſſes nearer to *k* than this, would terminate the picture at the top. Thus by diminiſhing the diameter of the hole the bottom and top of the picture will be brought nearer together, that is, the height of it will be diminiſhed. And the ſame might be proved of the breadth or of any other diameter of the picture. In like manner by encreaſing the diameter of the hole *ik* the height or any other diameter of the picture will encreaſe. For when *i* and *k* are farther aſunder the loweſt ray, that paſſes by *k*, will fall lower upon the paper than the ray *akf*, and the higheſt, that paſſes by *i*, will fall higher upon the paper than *cid*. But the height of the picture is the diſtance between the loweſt and higheſt ray, when they come to the paper. Therefore the height of the picture is encreaſed by encreaſing the diameter of the hole *ik*.

107. *The heat at the focus of a burning-glaſs, when the area of the glaſs is given, is as the ſquare of the focal diſtance inverſely.*

The burning ſpot, into which the rays of the ſun are collected, is a picture of the ſun, as was obſerved under propoſition 84. And the diſtance of this picture will always be equal to the focal diſtance of the lens, upon account of the remoteneſs of the ſun, which is the object. Now ſuppoſing the heat at the focus to be proportional to the number of rays or quantity of light that is collected there, ſince the brightneſs of the picture is likewiſe proportional to the quantity of light, it follows that the heat will be proportional to the brightneſs. The diſtance of the object is here given; for the object is the ſun; and the area of the lens is ſuppoſed to be given. Therefore the brightneſs of the picture is inverſely as the ſquare of its diſtance from the glaſs, by propoſition 102; and conſequently the heat at the focus, which is always as the brightneſs of the picture, will be inverſely as the ſquare of the focal diſtance of the glaſs.

Upon this account of two lenſes, which have equal ſurfaces, that lens will ſet fire to any thing the ſooneſt which has the ſhorteſt focus. Thus if the focal diſtance of one lens was 1 inch and the focal diſtance of any other lens, that had an equal ſurface, was 4 inches, the ſquares of their focal diſtances are to each other as 1 to 16; and the heat at the focuſes will be as thoſe ſquares inverted, or as 16 to 1; that is, the heat at the focus of the former lens will be 16 times greater than the heat at the focus of the latter.

108. *The heat at the focus of a burning-glaſs, when the focal diſtance is given, is directly as the area of the glaſs.*

The heat at the focus is proportional to the brightneſs of the picture, by what has been ſaid in demonſtrating propoſition 107. The diſtance of the ſun, which is the object, is given, and the focal diſtance of the lens, or diſtance of the picture, is ſuppoſed to be given. But in this caſe the brightneſs of the picture is as the area of the lens, by propoſition 101. Therefore the heat at the focus is likewiſe as the area of the lens. This is the reaſon why, if two lenſes have equal focal diſtances, that lens which is the broader of the two will ſet fire the ſooneſt to any thing that is placed at the focus.

109. *The heat at the focus of a burning-glaſs is to the common heat of the ſun, as the area of the focus to the area of the glaſs inverſely.*

The common heat of the ſun is the heat of the direct rays before they are refracted. Now when the direct rays fall upon the glaſs, they are ſpread over the whole area of it; and when they are collected at the focus they are reduced within the compaſs of the area of the focus. But the brightneſs ariſing from the ſame number of rays is inverſely as the area over which they are ſpread, or the ſame number of rays when they are ſpread over a large ſurface will not make that ſurface appear ſo bright, as they would make a ſmaller ſurface appear if they were brought cloſer together and were all of them made to fall upon this ſmaller ſurface. Therefore the brightneſs at the focus is to the brightneſs upon the ſurface of the glaſs, as the area of the focus to the area of the glaſs inverſely.

ly. But ſince the heat is proportional to the brightneſs, it follows that the heat at the focus will be to the heat at the glaſs, or to the common heat of the ſun, as the area of the focus to the area of the glaſs inverſely: or as much as the area of the burning ſpot is leſs than the area of the glaſs, ſo much greater the heat of that ſpot will be than the common heat of the ſun.

Indeed the heat at the focus will be ſomething leſs than in this proportion, becauſe in proving the truth of the propoſition we have ſuppoſed that there is the ſame number of rays at the focus, as there is upon the ſurface of the lens, that is towards the ſun. But this ſuppoſition is not exactly true: for though the greateſt part of the rays, that fall upon the glaſs, will paſs through it to the focus, yet all of them will not; ſome few rays, that fall upon the ſurface of the glaſs, will be reflected from it, and will therefore never come to the focus.

CHAP. IX.

Of plane viſion.

110. *When the rays, that come from the ſeveral points in any object, enter the eye, they will paint an inverted picture of that object upon the bottom of the eye.*

IN order to underſtand this propoſition it will be neceſſary to give ſome account of the ſtructure of the eye and of the principal parts that compoſe it. If an eye, when it is taken out of the head of an animal, is cut by a plane, that is parallel to the horizon and paſſes through the center of the eye; this plane will divide the ſphere or globe of the eye into two hemiſpheres; and the flat ſide of either hemiſphere is called a ſection of the eye, and will appear as in Plat. XI. fig. 4. In ſuch a ſection we have at once a view of all the humours and other principal parts of the eye. ABBA is the white coat, which encompaſſes the whole globe of the eye, except the fore part of it between A and A. This coat is called the ſclerotica. The fore part AA is covered by a tranſparent coat that is a little more protuberant than any other part of the eye and prevents the figure of it from being a perfect ſphere. This tranſparent coat is called the cornea. The cavity of the eye is divided into two parts by a convex lens CC. This lens conſiſts of a hard tranſparent ſubſtance, and is called the chryſtalline humour. It is kept in its place and fixed to the coats by certain ligaments all round it, as *ee*, which are called ligamenta ciliaria. Under the cornea, and at ſome little diſtance from it, is the iris *ſo*, *ſo*, which has a hole in it at *oo*. This hole is called the pupill of the eye. The iris is tinged with variety of colours; and from the difference of theſe colours ſome eyes are called blue ones, others brown, others black. It conſiſts of muſcular fibres, which can either contract or dilate the hole or pupill *oo*. So much of the eyes cavity as lies between the cornea and chryſtalline humour is filled with a thin tranſparent fluid like water, which is called the aqueous humour. *e*SN, *e*SN is a white coat, that conſiſts of the fibres of the optic nerve woven together like a net. This coat is called the retina. Between the ſclerotica ABBA, and the retina *e*SN, *e*SN, is another coat, which is called the choroides. The cavity of the eye between the chryſtalline humour and the retina is filled with a tranſparent ſubſtance, that is not ſo fluid as the aqueous humour, nor ſo hard as the chryſtalline: it has much the ſame conſiſtency with the white of an egg. This is called the vitreous humour. The optic nerve MBB is inſerted at N.

Now the cavity of the eye may be conſidered as a dark room, and the pupill is the hole through which rays that diverge from the ſeveral points of any object *abc*, Plat. XI. fig. 5, paſs into this cavity. The rays firſt paſs through the cornea *ik*, which is a meniſcus like a watch-glaſs; and therefore the effect of it upon the rays may be neglected. The fore part of the aqueous humour under the cornea is convex; and this humour being denſer than the air will make the ſeveral diverging beams from *a* and *b* and *c*, and any other points of the object, diverge leſs. After the rays, that come from the ſeveral points of the object, have paſſed through the pupill, which to keep the figure diſtinct is not here repreſented; they are to paſs through the chryſtalline humour *goh*, which is a convex lens and will make them converge to as many points upon the retina *lem*, by propoſition 83. And conſequently at *df* or ſomewhere upon the white retina, as upon a piece of white paper in a dark room, there will be painted an inverted image of that object, by propoſitions 84, 85.

This theory is agreeable to fact. For when the ſclerotica is carefully taken off, the back part of an animals eye *lem* is tranſparent. And if the eye thus prepared is placed in the hole of a window-ſhutter of a dark room with its fore part *ik* towards the objects abroad, a perſon in the room may ſee the inverted pictures of thoſe objects through the tranſparent coats where they are painted.

111. *The optic axis is the axis of the chryſtalline humour continued to the object, that we look at.*

The chryſtalline humour is a convex lens, and *po*, Plat. XI. fig 5, is the axis of it, by propoſition 19. This axis *po* continued from *o* to any point *b* in the object that we look at is the optic axis.

112. *By the middle of the retina we mean that point of it, upon which the optic axis continued back to it would fall.*

If the optic axis *bp*, Plat. XI. fig. 5, was continued back to the retina it would fall upon the point *e*; and this point we ſhall hereafter call the middle of the retina.

And here we may obſerve that N, Plat. XI. fig. 4, or the place where the optic nerve is inſerted is not in the middle of the retina but a little on one ſide towards the noſe.

113. *The pictures upon the retina are the cauſe of viſion.*

When the picture of an object is diſtinct upon the retina, the object is ſeen diſtinctly; and when the picture is confuſed, the object appears in confuſion. In like manner when the picture is bright, the object is clearly ſeen; and when the picture is faint, the object appears faint. Since therefore viſion ſo much depends upon theſe pictures, that the diſtinctneſs or confuſion of the picture will occaſion diſtinctneſs or confuſion in the appearance of the object, and the different brightneſs of the picture will make a proportionable alteration in the clearneſs or faintneſs of viſion; we may conclude that theſe pictures are the cauſe of viſion, or that by means of theſe pictures, the objects, which are painted upon the retina, are made viſible.

The only difficulty is, how we ſhall be able to know, when an object is ſeen by the eye, whether the picture on the retina is diſtinct or confuſed, bright or faint. For if the eye is in the head of an animal, ſo as to enjoy the uſe of ſight, we have no opportunity of making any obſervations upon the pictures on the retina. And if the eye is taken out of the head, ſo as to give us an opportunity of obſerving the pictures, then the eye has no viſion. So that when there is viſion, how ſhall we know whether the pictures are diſtinct or confuſed, bright or faint? and when we are able by obſerving the picture to know whether it is diſtinct or confuſed, bright or faint, how ſhall we be able to tell what ſort of viſion an animal would have in theſe circumſtances of the picture? Now to this we may anſwer that there is no occaſion to take the eye out of the head of the animal, and to remove the hinder coat, ſo as to ſee the picture upon the retina, in order to know whether it is diſtinct or confuſed, bright or faint. For ſince the picture on the retina is of the ſame ſort with that in a dark room, and ſince we know what will make the picture in a dark room diſtinct or confuſed, bright or faint, we know likewiſe what will have the ſame effect upon the picture on the retina. Thus for inſtance, if an object is placed too near the lens, the picture upon a paper in a dark room becomes confuſed; but its diſtinctneſs will be reſtored, if the area of the lens, or hole through which the rays are admitted, is very much contracted, by propoſition 99. The ſame cauſe will produce the ſame effect in the picture on the retina: if a book is held too near the eye, the pictures of the letters on the retina will be confuſed; but the diſtinctneſs of them will be reſtored by looking through a ſmall hole made with a pin in a piece of paper, which is an artificial contraction of the pupil. By this way of reaſoning we may find out in other inſtances when the picture on the retina is diſtinct and when it is confuſed. And as the letters are ſeen in confuſion, or viſion is confuſed, when the book is too near the eye, but are ſeen diſtinctly, or viſion is diſtinct, if we look through the pin hole; ſo likewiſe we ſhould find, in all other inſtances, that our diſtinct or confuſed viſion depends upon the diſtinctneſs or confuſion of the pictures on the retina. When the ſun ſhines upon an object, the picture of it in a dark room, and conſequently the picture of it on the retina, will be bright: but if the object is ſhaded, either of theſe pictures will for the ſame reaſon be faint or obſcure, by propoſitions 104, 105. And we find by our own experience that viſion is more clear, when the ſun ſhines upon any object that we look at, and more obſcure, when the object is ſhaded. The ſame holds good in all other inſtances. When we have found, by reaſoning about the picture on the retina as if it was a picture in a dark room, whether it is clear or obſcure, we ſhall always find by experience that our viſion will be clear or obſcure accordingly.

114. *That point in any object, towards which the optic axis is directed, is ſeen more diſtinctly than the reſt.*

The truth of this is confirmed by every ones own experience: if we turn our eyes directly towards one particular part of an object ſo as to look ſteadily at it, we may indeed, if the object is not very large, ſee all the reſt of it at the ſame time, but this part will appear more diſtinct than the reſt: and turning of our eyes towards it or looking at it ſteadily is turning our optic axes towards it. And indeed when the optic axis *eb*, Plat. XI. fig. 5, is directed towards any point *b* in an object, ſo that this point is painted on *e* the middle of the retina; if this part of the picture is diſtinct, the other parts will be a little confuſed: becauſe the retina in any other place, as *d* or *f*, is nearer to the lens or chryſtalline humour than *e* is; therefore if *e* is at a proper diſtance from the chryſtalline humour to make the picture diſtinct there, *d* or *f* will be rather too near, and upon that account the picture in theſe places will be a little confuſed, by propoſition 86.

115. *Objects appear erect, when their pictures on the retina are inverted.*

This is one objection to the account which has juſt been given of the cauſe of viſion. For if viſion is owing to the picture on the retina, and this picture is inverted, we are apt to enquire why the object does not appear in the ſame ſituation with the picture. This objection would be of ſome weight if it was the picture that we ſaw and not the object. But the picture is not ſeen at all: it is the inſtrument by means of which the object is perceived; but it is not perceived itſelf. When we perceive an object by the ſenſe of feeling, how do we know the ſituation of it? If for inſtance we were to ſhut our eyes and to lay hold on the middle of a ſtick, that is perpendicular to the horizon, how do we find out which end of the ſtick is the higheſt and which the loweſt? The way of finding this out is by feeling along the ſtick, and we know that end to be the higheſt, which we feel by moving our hand upwards, and that to be the loweſt, which we feel by moving our hand downwards. If we aſk the ſame queſtion about the ſenſe of ſeeing, we may give the ſame anſwer. When we perceive an object by the ſenſe of ſeeing, how do we know the ſituation of it? how do we find out, which end of it is the higheſt and which the loweſt? The way of finding this out is by looking along the object, which is done by moving the eye and is a ſort of feeling with the eye. For ſince, by propoſition 113, that point of an object is ſeen the moſt perfectly which we look at directly, we know that part of an object to be the higheſt, which we ſee diſtinctly by turning our eyes upwards, and that part to be the loweſt, which we ſee diſtinctly by turning our eyes downwards.

There is one remarkable inſtance of inverted viſion, which ſeems to confirm this account, and cannot well be explaned any other way. If any one holds a pin in an erect poſition ſo near to his eye that it appears confuſed; notwithſtanding the confuſion, the picture of the pin upon the retina is inverted, by propoſition 85; and the pin though ſeen confuſed will appear erect. But if he turns his face towards a window in order to have a ſtrong light, and holds a paper with a ſmall hole in it, ſuch as a pin-hole, at a little diſtance behind the pin, ſo as to be between the pin and the window, the pin will then appear more diſtinct, and will be inverted.

The difficulty here to be explaned is why the pin ſhould appear inverted; ſince certainly the putting a paper behind it will not alter the poſition of the picture upon the retina? Why therefore, as the pin appears erect, when there is not paper behind it, ſhould it appear inverted, when there is? Let *fg*, Plat. XI. fig. 6, be the paper, *o* the hole, *pſn* the pin, *rct* the eye, and *d* a candle, which, inſtead of the window, caſts a ſtrong light through the hole. The light, which comes through the hole *o*, diverges from thence afterwards as from a radiant, ſo that any two rays as *ao* and *bo*, when they have paſſed the hole, will diverge from thence in the lines *oe* and *ol*: but the croſſing of theſe rays at the hole, ſince this happened before they came to the pin, will make no alteration in the poſition of the picture. But here it is to be obſerved, that the ray *oe*, which paſſes by *p* the head of the pin, came from the loweſt part of the candle, and the ray *ol*, which paſſes by *ſ* the loweſt part of the pin that can appear oppoſite to the hole, came from the higheſt part of the candle. And as the hole, though it is a ſmall one, is very near the eye, the whole flame of the candle, or however the greateſt part of it, will appear through the hole. But it is plane that if the eye is turned downwards to look at the loweſt part of the flame, *p* the head of the pin will intercept the rays which come from that part, or will cover that part, and if it is turned upwards to look at the higheſt part, then *ſ* the loweſt part of the pin will intercept the rays which come from thence, or will cover that higheſt part. Now the pin is no otherwiſe viſible than as it intercepts the light of the candle, and ſo, properly ſpeaking, the picture of it upon the retina is nothing but a ſhadow in the ſhape of the pin. And ſince the head of the pin intercepts the light, which comes from the loweſt part of the candle, by turning the eye downwards we ſee the head; and ſince the loweſt part of the pin intercepts the light, which comes from the higheſt part of the candle, by turning the eye upwards we ſee the loweſt part of the pin. And for this reaſon the head of the pin will appear the loweſt, becauſe we turn the eye downwards to ſee it, and the bottom of it will appear the higheſt, becauſe we turn the eye upwards to ſee it. What is here ſaid of the flame of a candle may eaſily be applyed to a pane of glaſs in a window, or to any light ſpace beyond the hole: for the head of the pin will cover the bottom of the pane, and the loweſt part of the pin will cover the top of it.

From this particular inſtance and from the common inſtances of plane viſion we may conclude that the poſition, in which an object appears, does not at all depend upon the poſition of the picture upon the retina. For in this caſe the picture is inverted and the object appears inverted; in common caſes the picture is likewiſe inverted, and the object appears erect. But the apparent poſition of an object, or what point of the object ſhall appear the higheſt and what the loweſt, depends upon the direction in which the eye muſt be turned in order to ſee that point to the beſt advantage.

116. *An object may appear ſingle, though it is ſeen with both eyes at once.*

This is a ſecond objection to the cauſe of viſion, as we have juſt explaned it. For ſince, when we look at an object with both eyes, there is a picture of it upon each retina, it may be aſked, if the picture is the cauſe of viſion, why is viſion ſimple, ought we not to ſee as many objects as there are pictures? And ſince there are two pictures of every object why does not every object appear double?

In anſwer to this enquiry, we ſay that the ſame object may appear either double or ſingle, and by obſerving in what poſition the two eyes are, when it appears double, and in what poſition they are, when it appears ſingle, we may be led to the true cauſe of ſimple viſion. If, in common language, both eyes are turned directly towards an object, or, in the language of optics, if the optic axes are directed towards it and concur or meet at the place of the object, the object appears ſingle: but if, when we ſee an object, one eye only is directed towards it, and the other is directed another way, or if neither eye is turned towards it, but both of them are directed towards ſomething elſe; in either of theſe caſes the object appears double.

If both eyes are turned directly to the object C, Plat. XI. fig. 7, or, if the optic axes AC and AB concur in this object, it will appear ſingle. But if, whilſt one eye is turned towards C, the corner of the other is preſſed with the finger ſo as to turn it another way, then the object will appear double. Or if, whilſt both eyes are directed to C, there is any other object D placed at ſome diſtance beyond it; the object D will appear double, as the eyes are not directed towards it, but towards ſomething elſe, which is placed nearer, at C. Or if two objects *c* and *d*, Plat. XI. fig. 8, were placed as before, and both eyes were directed towards the remoter of them *d*, then the nearer, which is at *c*, will appear double, ſince therefore an object appears ſingle, when the optic axes concur in it, and double, when they do not, we may conclude that ſimple viſion is owing to the concourſe of the optic axes. But in what manner it is owing to this cauſe is the queſtion: as thoſe perſons ſee objects ſingle, who know nothing of either optic axes or their concourſe. It is true that common people know nothing of theſe technical terms: but then they know ſomething which is equivalent to what theſe terms ſtand for, ſomething which is the true reaſon why ſuch a concourſe of the optic axes ſhould produce ſimple viſion.

When one eye is turned towards an object, and the other is turned a different way, which is the caſe, if the corner of either is preſſed with the finger; then if the ſame object is ſeen by both the eyes, it is ſeen by each in a different direction, or one eye ſees it in one place and the other eye ſees it in another: this is obvious to every man who has the uſe of his eyes; and it is as obvious that when an object is ſeen in two places at once it muſt appear double. But, if both eyes are directed the ſame way or to the ſame place, and that is the place of the object; then, though it is ſeen by each eye, ſo that two objects may be ſaid to be ſeen, yet as both of them are ſeen in the ſame place they will appear but as one.

This is ſo certain, that even when there are really two objects, yet if they can by any contrivance be made to appear in the ſame place they will look like one. Let C and D, Plat. XI. fig. 9, be two candles. EH FG a piece of paſte-board with a hole in it at K. Place the candles at ſuch a diſtance from each other and from the paſte-board that, when the eyes are at A and B, if the right eye B is ſhut the left eye A may ſee only the candle D, and if the left eye A is ſhut the right eye B may ſee only the candle C. From this ſuppoſition it is evident that if both eyes are open both the candles will be ſeen, one of them by one eye and the other by the other eye. And yet if both eyes are directed to the hole K ſo as not to attend to the diſtance of the candles from the hole, then both will ſeem to be at K, or both of them will be ſeen in the ſame place, and then theſe two candles will appear as if there was only one.

If the eyes were both of them fixed upon the ſame object C, Plat. XI. fig. 7, any other object as D, which ſtands beyond it will appear double. For ſince the eyes ſee the object D without being turned directly towards it, the place of D is quite indeterminate; only it appears on the right hand ſide of C to the right eye at B, and on the left hand ſide of C to the left eye at A: and therefore it appears double becauſe it is ſeen on both ſides of C or in two different places at the ſame time.

If the eyes were both of them fixed upon the ſame object *d*, Plat. XI. fig. 8, any other object *c*, which ſtands nearer than *d*, will appear double. For ſince the eyes ſee the object *c* without being turned directly towards it, the place of *c* is quite indeterminate; only it appears to the right eye ſomewhere in the line *bce*, and to the left eye ſomewhere in the line *acf*. And becauſe the eyes are not directed towards *c* but towards *d*, no ſort of judgment is made about the diſtance of *c*, but it ſeems to be ſomewhere in the ſame plane *odt* where *d* is. Now theſe two lines *bce* and *acf* croſs each other at *c*; ſo that the line *bce*, which is drawn from the right eye falls on the left hand ſide of *d* at the plane *ot*, and the line *acf*, which is drawn from the left eye, falls on the right hand of *d*. Therefore the object *c* is ſeen on the left hand ſide of *d* by the right eye, and on the right hand ſide of it by the left eye; and conſequently ſince each eye ſees it in a different place, it muſt neceſſarily appear double.

From theſe two laſt caſes we might now draw two general concluſions about double viſion; but as they will be of ſome uſe to us hereafter and will more eaſily be referred to, if we make diſtinct propoſitions of them, they ſhall be the ſubject of the two next propoſitions.

People, that ſquint, never direct both their eyes to the object they are looking at: and for this reaſon if they uſed both their eyes to look at it, they would ſee it double. But when one of their eyes is directed towards the object, the diſtorted eye is turned ſo far away from it, towards their noſe commonly, that by this eye the object is not ſeen at all or elſe is ſeen too confuſedly to be taken notice of: ſo that their ſingle viſion ariſes from their uſing but one eye at a time. A Perſon, who can ſquint a little when he pleaſes, by a voluntary diſtortion of his eyes, may eaſily try this. For if he diſtorts them juſt enough to appear to ſquint to any one that looks at him, but not enough to turn either eye quite away from the objects before him, he will find that theſe objects appear double.

117. *When an object is more remote than the concourſe of the optic axes, it appears double; and of this double appearance the left-hand one is ſeen by the left eye, and the right-hand one by the right eye.*

If the eyes are both of them directed to C, Plat. XI. fig. 7, that is, if C is the concourſe of the optic axes; any object D, which is farther off than C, will appear double. This was proved and the reaſon of it was ſhewn in propoſition 116. Where we obſerved that D is ſeen by the left eye on the left hand of C, and by the right eye on the right-hand of C. And as this is the reaſon why D appears double, it follows that ſince in this double appearance the left-hand one is ſeen on the left hand of C it muſt be ſeen by the left eye; and for the ſame reaſon the right-hand appearance muſt be ſeen by the right eye. This will be made farther evident upon tryal. For if the left eye A was ſhut, the right one continuing open, then the left-hand appearance, would be ſeen no longer, but D would appear ſingle: from whence it is plane that the left eye, when open, ſees this left-hand appearance. In like manner if the right eye B was ſhut, the left one continuing open, then the right-hand appearance would vaniſh and D would appear ſingle: from whence it is plane that the right eye, when open, ſees this right-hand appearance.

118. *When an object is nearer than the concourſe of the optic axes, it appears double; and of this double appearance the left-hand one is ſeen by the right eye, and the right-hand one by the left eye.*

If both eyes are directed to *d*, Plat. XI. fig. 8, or if *d* is the concourſe of the optic axes, then this concourſe is farther from the eyes than an object placed at *c*, and the object *c* will appear double; as was ſhewn in propoſition 116: to the right eye *b* it will appear at *e* on the left-hand of *d*, and to the left eye *a* it will appear at *f* on the right hand of *d*: ſo that in this double appearance the left-hand one *e* is ſeen by the right eye, and the right-hand one by the left eye. This will be farther evident upon tryal. For if only the right eye *b* is ſhut the left-hand appearance at *e* vaniſhes; and if only the left eye *a* is ſhut, the right-hand appearance at *f* vaniſhes.

119. *The choroides is the principal ſeat of viſion.*

Let two ſmall circles of white paper be faſtened upon a dark-coloured wall at ſuch a height as to be level with the eyes and at the diſtance of ſomething leſs than two feet from each other. Then let the ſpectator place himſelf at a proper diſtance (what that proper diſtance is he will beſt find by trying.) And if he ſhuts his left eye and turns his right eye towards the paper on his left hand, he will not be able to ſee the right-hand paper at all. In like manner if he ſhuts his right eye and turns his left eye ſo as to look directly at the paper on his right hand, he will ſee nothing of the left-hand paper. In either of theſe caſes, in the firſt of them for inſtance, when his right eye is directed towards the left-hand paper, notwithſtanding the turn of his eye makes it very oblique to the right-hand paper, yet ſome rays from thence will enter it: for if this obliquity was great enough to prevent the rays from entering it, then certainly no rays could enter it from any other objects which are farther placed to the right than this paper is, and to which conſequently his eye is more oblique. But ſuch objects as are ſo placed will be viſible at the ſame time that the paper is not viſible. Therefore rays coming from thoſe objects and conſequently rays coming from the paper enter his eye. From whence we may conclude, that as the paper is inviſible, though rays, which come from thence, paſs into the ſpectators eye, thoſe rays or the picture of the paper muſt fall upon ſuch a part at the bottom of the eye as is inſenſible. The part where they fall is the point N, Plat. XI. fig. 4, where the optic nerve is inſerted. Now the retina, which is ſpread every where elſe over the bottom of the eye is not diſcontinued at N but is ſpread over that part as well as the reſt. Therefore if the retina was the ſeat of viſion the paper would be ſeen as well, when the picture of it falls upon the point N, as when it falls any where elſe. But in this place where the nerve is inſerted, the choroides, which encloſes and touches the retina in all other parts, does not touch it at the baſe of the optic nerve. From whence we may conclude that though the picture is painted upon the retina, yet the preception of this picture is in the choroides; for where this coat is cloſe to the retina, and can receive impreſſions from it, there the picture produces viſion; but notwithſtanding a picture is painted where it is not cloſe enough to receive thoſe impreſſions, that picture produces no viſion.

From hence we may ſee the reaſon why the optic nerve was not inſerted exactly in the middle of the retina: for if it had, any picture falling upon the middle of the retina would have produced no ſenſation, and conſequently that point of an object, towards which our eyes are immediately directed, inſtead of being ſeen diſtincter than the reſt, would have been inviſible.

120. *Objects, which are placed at different diſtances, could not be ſeen diſtinctly, unleſs ſome change was made in the diſtance between the chryſtalline humour and the retina, or elſe in the convexity either of the fore part of the eye or of the chryſtalline humour.*

If the figure of the eye and the ſituation of all its parts were to continue always the ſame at all diſtances of the objects that we are looking at; the picture of ſome objects upon the retina would be confuſed becauſe they are too far off, and of others becauſe they are too near.

For ſuppoſe an object was ſeen diſtinctly at the diſtance of 2 feet from the eye; then the picture of it, when it is at this diſtance, is diſtinct upon the retina; as was obſerved under propoſition 112. But ſince, by propoſition 111, this picture has the ſame properties with thoſe that are painted upon a paper in a dark room; if the object was nearer than 2 feet, and the eye did not alter its figure, the picture upon the retina and conſequently the appearance of the object would be confuſed. For, by propoſition 87, as the object approaches the cornea, or fore part of the eye, the diſtinct picture will depart from it: and conſequently if the retina does not move but continues at the ſame diſtance from the cornea, the picture of the object upon it cannot be diſtinct. But if either the retina could be moved farther from the cornea, and conſequently farther from the chryſtalline humour; or if the chryſtalline humour was moveable and could be brought forwards ſo as to be farther from the retina; that is, if the paper was moved farther from the lens, or the lens was moved farther from the paper, the picture, and, in conſequence of that, the appearance of the object would be diſtinct; by propoſitions 87, 98. In like manner, if the object was at a greater diſtance from the eye than 2 feet, becauſe as the object departs the diſtinct picture approaches, the retina in order to preſerve the repreſentation upon it diſtinct muſt be brought nearer to the chryſtalline humour, by propoſition 87, or elſe the chryſtalline humour muſt be moved nearer to the retina, by propoſition 98. Thus if either the chryſtalline humour or the retina or both of them are moveable, ſo that the diſtance between them may be varyed; objects, which are placed at different diſtances from the eye, may be ſeen diſtinctly.

The ſame advantage of diſtinct viſion, at different diſtances of the object, may be obtained another way, and that is by changing the convexity either of the chryſtalline humour or of the cornea. Where we are to obſerve that changing the convexity of the cornea will neceſſarily change the convexity of the fore part of the aqueous humour which is contained in this coat. The picture upon the retina is made diſtinct by either of theſe means, in the ſame manner as, in propoſition 97, the picture upon the paper in a dark room is preſerved diſtinct, when the diſtance of the object is altered, by changing the lens and uſing one either of a greater or of a leſs convexity, as the diſtance of the object requires. Thus if the object is remote the convexity either of the aqueous humour where the rays ſuffer the firſt refraction, or of the chryſtalline humour where they are refracted again, ſhould be diminiſhed; that the picture may be thrown back by thus diminiſhing the convexity, juſt as much as it is brought forwards by encreaſing the diſtance of the object. Or if the object is a near one, then the convexity of one or perhaps of both theſe humours may be encreaſed, that the picture may be brought forwards by thus encreaſing the convexity, juſt as it is thrown backwards by diminiſhing the diſtance of the object. By ſuch an alteration as this the picture may be preſerved diſtinct upon the retina, though the retina itſelf is immoveable.

Some of theſe changes are neceſſary. But which of them is the change that the eye has a power of making we muſt leave to the anatomiſts to determine.

121. *The defect of ſome eyes is of ſuch a ſort, that no objects will appear diſtinct unleſs they are very near.*

This is the defect in their eyes, who are ſaid to be ſhort-ſighted. If an object is at ſuch a diſtance, that a perſon with good eyes might ſee it diſtinctly, it will appear confuſed to them; but if the object is at a leſs diſtance or is almoſt cloſe to their eyes, then they are able to ſee it diſtinctly. If the eye, either in the fore part of it or in the chryſtalline humour is too convex it will have this defect. For the greater the convexity is, where the rays are refracted, the nearer the picture will be to the fore part of the eye, when the diſtance of the object is given, as appears from what was ſaid under propoſition 95. Now if the diſtinct picture is nearer to the fore part of the eye than the retina is, then the repreſentation upon the retina and conſequently the viſion will be confuſed; by propoſitions 86, 110, 113. But when the object is brought nearer, the diſtinct picture will be farther from the fore part of the eye, by propoſition 87. And therefore, if the object is brought near enough, the diſtinct picture may by this means be thrown ſo far back as to fall upon the retina and produce diſtinct viſion.

Some ſhort-ſighted perſons in reading or in looking at any object hold the book or the object oblique to their eyes, and this helps them to ſee it more diſtinctly. For if they hold it directly before their eyes the picture will fall upon the middle of the retina about the point *e*, Plat. XI. fig. 5. But if they hold it obliquely it will fall upon the ſide of the retina between *d* and *l*, or between *f* and *m*, according as they hold the object on one ſide or the other. Now the middle of the retina is farther from the fore part of the eye than the ſide of it is. Therefore though the picture is ſo near to the fore part of the eye as to be confuſed, if it was to fall upon the middle, it may be diſtinct, when it falls upon the ſide.

Perſons, who are not born with this defect, are frequently found to bring it upon themſelves by the manner in which they uſe their eyes. This is remarkably the caſe of watch-makers, engravers, and painters in miniature, who are obliged to look very near to their work. For when the eye has been long uſed to that figure, which is proper for ſeeing near objects diſtinctly, it ſettles into that figure, and cannot, as a good eye may, be altered at pleaſure ſo as to ſee any objects diſtinctly that are not near to it. For the ſame reaſon ſtudents, if they hold their book too cloſe to their eyes, will be apt to grow ſhort-ſighted.

From hence we may obſerve that this defect does not always mend as a man grows older: for by looking long and frequently at very near objects, whilſt he is young and his eyes are capable of being fixed in any ſhape by uſe and cuſtom, he will keep himſelf ſhort-ſighted, if he was ſo at firſt, or will make himſelf ſhort-ſighted, though he had naturally good eyes. This obſervation will point out a way either to remedy this defect or to prevent it, if our employment is of ſuch a ſort as to allow us to follow it. If thoſe, who are ſhort-ſighted, inſtead of uſing the artificial helps of glaſſes, will frequently look at remote objects, and in reading would hold their book as far from their eyes as they poſſibly can; the eye will change its figure by degrees and by cuſtom will acquire the power of adapting itſelf ſo as to ſee diſtant objects more diſtinctly. And good eyes may be prevented from becoming ſhort-ſighted, if in reading we are careful not to hold the book too cloſe to our eyes, and uſe ourſelves to look at diſtant objects as well as at near ones.

122. *The defect of ſome eyes is of ſuch a ſort that no objects are ſeen diſtinctly, unleſs they are at ſome diſtance.*

Old people are moſt liable to this defect, and thoſe more eſpecially who have been uſed to look much at very diſtant objects, and ſeldom at near ones, ſuch as travellers, ſailors, and ſportſmen. For as the eye grows flatter, either by age or by uſe, the rays of each pencil in paſſing through it are made to converge leſs, and the place of the diſtinct picture, if the object is near, will be more remote than the retina, and conſequently the picture upon the retina will be confuſed; by propoſition 87. But if the object is removed farther from the eye, this will bring the diſtinct picture nearer to the fore part of the eye; and by removing the object to a proper diſtance, the diſtinct picture may be brought near enough to fall upon the retina.

It is ſometimes impoſſible to prevent this defect by any care whatever: for age will ſometimes flatten the eye notwithſtanding all our endeavours. But it may frequently be prevented by uſing ourſelves to look often at near objects as well as at remote ones. For this will prevent the eye from ſettling to any particular figure. And indeed the muſcles, which are uſed for changing the figure of the eye in order to ſee objects at different diſtances, will beſt retain their uſe in all inſtances by frequent exerciſe; ſo that by accuſtoming ourſelves ſometimes to look at very near objects and ſometimes at very remote ones, we may commonly preſerve our eyes, and neither make ourſelves ſhort-ſighted, when we are young, nor be long-ſighted, when we are old.

123. *An object may be either ſo near or ſo remote, that the beſt eyes would be unable to ſee it diſtinctly.*

Though the eye has a power of changing its figure ſo as to ſee objects diſtinctly at different diſtances; yet this power is a limited one. If the object is either too near or too remote, the eye cannot change itſelf enough to make the picture diſtinct upon the retina. And for this reaſon near objects or very remote ones will appear a little confuſed to the beſt eyes. To ſee an object very diſtinctly the diſtance of it ſhould not be leſs than about 7 or 6 or at the leaſt 5 inches.

124. *The optic angle in viewing any object is an angle at the eye ſubtended by the diameter of the object.*

The angle *aoc*, Plat. XI. fig. 5, which in viewing the object *abc* is ſubtended by *ac* the diameter of this object, is called the optic angle.

125. *The apparent diameter of any object is proportional to the diameter of that objects picture upon the retina.*

The apparent diameter of an object is a quantity of viſible extenſion, that lies between the two extremities of that object. Thus the apparent diameter of an object *abc*, Plat. XI. fig. 5, to an eye placed at *o* is that viſible extenſion, which lies between the two extreme points of it *a* and *c*. But if two lines *ao* and *co* are drawn from theſe points ſo as to croſs one another at the eye, theſe two lines will be the axes of the extreme pencils, one of which comes from *a*, and the other from *c*; and each point will be painted upon the retina in theſe two lines continued thither, as was obſerved in proving propoſition 85. From hence it follows that the apparent diameter of the object is a viſible extenſion contained within two right lines *ao* and *co*, which are drawn from the extremities of the object and croſs each other at the eye; and *df* the diameter of the picture upon the retina is likewiſe contained between the ſame right lines continued to the retina. Therefore the angle *aoc*, which is ſubtended by the object, is equal to the angle *dof*, which is ſubtended by the picture: for they are vertical angles made by two right lines croſſing each other. Euc. b. I. prop. 15. But if theſe two angles are always equal, one of them can neither encreaſe nor decreaſe but the other will encreaſe or decreaſe in the ſame proportion: thus if *aoc* was to encreaſe ſo as to be double what it is, *dof* would likewiſe be doubled, if *aoc* was to be tripled, *dof* would be tripled; or if *aoc* was to be diminiſhed one half, *dof* would be halved at the ſame time, if *aoc* was reduced to $\frac{1}{3}$ of what it is, *dof* would be reduced to $\frac{1}{3}$. Now the viſible extenſion contained between *ao* and *co* will be greater as the diſtance between theſe lines is greater, or leſs as the diſtance between them is leſs; that is, the apparent diameter of the object will be proportional to the angle, which theſe two lines make with another at the eye, or to the angle *aoc*. And the diameter of the picture upon the retina which is contained between *do* and *fo* will be greater as the diſtance between theſe lines is greater, and leſs as the diſtance between them is leſs; that is, the diameter of the picture will be proportional to the angle which theſe lines make with each other at *o*, or to the angle *dof*. But the angle *aoc* is always equal and conſequently proportional to the angle *dof*. Therefore the apparent diameter of the object is proportional to the diameter of its picture upon the retina.

126. *When the diameter of an object is given, its apparent diameter is inverſely as its diſtance from the eye.*

The diameter of the ſame object does not appear equal, when it is viewed from different diſtances; but the apparent length of it will be inverſely as the diſtance of the eye from it, greater in proportion as this diſtance is leſs, and leſs in proportion as this diſtance is greater. For the apparent diameter of an object is proportional to the diameter of its picture upon the retina, by propoſition 125. And when the diameter of the object is given, the diameter of the picture, like that of a picture in a dark room will be inverſely as the diſtance of the object, by propoſitions 91, 113. Therefore the apparent diameter of the object is likewiſe inverſely as the objects diſtance.

What has here been proved of the apparent diameter or apparent length of an object is applicable to any other apparent length whatſoever. Thus if AE and BF, Plat. XI. fig. 10, are two parallel rows of trees; to a ſpectator ſtanding at G the two oppoſite trees E and F at the farther end of the row will ſeem nearer to one another than the two oppoſite trees A and B, which are at the end next to him. For the apparent lengths AB, CD, EF, or the apparent diſtances of any two oppoſite trees, will be inverſely as the ſpectators diſtance from them. And upon this account the two rows, though they are parallel, will ſeem to converge. In like manner if AE was the cieling and BF the floor of a large room, the diſtance between the floor and the cieling will appear to be leſs at E and F than at A and B, if the ſpectator ſtands at G. Or, if GH an imaginary line is conceived to be drawn parallel to both the floor and the cieling; the cieling will converge downwards and the floor upwards towards this line. Now this apparent riſing of the floor towards the line GH would evidently be the ſame whether there was any cieling or not. Let us therefore ſuppoſe that BF, inſtead of being the floor of a room, was the ſurface of the ſea; this ſurface would apparently riſe towards the imaginary line GH. And this is the reaſon why to a ſpectator upon the ſhore the ſurface of the water, as he looks out to ſea, ſeems gradually to aſcend, and the ſhips out at ſea ſeem to be higher than the land that he ſtands upon. Or if BF, inſtead of being a horizontal plane ſuch as the floor of a room or the ſurface of the ſea, was the tower of a church or any high building perpendicular to the horizon; then if a ſpectator ſtood at the bottom or at G, and we conceive an imaginary line GH drawn from his eye parallel to the building, however upright the building may ſtand, it will apparently converge to this imaginary line, and therefore will ſeem to lean towards that ſide, where the ſpectator is.

From this propoſition it follows that the apparent diameter of an object when we ſee it with the naked eye, may be magnifyed in any proportion that we pleaſe. For, ſince the apparent diameter is inverſely as the diſtance of the eye, in whatever proportion we have a mind to encreaſe the apparent diameter we need only diminiſh the diſtance of the eye from the object in the ſame proportion. Thus ſuppoſe there is an object AB, Plat. VIII. fig. 1, which to an eye at E appears under or ſubtends the angle AEB; we may magnify the apparent diameter in what proportion we pleaſe by bringing our eye nearer to it. If for inſtance we would magnify it in the proportion of FG to AB, that is, if we would ſee the object under an angle as great as FEG, or would make it appear of the ſame length that an object as long as FG would appear, this may be done by coming nearer to the object. For the apparent diameter is as the diſtance inverſely; therefore, if CD is as much leſs than CE as FG is greater than AB, by bringing the eye nearer to the object in the proportion of CD to ED, the apparent diameter will be magnifyed in the proportion of FG to AB. So that the object AB to the eye at D will appear as long as an object FG would appear to the eye at E. In the ſame manner we might ſhew that the apparent diameter of an object, when ſeen by the naked eye, may be infinite. For ſince the apparent diameter is reciprocally as the diſtance of the eye, when the diſtance of the eye is nothing, or when the eye is cloſe to the object at C, the apparent diameter will be the reciprocal of nothing or will be infinite.

There is indeed one great inconvenience in thus magnifying an object without the help of glaſſes only by placing the eye nearer to it. The inconvenience is, that we cannot ſee an object diſtinctly unleſs the eye is about 5 inches from it: therefore if we bring it nearer to our eye than 5 inches, however it may be magnifyed, it will be ſeen confuſedly. Upon this account the greateſt apparent magnitude of an object, that we are uſed to, is the apparent magnitude, when the eye is about 5 inches from it. For though by bringing it within 1 inch of our eye the apparent diameter would be 5 times as great, or at the diſtance of $\frac{1}{4}$ inch would be

20 times as great, or at the diſtance of $\frac{1}{10}$ inch would be 50 times as great; yet ſince in theſe near approaches of the eye the object will be ſeen in great confuſion, we never place an object ſo near us in order to look at it: becauſe though it might be magnifyed by this means, the confuſion would prevent us from making any uſe or advantage from ſeeing the object ſo large. The ſize of any object, when it is viewed through a convex lens, ſeems to be extraordinary; not becauſe it is impoſſible to make it appear of the ſame ſize to the naked eye, but becauſe in ſo near an approach of the naked eye, as would be neceſſary for this purpoſe, it would appear very much confuſed; for which reaſon we never do bring our eye ſo near to it: and conſequently, as we have not been commonly uſed to ſee the object of this ſize, we think it an extraordinary one.

127. *The apparent diameter of an object is directly as its real diameter, when its diſtance from the eye is given.*

The apparent diameter of an object is proportional to the diameter of the picture of it upon the retina, by propoſition 125. And the diameter of the picture upon the retina, when the diſtance of the object is given, is proportional to the real diameter of the object, by propoſitions 93, 113. Therefore the apparent diameter of an object, whoſe diſtance is given, is proportional to its real diameter.

128. *The apparent diameters of different objects at different diſtances from the eye will be equal, if their real diameters are proportional to their diſtances.*

If a man 6 feet high is at 40 yards diſtance from us, and a boy 3 feet high is at only 20 yards; then their heights are proportional to their reſpective diſtances for 6 is to 3 as 40 is to 20. In this and in all other inſtances of the ſame ſort, their apparent heights will be equal. This follows from propoſitions 94, 125. For the diameters of their reſpective pictures upon the retina will be equal; and their apparent diameters are proportional to the diameters of theſe pictures.

129. *The apparent diameter of an object ſeen obliquely, is proportional to the apparent length of a ſubtenſe of the optic angle perpendicular to the optic axis.*

We have hitherto ſuppoſed that the object is ſeen directly, or that the optic axis *ob*, Plat. XI. fig. 5, is perpendicular to the object *ac*. But let the object AB, Plat. XI. fig. 11, be ſeen obliquely, that is, let it be inclined to the optic axis OC; and then if AOB is the optic angle the ap-

parent

parent diameter of the object will be proportional to the apparent length of DB, which is a ſubſenſe of AOB perpendicular to OC the optic axis. For ſince the apparent diameter of the object is the viſible extenſion which lies between A and B, or which is included between AO and BO the two legs of the optic angle, by propoſition 125; and ſince the ſubtenſe DB is included between AO and BO; it follows that the viſible extenſion or apparent length of this ſubtenſe is proportional to the apparent diameter of the object. Or otherwiſe. If AB is an object ſeen obliquely, and DB is an object ſeen directly; that is, if AB is oblique to the optic axis and DB is perpendicular to it, then ſuppoſing both of them to ſubtend the ſame angle AOB, their pictures upon the retina and conſequently their apparent diameters will be equal, by propoſition 125. That their pictures upon the retina will be equal is evident: for the ſame right lines BO and ADO continued to the retina determine the length of both their pictures, as appears from what was ſaid in proving propoſition 125.

From hence we ſee that an object appears ſhorter for being ſeen obliquely. For though the object AB is longer than DB, yet becauſe AB is ſeen obliquely and DB is ſeen directly, AB appears no longer than DB. If AB was more oblique it would appear ſtill ſhorter. And if it was ſo oblique as to lie ſtrait out from the eye, ſo that the end B was towards the eye and the end A was in the ſame right line directly from it, the apparent length of BA would then be nothing. For in this caſe a line AO drawn from A the farther end of the object to O the eye, and a line BO drawn from the nearer end B to O would coincide, ſo that the angle AOB contained between them, and conſequently DB the ſubtenſe of that angle, would be nothing.

130. *When equal objects are ſeen very obliquely, the apparent lengths of them are inverſely as the ſquares of their diſtances from the eye.*

Let the eye be at O, Plat. XII. fig. 1, and any where in the line BC at a conſiderable diſtance from the eye take equal ſpaces DF and *df*. The apparent lengths of theſe equal ſpaces DF and *df* will be inverſely as the ſquares of their diſtances, that is, they will be as the reciprocals of the ſquares of thoſe diſtances. Thus if O*f* is double OF, or if the diſtances are as 2 to 1, the apparent length of *df* will be $\frac{1}{4}$ the apparent length of DF. If O*f* was to OF as 3 to 1, the apparent length of *df* would be $\frac{1}{9}$ that of DF. If their diſtances were as 4 to 1, their apparent lengths would be as $\frac{1}{16}$ to $\frac{1}{1}$.

The apparent length of DF is proportional to the apparent length of GD, by propoſition 129. For DF is ſeen obliquely, OR is the optic axis, DOF is the optic angle, and GD is a ſubtenſe of this angle perpendicular to OR. For the ſame reaſon the apparent length of *df* is proportional to the apparent length of *gd*. But if *Of* is double OF than *gd* is $\frac{1}{2}$ GD, if *Of* is triple OF, *gd* is $\frac{1}{3}$ GD, and univerſally, *gd* is to GD as *Of* to OF, or as the diſtances of *f* and F from the eye inverſely. In order to prove this, we muſt obſerve that GD is the ſubtenſe not only of the angle DOF, but likewiſe of the angle GFD; and *gd* is the ſubtenſe not only of the angle *dOf*, but likewiſe of the angle *gfd*. Now if *Of* is double OF, the angle *Of*F is $\frac{1}{2}$ the angle OFB. For ſince OF*f* is a plane triangle the ſides are to one another as the ſines of the oppoſite angles, by propoſition 32, or *Of* is to OF as the ſine of the angle OF*f* to the ſine of the angle *Of*F. But OF*f* and OFB have the ſame ſine, becauſe one of theſe angles is the complement of the other to two right ones. Therefore *Of* is to OF as the ſine of the angle OFB to the ſine of the angle *Of*F or *Of*B, ſince *Of*F and *Of*B are one and the ſame angle. Now *Of*, by the conſtruction, is double OF, and conſequently the ſine of the angle OFB is double the ſine of the angle *Of*B. But ſmall angles are to one another nearly as their ſines, by propoſition 26. Therefore if theſe are ſmall angles OFB will be double *Of*B; and the ſubtenſe GD will be double *gd*, or GD will be to *gd* as 2 to 1; that is, ſince OF is to *Of* as 1 to 2, GD will be to *gd* as OF to *Of* inverted; or the ſubtenſes to which the apparent diameters of DF and *df* are proportional will be inverſely as the diſtances of DF and *df* from the eye, upon account of the different obliquities at which the eye ſees theſe two equal ſpaces DF and *df*. But then beſides this, without conſidering the difference of obliquity, or if their obliquities were the ſame, the apparent length of DF is to the apparent length of an equal line *df*, as their diſtances inverſely, by prop. 126. Since then the apparent lengths of DF and *df* are inverſely as their diſtances upon account of their different obliquities; and are likewiſe inverſely as their diſtances upon account of their different diſtances; it follows that upon both accounts together they will be in a ratio compounded of the inverſe one of their diſtances, or will be inverſely as the reſpective diſtances multiplyed into themſelves, or inverſely as the ſquares of their reſpective diſtances. Thus if *Of* is twice OF or if *Of* is to OF as 2 to 1, the apparent length of *df* will be to the apparent length of DF in the compound ratio of $\frac{1}{2}$ to $\frac{1}{1}$ and $\frac{1}{2}$ to $\frac{1}{1}$ or as $\frac{1}{4}$ to 1. For *df* appears but $\frac{1}{2}$ as long as DF upon account of their different obliquities, and but $\frac{1}{2}$ as long upon account of their different diſtance, therefore, the apparent length of *df* being twice halved in reſpect to DF, upon both accounts together it will be only $\frac{1}{4}$ of the apparent length of DF.

From whence we ſee the reaſon, why in walking through a double row of trees, the apparent diſtance between the two neareſt trees in the ſame row encreaſes much faſter than the diſtance between the two oppoſite trees in the different rows. Thus, Plat. XI. fig. 10, if trees were planted in a row along the line ACE, and other trees along the line BDF, any perſon walking between the rows in the direction GH will ſee the apparent diſtance DF between the two trees D and F, that are neareſt to one another in the ſame row, encreaſe much faſter than the apparent diſtance EF, which lies between two oppoſite trees in the different rows. For the diſtance EF is ſeen directly and conſequently the apparent length of it encreaſes, when the ſpectator approaches it, in the ſame proportion that his diſtance from it decreaſes; for the apparent length of it is inverſely as his diſtance, by propoſition 126. But, by the propoſition now before us, the apparent length of DF which is ſeen obliquely encreaſes in the ſame proportion that the ſquare of the ſpectators diſtance decreaſes, or is inverſely as the ſquare of his diſtance. Therefore ſince the ſquare of the ſpectators diſtance decreaſes faſter than the diſtance itſelf, the apparent length of DF will encreaſe faſter than the apparent length of EF. If the reader is not aware that the ſquares of a quantity decreaſe faſter than the quantity itſelf, he may ſee it in the following inſtance. Suppoſe a ſpectator was walking in the direction GH, his diſtance then from EF and from DF would keep decreaſing: ſuppoſe him to view EF and likewiſe DF at three ſeveral diſtances, and let theſe diſtances be to one another as 3, 2, and 1; then the ſquares of theſe diſtances will be 9, 4, and 1: and it is evident that theſe latter numbers decreaſe faſter than the former. Now the apparent length of EF is as his diſtances inverted; and the apparent length of DF is as the ſquares of his diſtances inverted. Therefore at the diſtances 3, 2, and 1, the apparent length of EF will encreaſe in the proportion of 1, 2, and 3; and the apparent length of DF will encreaſe in the proportion of 1, 4, and 9. But theſe latter numbers encreaſe faſter than the former; and conſequently the apparent length of DF will encreaſe faſter than the apparent length of EF.

131. *The apparent diameter of an object is not at all changed either by contracting or dilating the pupil.*

When the diſtance of the object is given, the diameter of its picture upon the retina will be the ſame, whether the area of the pupill is large or ſmall, by propoſition 100. And conſequently the apparent diameter of the object will be the ſame, whatever is the area of the pupill, by propoſition 125.

132. *If all other circumſtances are the ſame an object will appear larger when it is ſeen confuſedly than when it is ſeen diſtinctly.*

When an object is ſeen confuſedly the picture upon the retina is confuſed, by propoſition 113. But the confuſed picture is bigger than the diſtinct picture of the ſame object would be, when all other circumſtances are the ſame, by propoſition 89. And the apparent magnitude of the object is always proportional to the magnitude of its picture upon the retina. Therefore any object, when it is ſeen confuſedly, will appear bigger than when it is ſeen diſtinctly.

This is the reaſon why any object, as a man, or a houſe, or a tree, in very miſty weather ſhould ſeem larger, than they do when the weather is clear. For when the rays that are to make the objects viſible are to paſs through ſo many ſmall drops of water, as they do in miſty weather, ſuch a great variety of ſurfaces will ſo differently refract them, that thoſe of the ſame pencil will not be collected into one and the ſame focus upon the retina; and this will make the picture at the retina a confuſed one.

Upon this principle any luminous body, when the picture of it upon the retina is confuſed or imperfect, ſhould appear larger, than when that picture is diſtinct or perfect. And the leſs perfect the picture is, ſo much the larger the body ought to appear. Now when the ſun or the moon are in the horizon, that is, when they are either riſing or ſetting; all the rays, which come from them, paſs along very near the ſurface of the earth, through all the vapours that float in the lower part of the atmoſphere; whereas when they are in the meridian or at their greateſt height, the rays come more directly to the ſpectators eye without paſſing ſo cloſe to the earths ſurface, or without paſſing through ſo many vapours or ſo much miſt. For this reaſon the picture of the ſun or moon upon the retina, when they are in the horizon, will be leſs perfect and conſequently will be bigger, than when they are in the meridian; and this makes them appear larger at riſing and ſetting than when they are at their greateſt height. The apparent magnitude of the ſun or moon is the ſame, when they are in the horizon, as when they are in the meridian, if they are ſeen through a teleſcope: for the picture, which is imperfect when they are viewed by the naked eye, is made a perfect one by the refraction of a teleſcope: and if the difference of apparent magnitude in the horizon and meridian was owing to the imperfection or confuſedneſs of the picture, then, by making the picture perfectly diſtinct, the apparent magnitude ought to be the ſame. From hence we may ſee the reaſon, why the horizontal moon ſhould at ſome times appear much larger than at others: for the more imperfect the picture is the larger the moon will appear; and the picture will be more imperfect, when the rays have a thicker miſt to paſs through, that is, when the lower parts of the atmoſphere are filled with a greater quantity of vapours. Whether the cauſe here aſſigned is the only one, upon which the difference of apparent magnitude of the ſun or moon in the horizon and in the meridian depends, is more than I will venture to affirm: but perhaps we may find as we go on that the principal cauſes which have commonly been aſſigned, are at leaſt as unſatisfactory.

133. *Bright objects ſeem to be larger than obſcure ones, though they are really of the ſame ſize.*

This is remarkably the caſe in the appearance of the moon, juſt before and juſt after the new. At that time the whole circular face or diſk of the moon is viſible; but only one part of it, which is the creſcent or horns, is bright; the reſt of the circle has barely light enough to make it viſible and is very faint in compariſon of the creſcent. Now though the creſcent and the faint part of the diſk are portions of the ſame circle, yet the creſcent ſeems to be a portion of a greater circle than the faint part. There are many inſtances of the ſame ſort: and it is well known that a white or light-coloured object will appear larger than a black or dark-coloured one, more particularly if they are at ſome diſtance from the eye.

When an object is remote enough from the eye to make the picture of it upon the retina a little confuſed, the rays of each pencil, inſtead of being collected into a ſingle point upon the retina, are ſcattered over a ſmall circular ſpot upon it; and a picture, that is made up of ſuch ſpots, will be ſomething larger than one, that is made up of exact focuſes or points, as we obſerved in propoſition 89. Upon this account, by propoſition 130, all objects whether bright or obſcure will appear the bigger for being ſeen a little confuſedly. But we are to aſſign a reaſon why this ſhould be more perceivable in bright than obſcure ones. Now the edges of a confuſed picture will always be fainter than any other part of it. Thus for inſtance, if the picture *df*, Plat. XI. fig. 5, was confuſed, the ends of it at *d* and *f* would be fainter than the middle of it at *e*. The rays indeed of the middle pencil, which ſhould be all of them collected into one and the ſame point at *e*, are ſpread over a ſmall circular ſpot there of which ſpot *e* is the center: and for this reaſon any part of that ſpot will be leſs enlightened, and conſequently will be fainter than the point *e* would have been, if they had all been collected into that ſingle point. But then as the rays of this pencil are thus ſcattered, ſo likewiſe are the rays of each pencil all round it; ſo that ſome rays of all theſe pencils, when they are are thus ſpread upon the retina, will fall upon the ſame part of it with thoſe of the middle one: and conſequently what the middle ſpot would want in brightneſs, if it had been enlightened by no rays but thoſe of the middle pencil, will be made up by the other rays, which are extended from the neighbouring pencils and fall upon the ſame place. This is equally true of any other part of the picture as well as of the middle, except only its edges. For at the two ends *d* and *f*, or any where elſe at the edge, the rays of each pencil are ſcattered, and ſpread themſelves over a ſmall circular ſpot, inſtead of being collected into a ſingle point. The picture extends no farther upon the retina than the extreme focuſes, when it is diſtinct; but it is extended as far as the extreme ſpots reach, when it is confuſed. But as all the rays of any pencil will enlighten a ſingle point more, than they will a circular ſpot which is bigger than a point, the edges of a confuſed picture will neceſſarily be fainter than the edges of a diſtinct one: unleſs where the rays of the neighbouring pencils are ſpread over the ſame ſpace with thoſe of the extreme ones; which, as we have already ſeen, is the caſe in the middle of the picture. But at the edges this is impoſſible: becauſe there are no other pencils beyond the extreme ones. Therefore the edges of the confuſed picture will be fainter than the middle or than any other part of it. Now if the object is a bright one, the edges of the confuſed picture, though they are faint, will however be bright enough to be perceived; and conſequently the object will appear larger, than if the picture had been diſtinct. But if the object is very obſcure, the edges of the confuſed picture will be too faint to be perceived, and the object will ſcarce appear larger than if the picture had been diſtinct. Therefore a bright object will ſeem bigger, when ſeen a little confuſedly, than an obſcure one of the ſame ſize.

This is no objection to the account that was given of the horizontal moon. For though the moon, when in the horizon, is not ſo bright, as when in the meridian; yet in the horizon it is bright enough for the whole confuſed, or rather imperfectly diſtinct, picture upon the retina to be perceived.

134. *Objects would appear equally bright at all diſtances, if no rays were intercepted in their paſſage from them to the eye.*

This follows from propoſitions 103, 113. For the brightneſs of viſion depends upon the brightneſs of the picture upon the retina: and this picture would be equally bright at all diſtances of the object, if no rays were intercepted in coming from the object to the eye of the ſpectator.

The ſun and the moon appear fainter when they are riſing or ſetting, than they do when they are at their greateſt height. This faintneſs is not owing to the remoteneſs of either of them, as has been already proved. But when they are in the horizon, the light that comes from them paſſes along near the ſurface, where the vapours are thick and will intercept many of the rays: whereas when they are in the meridian, the light comes to the eye more directly, and does not paſs ſo far through the loweſt, which are alſo the denſeſt, parts of the atmoſphere, and conſequently fewer rays will be intercepted.

Some have imagined that this faintneſs of the moon in the horizon may be the reaſon, why it ſhould ſeem bigger, when it is there, than it does, when it is in the meridian. Becauſe when an object is faint, the pupill in looking at it is dilated, and this, as they affirm, will make the picture upon the retina ſo much the bigger. But this has already been ſhewn to be a miſtake, ſince we have proved, in propoſition 131, that the apparent diameter of an object is not changed either by contracting or dilating the pupill. To this indeed it is replyed, that the merely contracting or dilating the pupill will not alter the diameter of the picture upon the retina, and conſequently will not alter the apparent diameter of an object; yet when the pupill is dilated the conſequence is that the ligaments *e* and *c*, Plat. XI. fig. 4, are drawn outwards, or towards A and A, by which means the chryſtalline humour will be brought forwarder or will be removed farther from the retina; and when the pupill is contracted this humour will be removed nearer to the retina: and the farther the chryſtalline humour is from the retina ſo much bigger the picture will be, or the nearer this humour is to the retina, ſo much leſs the picture will be, by propoſition 95. Now if this was true, if a dilated pupill made the picture larger and a contracted one made it ſmaller, all faint objects, which are looked at with a dilated pupill, ought to appear larger than bright objects of the ſame ſort, which are always looked at with a more contracted one: but ſince we have ſeen, in propoſition 133, that juſt the reverſe of this happens, we may conclude that it is not the contraction or dilatation of the pupill, nor any conſequences of ſuch contraction or dilatation, which makes the moon appear bigger in the horizon and leſs in the meridian.

135. *The judgment that we make about the diſtance of objects does not depend upon any ſingle principle.*

The diſtance of an object is the extenſion that lies between the eye and the object, or is the length of a line drawn from the eye to the object.

Thus, Plat. XI. fig. 10, the diſtance of EF the end of a room from the ſpectator at G is the extenſion that lies between G and EF or is the length of the imaginary line GH. Now this line is inviſible in itſelf, or if it was not ſo but was a real line, ſuch as a pack-thread, extended from G to H, yet the length of it would be inviſible from its ſituation, by propoſition 129; becauſe the end of it or G is at the eye and it is drawn directly from thence in the direction GH. Therefore the diſtance of EF, if we look at EF directly, without conſidering any thing elſe, is not to be ſeen or is not the object of ſenſe. But though GH is inviſible, yet BF the floor of the room, or AE the cieling of it, or the walls on each ſide, are any of them viſible extenſions, which lie between the eye at G and the end of the room EF, therefore by looking at any of theſe we may judge of the diſtance of EF or may make this diſtance in ſome ſort the object of ſenſe. By the ſame method we perceive or judge of the diſtance of other objects in many inſtances. The viſible length of road, which is between us and the place we have in view, is our apparent diſtance from that place. The viſible length of the fields, hedges, woods, hills, and buildings, between us and a houſe at the end of the proſpect, is the apparent diſtance of that houſe from us.

This method of eſtimating apparent diſtances will frequently fail us and ſome other muſt be made uſe of. When the diſtances of objects are very great, the farther parts of theſe diſtances are inviſible. The apparent magnitudes of the remoter parts of long lines decreaſe very faſt, by propoſition 130; in the line BC, Plat. XII. fig. 1, the apparent magnitudes of DF and *df* are inverſely as the ſquares of their reſpective diſtances from the eye; and in all other caſes the apparent length of any given part in the ſame line BC continued will decreaſe as faſt as the ſquare of its diſtance from the eye encreaſes. But ſince this quick decreaſe of apparent magnitude makes the remoter parts of long lines inviſible on account of the ſmallneſs of the angles which they ſubtend; it follows that though two remote objects are at different diſtances, yet the eye cannot perceive this, becauſe the difference of their diſtance may ſubtend too ſmall an angle for the eye to diſcern it. This is the reaſon why a perſon, who is unuſed to a ſea-proſpect, and has not by habit acquired ſome other method of judging how far an object is from him, beſides this of looking at the apparent ſpace that lies between him and the object, will be much miſtaken in his judgment of diſtances. A ſhip that is eight or ten miles from the ſhore, will ſcarce ſeem to him to be a mile off. For at land, when we have an extenſive proſpect we commonly ſtand upon ſome eminence, and the country about us from the valley beneath or from the foot of the eminence riſes gradually; ſo that there the variety

of

of viſible objects riſing one above another in the proſpect helps us to perceive and obſerve the intermediate extenſion. But upon a continued uniform level, as the ſea is, we have no ſuch helps: and if we attend only to the length of a viſible ſpace lying between us and the object, we ſhall be miſled in our judgment: becauſe the ſame diſtance, when ſeen upon an uniform level, will appear much ſhorter than it uſed to do at land. This method of judging how far an object is from us will likewiſe fail us at land, when there are hills between us and the object, which do not riſe one above another, ſo that neither the valleys between them are ſeen, nor moſt of the hills which lie beyond the firſt of them. If at the end of a flat proſpect there are two hills one much farther off than the other but of nearly the ſame apparent height, and a wind-mill ſtands upon each of them; the eye could not perceive at once which of the two wind-mills was the moſt remote: for the ſpace that lies between them cannot be ſeen, becauſe the nearer of the two hills will hide it. In any of theſe caſes we find out the diſtance of an object, or rather the difference between the diſtances of different objects, by reaſoning upon it. How we reaſon about it and form our judgment will beſt appear from conſidering what anſwer a man would make, who can tell which of two ſhips out at ſea or which of two wind-mills in the ſituation laſt deſcribed appears to be the fartheſt off, if he was to be aſked why he thinks ſo. He might ſay that one of the ſhips ſeems leſs than the other, and that he judges that to be the nearer of the two, which appears the bigger. This would be a rational anſwer. But the reader muſt obſerve that this principle of judging might fail or might miſlead the ſpectator: for if the ſhips were of different ſizes, and the remoter of the two was the larger, it might appear as big or bigger than the nearer, by propoſitions 126, 127, 128. There is another thing worth taking notice of in this principle of judging, and we ſhould be led to it by aſking the ſpectator why he thought either of the two ſhips was very far from him. His anſwer would probably be that even that which appears the bigger of the two ſeems to be very ſmall. From this anſwer it is evident that he has in his mind a notion of the real magnitude of a ſhip, and that he has been uſed to ſee it at different diſtances: ſo that ſince it appears much ſmaller than it really is and than he has been uſed to ſee it, when it was near to him, he concludes that it muſt be at a great diſtance. Upon this account he will make falſe judgments of the diſtance of ſuch objects, as he is before unacquainted with and has not frequently viewed at different diſtances, if he depends wholly upon this principle. In travelling towards a large city, or a caſtle, or a cathedral church, or a mountain larger than ordinary, if he has not been uſed to ſee ſuch large buildings or ſo large a mountain, he will think himſelf nearer to them than he is. But when this principle fails, or even in inſtances when we might make uſe of it, we ſometimes have recourſe to another. For if the ſpectator was to be aſked, in the caſe juſt now propoſed, why he thinks that one of the wind-mills is nearer to him than the other, it would be a very rational anſwer to ſay he can ſee all the parts of one more diſtinctly than of the other; and ſince by experience he has found that remote objects appear more confuſed than ſuch as are nearer, he concludes that the more diſtinct one of the two is nearer than the other. If the ſpectator was to change his place he might judge which of the two was the more remote, and likewiſe whether any ſingle object was near to him, or far from him, by the parallax of the object, that is, by the change of the objects apparent place, whilſt he has been changing his ſtation. Let L, Plat. XII. fig. 2, be one object and H another object more remote than L from the ſpectator. Then if the ſpectator is at A the remoter object H will be ſeen in the line FA, and the nearer object L will be ſeen in the line DA: and ſuppoſing that AD is on the left-hand ſide of AF the nearer object L will appear on the left-hand ſide of the remoter H. But if the ſpectator changes his place and removes to B, then the nearer object ſeen from this ſtation will appear in the line BC. and the remoter in the line BE; ſo that the nearer one L will be on the right-hand of the remoter one H. To thoſe, who are uſed to judge of diſtances, this change of the apparent place of objects, when viewed in different ſtations, is as ſure a principle as any: for ſuch objects, as by altering the ſpectators ſtation change place the moſt, are always the neareſt, and ſuch as change it the leaſt are always the moſt remote. There is yet another principle, beſides thoſe already mentioned, by which we ſometimes make a judgment of the different diſtances of remote objects; and that is their apparent faintneſs. When a man ſees two mountains at a great diſtance from him, though he cannot meaſure with his eye the viſible extenſion, that lies between him and either of them; though he is unacquainted with their real magnitude and therefore cannot judge of their diſtance by their apparent magnitude; though they both of them appear confuſed; and though he has no opportunity of changing his ſtation ſo as to try which has the greater parallax; yet if one of them is ſo faint as to look almoſt like a blue cloud in the horizon, and the other is ſomething brighter, he would ſcarce be at a loſs to determine which is the remoter of the two.

All theſe principles, except one, are obſerved in painting a landſkip. The parts, that are to appear the fartheſt off, are drawn of a leſs ſize, are made more confuſed and more faint, than thoſe that are to appear nearer. But where objects at different apparent diſtances are to be repreſented upon the ſame plane ſurface of the canvas, it is impoſſible to give them a different parallax, or to make thoſe, which are nearer, change their apparent place in reſpect of thoſe, which are more remote, from the ſpectators changing his ſtation. And indeed as a perſon looking at a landſkip does not change his ſtation much, ſuch a parallax in the ſeveral parts of the picture is neither expected nor wanted.

Now we have explaned the ſeveral principles, by which we judge of diſtances, when the objects are very remote, we may the better underſtand how we judge of diſtances when the objects are nearer. I choſe this method, becauſe when objects are very remote we have not ſuch frequent occaſion to judge of their diſtances, as we have when they are nearer. Upon this account we are not much practiſed in judging of the diſtances of remote objects, and therefore the mind proceeds more ſlowly for want of frequent exerciſe and habit, the ſeveral ſteps that it takes are more evident, the principles, upon which it reaſons, are more eaſily diſcerned, and the concluſion of the judgment can ſeldom be miſtaken for the perception of ſenſe. Whereas, when objects are nearer to us, though we judge of their diſtance by nearly the ſame principles, yet by frequent exerciſe the mind is too quick in the ſeveral ſteps that it takes for thoſe principles to be diſcerned and attended to: ſo that the concluſion of the judgment being made at once is commonly imagined to be a perception of the ſenſes; we imagine that we immediately ſee the diſtance of an object, when in fact we have determined it by a deduction of reaſoning; but by ſuch a deduction as frequent practice has made ſo obvious and familiar to us that we have no ſooner turned our eyes towards an object, but we can tell at the firſt view whether it is near to us or far from us.

If an object is at no great diſtance from us, and we attend to the viſible extenſion between us and that, ſuch as the floor of a room, the walk of a garden, the breadth of a bowling-green, we may then in ſome ſort be ſaid to ſee the objects diſtance. But if it is held before the eye in ſuch a manner as to be detached from all viſible extenſion between the eye and that, if we look ſo directly at the object as either not to ſee or not to attend to any thing elſe, then we muſt judge of its diſtance upon ſome other principles. We judge of the diſtance of objects by their apparent confuſion, when they are near enough to appear confuſed. When we know beforehand that two objects are very remote from us, and one appears more confuſed than the other, we conclude that the more confuſed one is the more remote one of the two. But if this principle is made uſe of in one caſe, it ſeems likely that we ſhould make uſe of it in another. And when we know beforehand that an object is near to us, and by that means are convinced that its apparent confuſion cannot be owing to its remoteneſs but muſt be owing to its nearneſs, we have a good reaſon for concluding, that thoſe near objects, which appear moſt confuſed, are at a leſs diſtance than thoſe, which are ſeen more diſtinctly. We judge of the diſtance of remote objects from their parallax, or their change of apparent place by the ſpectators changing his ſtation. Upon the ſame principle we may judge of an objects diſtance when it is nearer to us, without moving out of our place. For if the diſtance between the two eyes bears any proportion to the diſtance of the object, then one eye views it from one ſtation and the other views it from a different ſtation; ſo that by opening each eye alternately a parallax or change of apparent place will be produced in the object. If *c*, Plat. XI. fig. 8, is an object at ſuch a diſtance from us that the diſtance between the two eyes *a* and *b* bears ſome proportion to it, and if *d* is an object a little farther off, then by ſhutting the left eye at *a* the object *c* will appear to the right eye in the line *be*, on the left-hand ſide of *d*, and by ſhutting the right eye at *b* it will appear to the left eye in the line *af* on the right-hand ſide of *d*. This change of apparent place, when each of the two eyes views the object *c* from a different ſtation, is a certain mark that the object is near to us, and ſo much the nearer as this parallax is greater. Indeed we ſeldom proceed in this method when we judge of an objects diſtance; for ſome time will be taken up in making this tryal: the mind muſt therefore proceed ſlowly in determining, and the concluſion of our judgment could not be miſtaken for the perception of our ſenſes. But though we do not uſe this method yet we proceed upon the ſame principles: for where the ſame principles for judging of diſtances, that are applyed in one caſe, can be applyed in another, there ſeems to be little reaſon to doubt of their being made uſe of in both. It is this parallax or difference of apparent place to the two eyes which makes an object ſeem double, unleſs we turn both eyes directly towards it, as has been already ſhewn in propoſitions 116, 117, 118. And when to avoid this double appearance the eyes are both of them directed to the object, if the diſtance of it is ſmall, they are turned very much towards each other, and if its diſtance is greater they are turned towards each other in a leſs degree. Thus in looking directly at the nearer object *c*, or in directing the optic axes towards *c* the eyes muſt be turned ſo much towards each other as to look along the lines *ac* and *bc*; but in directing them to a remoter object *d*, they are leſs turned towards each other; for then they look along the lines *ad* and *bd*. And as this different turn of the two eyes is neceſſary to ſee objects ſingle at different diſtances, when we uſe both eyes at once, it is by experience and habit made a ſettled mark for determining the diſtance of an object. We no ſooner find it neceſſary to turn our eyes very much towards each other in order to look at an object, but the mind concludes the object to be near; and by the degree in which they are thus turned determines how near. After we have long uſed ourſelves to judge of the diſtance of an object by this method, in which both eyes are neceſſary, we ſhould be at a loſs for ſome time, if we were to loſe one of our eyes, to make any exact judgment of diſtances, till by uſe and experience we had ſettled ſome other method and had made it habitual. For this reaſon a perſon, who loſes an eye, is at firſt apt to miſtake the place of an object, that is not far from him, as will appear in his ſnuffing a candle or pouring liquor out of one veſſel into another. A perſon, who plays well at tennis, will find himſelf ſubject to the ſame miſtakes, the firſt time that he plays with one eye hood-winked. But in either inſtance experience and uſe will prevent theſe miſtakes. Thoſe, who cannot apply the principles of judging of diſtances by the help of both eyes, will by practice eſtabliſh a method of judging with one eye only, upon the principles of apparent magnitude. For theſe, which are principles of judging of diſtances, when objects are remote, will likewiſe be made uſe of, when the objects are near; where it is out of our power to make uſe of ſuch as are better. But from the miſtakes that are made by a man, when he firſt loſes the uſe of one eye, it is evident, that this is not the firſt principle that we go upon in forming our judgments, for if it had, he would determine the place of an object immediately with one eye as readily and as certainly as he uſed to do with both. From hence we may ſee the reaſon, why in looking through a lens either convex or concave, we ſhould commonly judge of an objects diſtance, that is ſeen through it, by the apparent magnitude only. For the object is ſeen through the lens detached from every thing elſe, ſo that no extenſion between the eye and the object can be ſeen: and ſince we ſeldom make uſe of more than one eye in any experiments of this ſort, we have no other way to judge of the objects diſtance but by its apparent magnitude. Therefore when it is proved that in theſe experiments we always do judge of diſtances upon this principle; yet this is very far from proving that this is either the only principle, or the principle that is in all caſes moſt attended to: all that ſuch proof amounts to is this, that we make uſe of this principle where it is not in our power to make uſe of any other. For even in looking through a lens we may find by a very eaſy experiment that the apparent diſtance of an object is not always determined by its apparent magnitude. Take a convex lens ſo large that you may look through it with both eyes at once, and hold a long ſlender object, ſuch as an ivory folding knife, behind it in ſuch a manner that half the knife may appear through the lens, and the other half may be ſeen below the lens by the naked eye: let the knife be ſo far behind the lens that the part, which is ſeen through it, may appear diſtinct. In this ſituation, that part, which is ſeen through the lens, will be magnifyed, and the reſt of the knife, which is ſeen by the naked eye, will not. And if the ſpectator looks directly with both his eyes at theſe two parts of the object, firſt at the magnifyed part and then at the other; he will find, after having with ſome attention compared the apparent diſtances of the two parts, that the magnifyed part, which is ſeen through the glaſs ſeems to be farther off than the other, which is ſeen below the glaſs and is not magnifyed. At leaſt I know that they appear ſo to me. Whereas, if he had judged of diſtance from apparent magnitude only, juſt the reverſe muſt have happened, the part which appears to be the larger of the two muſt likewiſe have appeared to be the nearer. When an object is very remote from us, we ſometimes judge of its diſtance, when compared with that of other objects, by the apparent faintneſs of it. But in near objects this principle of judging is of little or no uſe. For the brightneſs of an object is no otherwiſe altered by the greater diſtance of it, than as the rays have ſo much more of the atmoſphere to paſs through, by propoſition 134: and this will make but little difference, unleſs when objects are very remote and their reſpective diſtances differ very much from one another. Indeed in objects, that are near us, there are innumerable accidents which may vary their apparent brightneſs: for which reaſon the idea of a different diſtance is not ſo ſteadily connected by uſe and habit with the idea of a different brightneſs, as to make us conclude at once, when an object appears faint, that it is therefore at a great diſtance from us. Nay even in the caſe of very remote objects, this concluſion is not made at once without ſome previous doubt; and an object is not judged from the faintneſs of it to be at a very unuſual diſtance from us, unleſs it appears fainter than almoſt any accident can make an object appear, when it is nearer to us. This principle is commonly the laſt we have recourſe to, and is ſcarce ever made uſe of, where we can poſſibly form a judgment upon any other. This apparent faintneſs is indeed imitated in painting, but it is done rather to preſerve a likeneſs between the picture and the real appearance of things, than to ſuggeſt to us the idea of a greater diſtance of thoſe parts which are drawn the moſt faint. For though remote objects always appear fainter than near ones, and therefore, in imitation of nature, thoſe parts of the landſkip, which are deſigned for the moſt diſtant ones, ſhould be made leſs bright, than thoſe, which are nearer; yet faint objects do not always appear the moſt remote: and conſequently though the parts of the picture which appear the moſt remote are made the moſt faint, this apparent remoteneſs is not ſo much owing to their faintneſs as to their ſmallneſs and apparent confuſion. When the moon or the ſun are in the horizon, they appear more faint, than when they are in the meridian. But this does not make either of them appear more remote. For notwithſtanding they are fainter in the horizon, yet even there they are either of them as bright, and the ſun in particular is more bright than moſt other objects are, which are very near to us.

The heavenly bodies, the ſun, the moon, the other planets, and the fixed ſtars, though they are at very different diſtances from us, ſeem to be all at the ſame diſtance and appear as if they were in one and the ſame concave ſurface of the ſky. For here all our principles of making any quick judgment of diſtances fail us. There is no viſible extenſion, whoſe length we can judge of, between them and us. The apparent confuſion is the ſame in all of them. Their real magnitudes are unknown to us, and therefore cannot be compared with their apparent magnitudes. The parallax of thoſe, which have any, is not obvious, nor can it be found without taking up ſome time. And their apparent faintneſs is not ſufficent to ſhew us at once that they are very remote from us, and much leſs is it ſufficient to ſhew us that one of them is more remote than another. Therefore, as our judgment cannot help us to diſtinguiſh their ſeveral diſtances, they all ſeem to be at the ſame diſtance from us.

Indeed the moon or the ſun or any other of the heavenly bodies, will ſeem farther from us, when it is in the horizon, than when in the meridian. And the concave ſurface of the ſky in an open horizon does not ſeem a hemiſphere, but the parts that are directly over our heads ſeem nearer than thoſe parts which are directly before us about the horizon, where the ſky all round ſeems to touch the ſurface of the earth. So that this concave ſurface appears like a leſs ſegment of a ſphere than a hemiſphere is. And an arc drawn from the point L, Plat. XII. fig 3, in the horizon through R to the oppoſite point H will not appear to be a ſemicircle as LRM having its center at S, which is the place of the ſpectator, but will ſeem to be a ſegment of a circle having its center below the horizon, and ſo will be leſs than a ſemicircle. For the only viſible extended line between the ſpectator and the apparent ſky, if his face is towards L, will be SL, a line drawn along the viſible plane of the horizon, and by this viſible line he is to eſtimate, as well as he can, both the diſtance of the point L from him and of all the other points between L and R. Now when he looks directly before him towards L, he ſees this line to the greateſt advantage as to the apparent length of it, that he poſſibly can; for if he raiſes his eye to B this common meaſure SL will be ſeen more obliquely; and conſequently, when it is referred to the point B as the meaſure of its diſtance from him, it will appear ſhorter than when referred to L; upon which account he will ſuppoſe the point B nearer to him than the point L. In like manner, when he would judge of the diſtance of the point C from him, his common meaſure SL is ſeen ſtill more obliquely, and therefore will appear ſtill ſhorter, or the diſtance of C will ſeem to be leſs than the diſtance of B. And when this common meaſure is referred to R it is ſeen moſt oblique of all; ſo that R will appear nearer than any other point of the ſky. And ſince, as the eye riſes from L towards R, every point appears nearer and nearer; the whole apparent figure of this quarter of the ſky will be L*bcr*; and for the ſame reaſons the whole apparent figure of the other quarter will be *rm*H.

136. *In viewing objects that we are much uſed to, if they are at ſuch diſtances as we can readily allow for, the judgment of the mind about their magnitude is commonly more attended to than the perception of the eye.*

If a man 6 feet high is 40 yards from us, and a boy 3 feet high is 20 yards from us, their apparent heights, as perceived by the eye, will be equal, by propoſition 128. And yet when we look at them, we are apt to think otherwiſe; the man ſeems to be as much bigger than the boy, as he would if they were both at the ſame diſtance. The two objects are ſuch as we ſee frequently, we have often viewed them at different diſtances, and know by experience how much an encreaſe of diſtance will diminiſh their apparent magnitudes: the diſtances, at which they are ſuppoſed to be placed, are likewiſe ſuch as we are uſed to, and therefore are ſuch as we can readily allow for. And by a quick and imperceptible act of the judgment we determine which of the two is the more remote and what is the real magnitude of each. And having thus at once found out their real magnitude we attend ſo much to this, as to neglect the apparent magnitude perceived by the eye, and then think that we ſee each of them of the ſame ſize that it really is, and of the ſame ſize that they would appear if they were leſs remote.

The reader will do well to obſerve here, that this can never be the caſe unleſs the objects are placed at ſuch diſtances as we are uſed to and

can

can readily allow for, and unleſs the objects themſelves are ſuch as have been familiar to us, ſuch as we have frequently viewed at different diſtances, till we are well acquainted with their real magnitudes. Some of the principal inſtances to ſupport theſe obſervations, I ſhall borrow from the ingenious and truly learned doctor Jurin. The uſe, that is to be made of the obſervations themſelves, will be ſeen hereafter.

Firſt; in diſtances that we know not how to allow for readily, the perception of the eye is all that we attend to in judging of the magnitude of an object. " Children of three or four years of age, not having yet learned how to allow for the leſſer apparent magnitude of diſtant objects, take every thing ſeen at a diſtance to be leſs than it really is. If they ſee a man or a woman thirty or forty yards off, they take the one for a boy and the other for a girl. And though experience teaches them in a little time to correct this error in objects ſeen upon the level, or from a moderate height, yet if they happen to look upon them from any high building, they are apt to make the ſame miſtake as before. Let a boy, who has never been upon any high building, go up to the top of the monument, and look down into the ſtreet; the objects ſeen there, as men and horſes, will appear ſo ſmall as greatly to ſurprize him. But ten or twenty years after, if in the mean time he has uſed himſelf now and then to look down from that and other great heights, he will no longer find the ſame objects to appear ſo ſmall. Becauſe now, having had frequent experience of the leſſening of the apparent magnitude by the encreaſe of diſtance, he does by a quick and imperceptible judgment conceive the objects to be of the ſame magnitude as if they were leſs remote: which judgment of the mind not being diſtinguiſhed from the perception of the eye, he thinks that he actually ſees them larger than before. And if he were to view the ſame objects from ſuch heights, as frequently as he ſees them upon the ſame level with himſelf in the ſtreets, I ſuppoſe, they would appear to him juſt of the ſame magnitude from the top of the monument, as they do from a window one ſtory high. For in the ſtreet a man at a hundred yards diſtance appears of the ſame height as another at ten, though the picture of the former in the eye is but a tenth part of the length of the other, the ſmallneſs of the picture being compenſated by the knowledge of the greatneſs of diſtance." But though we think that a man at a hundred yards diſtance ſeems as big as if he was only ten yards from us, yet in greater diſtances than theſe we ſhould think otherwiſe. A man at the diſtance of a mile ſeems leſs than at the diſtance of ten yards: and as children three or four years old, who are unable to eſtimate and allow for ſmaller diſtances, take a man at thirty yards diſtance

ſtance for a boy; ſo when the diſtance of an object is too great, as by propoſition 130, it ſoon may be too great, for the eye of a grown perſon to eſtimate it, if he does not fall into the ſame miſtake that children do, yet the greatneſs of the diſtance, becauſe it is unknown, cannot compenſate for the ſmallneſs of the picture upon his retina: he cannot therefore perſuade himſelf by any quick and imperceptible judgment, that the objects at ſuch great diſtances appear of the ſame magnitude, as if they were leſs remote; ſince for want of a ready way to eſtimate and allow for thoſe diſtances, he has no opportunity of forming ſuch a judgment. And thus, though in travelling upon the road, a man ten yards before us does not ſeem larger than another a hundred yards before us; yet a mile-ſtone only ten yards from us, if we can ſee the next beyond it at the ſame time, will ſeem much larger than that which is a mile and ten yards from us. A church, at the diſtance of a furlong may ſeem no leſs than at the diſtance of half a furlong, becauſe theſe diſtances are ſmall enough for the eye to eſtimate and allow for readily; the viſible extenſion between us and the object is the meaſure of its diſtance. But if we were to ſee two churches of nearly the ſame ſize one at five miles diſtance and the other at ten, and were aſked which of thoſe two was the fartheſt off, and the reaſon why we think ſo; the anſwer would ſhew that they do not ſeem to be of the ſame ſize; for it would moſt probably be that we think one of them farther off than the other, becauſe it ſeems to be leſs. And I fancy, no one imagines, that either of them ſeems to be as big as it does, when he is within half a furlong of it. The reaſon is obvious. For in very great diſtances the length of the viſible extenſion between us and the object, decreaſes too faſt, by propoſition 130, for the eye to make any eſtimate of it; and upon this account our judgment of the objects diſtance is commonly made by attending to its apparent magnitude as perceived by the eye; and when we have attended enough to this apparent magnitude to determine the diſtance of the object, we cannot immediately allow for the diſtance thus eſtimated, forget the apparent magnitude as perceived by the eye, and then attend only to the dictate of our judgment about the real magnitude. If we attend only to this dictate of the judgment, we cannot eſtimate and allow for the objects diſtance; if we do eſtimate and allow for the diſtance, we muſt attend to the apparent magnitude as our ſenſes perceive it.

Secondly; unleſs the objects, that we are viewing, are ſuch as have been familiar to us, ſuch as we have frequently viewed at different diſtances, till we are well acquainted with their real magnitudes, the perception of the eye is all that we attend to in judging of their magnitude.

" Tho' ſome objects, when ſeen at a diſtance, are by a quick and imperceptible judgment conceived to be of the ſame magnitude as if they were leſs remote; yet the famous monſieur Perrault ſeems to be greatly miſtaken, when he applyes this principle to overthrow an opinion, which has generally been received by all architects and ſculptors ſince the time of Phidias, that ſtatues placed upon very high buildings ought to be made of a larger ſize, and to be heightened with bolder ſtrokes than thoſe, which are to be ſeen at a nearer diſtance. He ſays the judgment made by the eye is ſo perfect and ſo accuſtomed to make the proper allowance for the diſtance of the object, that it appears of the ſame magnitude to us whether at a greater or a leſſer diſtance. In things very familiar to us he is certainly right: but ſtatues upon very high buildings are not ſo conſtantly ſeen as men and horſes in the ſtreets; and therefore we cannot make ſo juſt an allowance for the diſtance in one caſe as in the other. Very few ſpectators are able to make a near eſtimate of the length of the dragon upon Bow-ſteeple, or of the height of the St. Michael at Bruſſels, except ſuch as ſaw thoſe figures before they were put up." It is for want of being acquainted with the real magnitude of the object, for want of having frequently viewed it at different diſtances, that a perſon, as has been already obſerved in propoſition 135, when he is travelling towards a large city, or a caſtle, or a cathedral church, or a mountain larger than ordinary, is not able to allow in his mind for the diſtance and to determine the real magnitude; but attending only to the apparent magnitude, as his eye perceives it, judges of the diſtance from thence and concludes himſelf to be nearer to them than he finds himſelf upon tryal.

We may now be able to gueſs whether the moon, when it is in the horizon, appears to be bigger, than when in the meridian, upon account of its apparent diſtance being greater, when in the former ſituation, than when in the latter. If this is the true reaſon of this remarkable appearance, we muſt explane it in the following manner. The moon is really at the ſame or at nearly the ſame diſtance from us in either place; and conſequently the picture of it upon the retina, and the apparent magnitude, as perceived by the eye, is the ſame in both, by propoſitions 127, 125. But the apparent diſtance of the moon, when in the horizon, is greater than when in the meridian. Therefore the mind by a quick and imperceptible act allows for this apparently greater diſtance and concludes that the moon, which at this greater diſtance ſeems as big to the eye, as when it is in the meridian at an apparently leſſer diſtance, muſt neceſſarily be bigger in itſelf. And when this concluſion is once made, we attend to it more than to what the eye perceives, and fancy that it looks bigger in the horizon than it does in the meridian..

It is, upon theſe principles, a ſort of perſpective view of the moon: for as in a picture, if two objects ſeem at different diſtances, and are drawn equal to each other, the imagination will immediately conceive that object to be the larger of the two, which appears to be at the greater diſtance, and this dictate of the imagination will be more attended to than the preception of the eye; ſo if the moon ſeems to be more remote in the horizon than in the meridian, and the apparent magnitude, as perceived by the eye, is the ſame in both ſituations; the imagination will immediately conceive that the moon is larger in the horizon than in the meridian; and our attention to this concluſion of the mind is all that we attended to, the perception of ſenſe ſerves only to lead us to this concluſion and then is forgotten. Thus ſuppoſe that the eye is at S, Plat. XII. fig. 3, the moon in the horizon at H and in the meridian at M; ſo that the angle, which it ſubtends at S, when in either place, is equal. If the moon in the meridian was to appear as if it was at M, that is, if its apparent diſtance SM when in the meridian, was equal to SH its apparent diſtance when in the horizon, then in both ſituations it would appear of the ſame ſize. But the moon in the meridian does not ſeem to be at the diſtance SM but at a leſs diſtance S*m*. Now ſince a moon at the greater diſtance SH muſt be larger in itſelf, than one, which at the leſſer diſtance S*m*, appears of the ſame ſize, as much larger as H is than *m*; and ſince the eye does perceive the moon of the ſame ſize when it appears at *m* as when at H; therefore the mind by a quick and imperceptible act conceives it, when at H, to be bigger, than when at *m*, and we attend ſo much more to this concluſion of the judgment, than we do to the perception of ſenſe, as to think, not only that it is bigger, but that we ſee it bigger in the horizon at H than in the meridian at *m*.

Now here we may apply the two foregoing obſervations. Firſt; this concluſion cannot be made, except in inſtances, where the mind can readily allow for the apparent diſtance of the object. The concluſion of the judgment is never ſubſtituted inſtead of the perception of ſenſe, except where this concluſion can be made at once. But the mind cannot act in ſo quick and imperceptible a manner, unleſs where the eye can eaſily eſtimate the different diſtances of the object. And though the moon does ſeem farther off, when in the horizon, than when in the meridian, yet both theſe apparent diſtances are much greater than the eye can eaſily eſtimate and readily allow for. If in a proſpect from any eminence we could ſee two buildings of the ſame ſize, one at 5 miles diſtance and the other at 10; though we might perceive a long tract of country lying between one and the other, and ſo might be ſure that one is more remote than the other; yet we ſhould never fancy that one ſeemed to be as large as the other. This is only done, where the objects are near, and the eye can, not only perceive that they are at different diſtances, but can likewiſe readily eſtimate that difference and allow for it at once. But in ſuch remote objects, the difference of diſtance, though it is perceived, can be eſtimated no otherwiſe than by attending to the apparent magnitude of the objects; and when this is attended to in eſtimating the diſtance, it is impoſſible immediately to allow for that diſtance, forget the apparent magnitude, as perceived by the eye, and ſubſtitute inſtead of it an imagined real magnitude, as dictated by the judgment. Though the diſtance of the moon in the horizon is apparently greater than in the meridian, yet as both of them are too great for the eye to eſtimate, we can form no concluſion of judgment to ſtrike our imagination and take off our attention from what the eye perceives: and if the apparent magnitude as perceived by the eye, is the ſame in either ſituation, we cannot fancy that it ſeems bigger when in the horizon, than when in the meridian.

Secondly; the concluſion of judgment, that we have been ſpeaking of, cannot be made, except in ſuch objects as are familiar to us, and have been often viewed at different diſtances. In objects that we are ſtrangers to, or that we have never ſeen but at one and the ſame diſtance, we attend only to the perceptions of ſenſe; and inſtead of judging what their real magnitudes are from their apparent diſtances, we eſtimate their diſtance from their apparent magnitudes as perceived by the eye. This is the caſe of the moon; it has never been viewed by us at more diſtances than one; its real magnitude is what common ſpectators have no conception of, and we cannot tell by an alteration in its diſtance, how much or how little its apparent magnitude will be altered: ſo that though its apparent diſtance, is greater in the horizon than in the meridian, the mind has not been uſed to make allowance for difference of diſtance and conſequently has never acquired ſuch an habit as is requiſite, where, in looking at an object, a concluſion of judgment is ſubſtituted in the place of the perception of the ſenſe.

We may now examine the inſtance of a picture in perſpective, which we before made uſe of in explaning the opinion, that we are now endeavouring to confute. In a perſpective picture, the apparent diſtance of the ſeveral parts depends upon the magnitude of them compared with one another: ſo that one part will appear remoter than the reſt by being made ſmaller

PLATE XI.

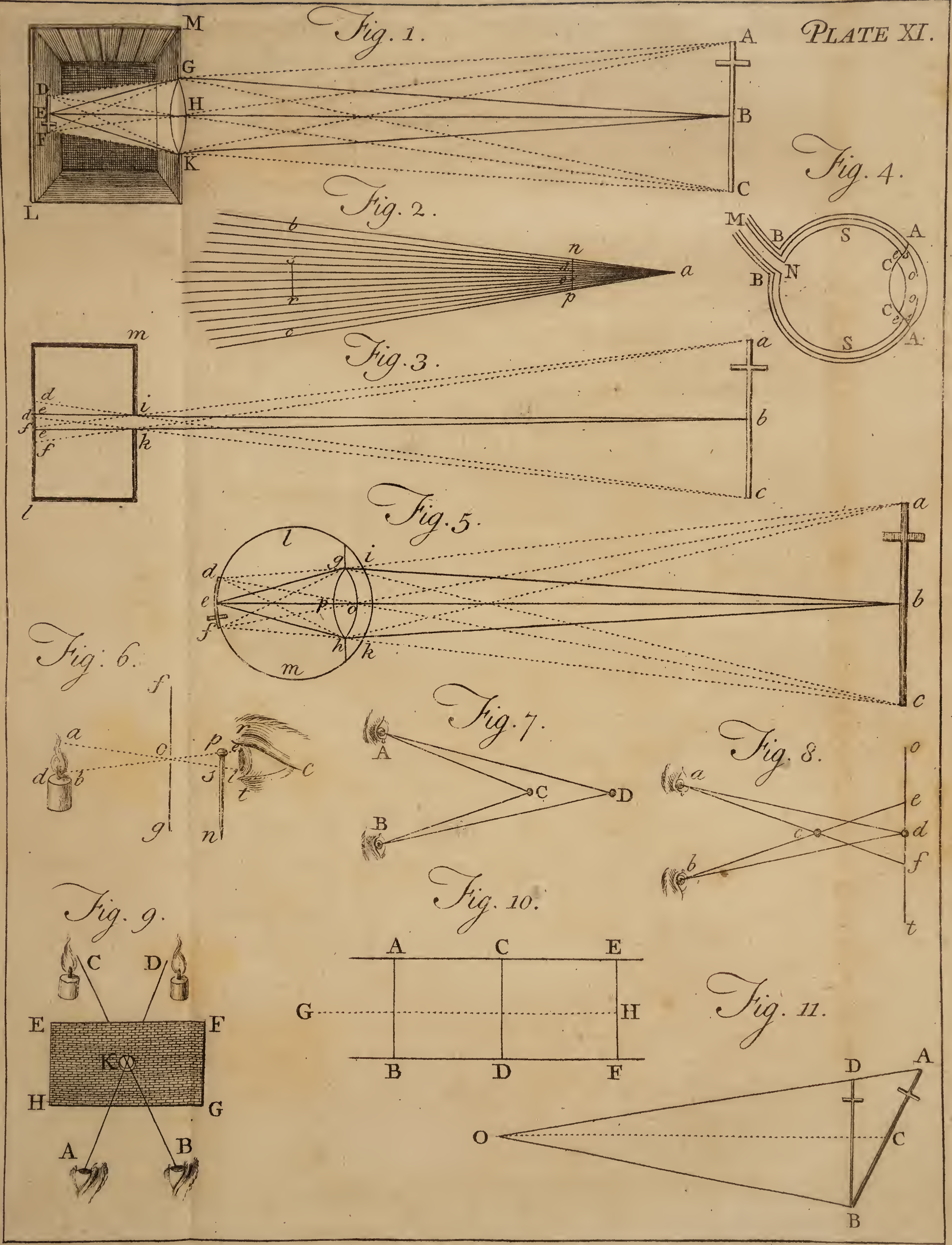
Fig. 1.
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Fig. 2.
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Fig. 3.
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Fig. 4.
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Fig. 5.
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Fig: 6.
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Fig. 7.
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Fig. 8.
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Fig. 9.
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Fig. 10.
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Fig. 11.
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ſmaller than the reſt; that is, the apparent diſtance is judged of by the apparent magnitude: which agrees very well with what has been proved in propoſition 135; that in very remote objects the apparent magnitude is the leading principle, by which we eſtimate the diſtance. But if we conſider the moon in the horizon and the moon in the meridian as two perſpective views of an equal object at different diſtances, the judgment is formed juſt the contrary way. Of two objects, that are equal as to ſenſe, we fancy that appears bigger which ſeems to be the moſt remote: ſo that in this inſtance there could be no perſpective view, unleſs the mind in very remote diſtances is able to judge of magnitude from apparent diſtance. How conſiſtent this is with the principles laid down in propoſition 135, how in very great diſtances, ſuch as we cannot eſtimate without attending to the objects apparent magnitude, we can judge of the real magnitude from the apparent diſtance, let the reader determine. But let this be as it will. In a perſpective picture the fancy determines the apparent diſtance of the ſeveral parts from the eye by the comparative magnitudes of thoſe parts; and if in a perſpective view of a portico, all of it was to be covered except one pillar, for want of the reſt to compare it with, the ſpectator would not be able to judge whether that pillar was a near one or a remote one. In like manner in the perſpective view of the moon, where the fancy is ſuppoſed to determine the magnitude from the comparative diſtances of the moon in the horizon and the moon in the meridian, if thoſe diſtances are hid, as they will be if we look through the tube of a ſhort teleſcope with the glaſſes taken out of it, for want of an opportunity of comparing the different apparent diſtances, the ſpectator ought not to ſee the moon at all larger in one ſituation than in the other. Whereas upon tryal he will find the appearance juſt the ſame as if he had made uſe of no tube at all.

CHAP. X.

Of refracted viſion through ſingle glaſſes.

137. *When any ſmall object or any point of an object is ſeen by refracted light, it appears in the direction of that line, which the rays deſcribe after their laſt refraction.*

IF the rays, that come from any ſmall object D, Plat. XII. fig. 4, paſs through a glaſs priſm, the end of which is ACB, the ray DE will be refracted towards a perpendicular, when it enters the priſm, and will deſcribe the line EF, and when it goes out of the glaſs, it will be refrac-

ted

ted from a perpendicular into the line FG; which line FG is the direction of it after its laſt refraction, and the object D will be ſeen in the direction of the line FG, and will appear to be at L inſtead of D. For in this and in all other caſes of the ſame ſort, the picture of the object upon the retina will be in the ſame place, that it would have been in, if the naked eye had been looking at an object really placed at L; for the refraction gives the rays the ſame direction, as if they had come originally from L.

From hence we may underſtand why an object ſeen through a multiplying glaſs or through a glaſs that is cut into different ſurfaces inclined to one another, appears at one view in many different places. If the object *f*, Plat. XII. fig. 5, is ſeen through the glaſs *abcd*; by the ray *fo* that paſſes through the ſurface *cb* the object by the eye at *o* will be ſeen at *f*: the ray *fx* paſſes through the ſurface *cd*, and, when it is refracted, comes out in the direction *xo* as if it had come from *l*; therefore through the ſurface *cd* the object appears at *l*: and for the ſame reaſon, through the ſurface *ab* it appears at *m*. Conſequently there muſt be the appearance of as many objects as there are ſuch ſurfaces on the glaſs; for each of them ſhews the ſame object in a different place.

Hence it is that if ſuch a glaſs as *abcd*, Plat. XII. fig. 5, or a lens, ſuch as *gil*, fig. 7, 8, was to be ſhaken before the eye, the objects on the other ſide would appear to ſhake: for the rays from the ſame point of the object, in different ſituations of the glaſs, will fall with different obliquities upon its ſurface, and therefore will be differently refracted. Now as the glaſs is ſhaken, its ſituation is conſtantly changing, upon which account the refracted rays will be ſhaken, and the apparent place of the object will by this means be conſtantly varyed. In viewing an object through the plane glaſs AB, fig. 6, it will be otherwiſe: for in all ſituations of the glaſs, the rays fall in the ſame manner on any part of that plane ſurface which is towards the object, and the refracted rays come out in the ſame manner from the other plane ſurface which is towards the eye: ſo that by ſhaking the glaſs, the refracted rays will not be made to tremble, and conſequently the object ſeen through it will not vary its apparent place, or will not appear to ſhake.

138. *In refracted viſion it is not the object itſelf that we ſee, but the laſt image of it, which conſiſts of all the imaginary radiants, from whence the rays appear to diverge after their laſt refraction.*

It will be neceſſary to deſcribe particularly what is here meant by the laſt image. Plat. XII. fig. 7. Let *ac* be an object nearer to a convex lens than its principal focus; the rays that diverge from any point *b* in this object, will by paſſing through the lens be made to diverge leſs, and the imaginary radiant will be more remote than the real one, by propoſitions 67, 28. Thus the rays *bg*, *bl*, when they have gone through the lens will not proceed ſtrait forwards in the lines *gk*, *lp*, but will be refracted into the leſs diverging directions *gm*, *ln*, as if they had come from the imaginary radiant *e*, which is more remote than the real one *b*. The ſame will happen to the rays, which come from *a* or *c* or any other point in the object, ſo that there will be ſomewhere behind the lens, as at *df*, as many imaginary radiants as there are real ones in the object, and theſe imaginary radiants taken altogether compoſe the laſt image. And ſince all the rays fall upon the eye as if they had diverged from this laſt image, the eye will be affected by the object *abc* juſt in the ſame manner, when it looks through the lens, as it would be without the lens by an object in all reſpects like *def*, or as it would be by the laſt image, if without the lens the laſt image could be made viſible: and becauſe the eye is affected when it looks through the lens as if *def* was the object and not *abc*, therefore we ſay that it is not the object itſelf that we ſee but its laſt image *def*. This is univerſal, in refracted viſion the eye is affected by the rays of light after refraction as if they had come, not from the object itſelf, but from its laſt image, which conſiſts of all the imaginary radiants from whence the refracted rays appear to diverge at the time they fall upon the eye. We will apply this to other particular inſtances as we have occaſion.

139. *The apparent place of a ſtar, when it is near the horizon, is higher than its real place.*

If *a*, Plat. XII. fig. 9, is a ſtar not very high above the horizon; any ray ſuch as *ab*, which falls obliquely upon the atmoſphere, becauſe it paſſes out of a vacuum into the air, will be refracted towards a perpendicular, by propoſition 37, and will not go ſtrait forwards, but will deſcribe the line *bc*. Now the atmoſphere is not ſo denſe at the top as it is near the earths ſurface, and the denſity of it keeps encreaſing from the top down to the earths ſurface: upon which account this ray as it goes farther into the atmoſphere keeps paſſing conſtantly into a denſer medium, and will be conſtantly refracted towards a perpendicular. Thus in paſſing through the firſt lamina it deſcribed *bc*, in paſſing through the next below it, which is ſomething denſer, it will deſcribe *cd*, in the next lamina below that it will be refracted into the line *de*, and in the loweſt, where the ſpectators eye is at *f*, the direction of the ray will be *ef*. But the ſtar will be ſeen, by propoſition 137, in the laſt direction of the refracted ray, or in the line *gf* ſo as to appear at *g*, inſtead of *a*: and becauſe the ray in its deſcent through the atmoſphere, has been conſtantly refracted towards a perpendicular, its laſt direction *ef* continued back, or *fg* will be nearer perpendicular to the ſurface of the atmoſphere than *ba*, that is, *g* the apparent place of the ſtar will be higher, or nearer to the zenith, than its real place *a*.

From hence it is plane, that a ſtar may be ſeen before it is above the plane of the horizon: for when its real place is below the horizon, its apparent place by the refraction may be above it.

140. *The nearer a ſtar is to the horizon the more its apparent place is changed by the refraction of the atmoſphere.*

If ABC, Plat. XII. fig. 10, is the earths ſurface, LDM the ſurface of the atmoſphere, T the common center both of the earth and of the atmoſphere, that ſurrounds it; when a ſtar Z is in the zenith of a ſpectator at B, or is directly over his head, a ray as ZD coming from the ſtar to his eye will be perpendicular to the atmoſphere, for its direction ZD continued paſſes through T the center of the atmoſphere, therefore this ray, by propoſition 41, will not be refracted, and conſequently the apparent place of a ſtar in the zenith will not be altered by the refraction or does not differ from the real place. But when the ſtar is at G in the horizon, as far off from Z as it can be, if it is ſeen at all, the incident ray GP will be as far from a perpendicular OT as it can be, and conſequently as the angle of incidence GPO is there the greateſt, the refraction will be the greateſt, by propoſition 40. And therefore the ſtars apparent place will be more altered when it is at G than when it is any where elſe between G and Z. Thus for inſtance when the ſtar is at H a ray HR is leſs oblique to the ſurface of the atmoſphere, than the ray GP when the ſtar was at G, or the angle of incidence HRN, is leſs than the angle GPO. For ſince, when it is at Z, this angle made with the perpendicular, or angle of incidence, is nothing, the nearer it approaches to Z, ſo much the leſs this angle will be, and the leſs the angle of incidence is the leſs will be the refraction, and conſequently the leſs likewiſe will be the change of apparent place.

Indeed when the ſtar is at G, or any where elſe, as well as when at Z, there will be a ray coming from it, which will fall perpendicularly on the ſurface of the atmoſphere; but this perpendicular ray, as GT paſſes towards the earths center; and therefore is not one of the rays which makes the ſtar viſible to a ſpectator at B on the earths ſurface.

141. *The ſun, or the full moon, appears of an oval figure, when they are in the horizon.*

As the apparent place of a ſtar, by propoſition 139, ſo likewiſe the apparent place of any point in the ſun, will by the refraction of the atmoſphere be higher than its real place. Now ſuppoſe a diameter drawn acroſs the diſk or face of the ſun perpendicular to the horizon; the loweſt point of this diameter will be apparently raiſed by the refraction, and ſo likewiſe will the higheſt point of the ſame diameter. But then as the apparent place of a ſtar, by propoſition 140, is more raiſed the nearer it is to the horizon, ſo for the ſame reaſon the loweſt point of this perpendicular diameter will be apparently more raiſed than the higheſt point of it, becauſe the loweſt point is nearer to the horizon than the higheſt. Therefore the two extremities of this diameter will be made to appear nearer to each other, or the perpendicular diameter will be apparently ſhortened by the refraction. The diameter, which croſſes this at right angles, or which is parallel to the horizon, is no otherwiſe affected by this refraction than that both ends of it are equally raiſed higher. So that ſince the perpendicular diameter of the ſun is ſhortened and the parallel one continues the ſame; this will apparently change the circular figure of the ſun into an oval or elliptical one, the ſhorter diameter of which will be perpendicular to the horizon.

142. *A veſſel ſeems to be ſhallower, when it is full of water, than when it is empty.*

When the veſſel is full of water the bottom of it is ſeen by refracted light: for the rays, which make it viſible, paſs out of the water into the air, before they come to the eye. Therefore, by propoſition 138, it is not the bottom itſelf that is ſeen, but its laſt image. And ſince, by propoſition 45, the imaginary radiants will be nearer to the eye than the real ones, the laſt image conſiſting of theſe radiants, which is ſeen when the veſſel is full, will be nearer than the bottom itſelf, which is ſeen when the veſſel is empty.

143. *In viſion through any glaſs the object will appear erect if the object and its laſt image are on the ſame ſide of the glaſs; but inverted if they are on contrary ſides.*

Since the laſt image is what we ſee in refracted viſion inſtead of the object, by propoſition 138, an object ſeen through a glaſs will appear erect, if its laſt image is ſo. And the image will be erect, or in the ſame poſition with the object if they are both on the ſame ſide of the glaſs, but it will be inverted, or the object and image will be in contrary poſitions,

tions, if they are on contrary ſides. If *abc*, Plat. XII. fig. 7, is the object, and the glaſs, through which it is viewed by the eye at *o*, is a convex lens *gil*; then ſuppoſe the object to be nearer to the lens than its principle focus, and all rays which diverge from any one point in it as *b*, will diverge leſs after refraction, and the imaginary radiant *e* will be more remote than the real one *b*, by propoſition 67. However, all the rays, which diverged from one and the ſame point *b* before refraction, will apparently diverge from one and the ſame point *e* after refraction. So likewiſe all the rays which diverged from the ſame point *a* in the object, will diverge from the ſame point *d* in the image; and all, which diverged from *c*, will after refraction diverge from *f*. Now *ai* is the axis of the beam that comes from *a* the higheſt point in the object, and this axis paſſes through the lens unrefracted, by propoſition 59. But the ray *ai* before refraction diverged from *a* as well as any other ray of the beam, and conſequently after refraction will appear to diverge from the ſame point *d* with the reſt of the rays. If therefore it paſſes ſtrait through the lens and yet appears on the other ſide to come from *d*, the point *d* muſt be in the ſame right line with *a*, or *dai*, which is drawn from the higheſt point in the image through the higheſt point in the object to the vertex of the lens, is always a right line. In the ſame manner we might prove that *fci*, which is drawn from the loweſt point in the object through the loweſt point in the image to the ſame vertex, is likewiſe a right line. Now theſe right lines croſs each other in the lens and no where elſe. And ſince the higheſt point both of the object and image is ſomewhere in the line *di*, and the loweſt point of both is ſomewhere in the line *fi*, therefore if the object and its laſt image are on the ſame ſide of the lens, that is, if the lens is not between them, theſe extreme axes will not croſs each other between the place of the object and the place of the image, but will be in the ſame ſituation in reſpect of each other at both places, *dai*, which is the higheſt at the object, will be the higheſt at the image, and *fci*, which is the loweſt at the one will be the loweſt at the other: and conſequently the point *a*, which is higheſt at the object, appearing at the image in the line *dai*, will be the higheſt there, and the point *c*, which is the loweſt at the object, appearing at the image in the line *fci*, will be the loweſt there; that is the image will be erect or in the ſame poſition with the object.

This may be made univerſal. For by what has been ſaid it appears that the higheſt point of an object, when ſeen by refraction, is always ſeen ſomewhere in the axis of the beam, which comes from the higheſt point; and the loweſt point is always ſeen ſomewhere in the axis, which comes from the loweſt point. Therefore if theſe two axes have not croſſed each other between the object and its laſt image, the higheſt point in the object will appear higheſt in the image and the loweſt point of the object will appear loweſt in the image; that is, the object and its laſt image will be in the ſame poſition.

In a concave lens *gil*, Plat. XII. fig. 8, if the object is *abc* the rays coming from every radiant in the object are made to diverge more, and the imaginary radiants or laſt image will be nearer than the object, by propoſition 76. The higheſt point *a* will appear at the image ſomewhere in *ai*, or in the axis of the cone of rays which comes from that point; and the loweſt point *c* will appear in the image ſomewhere in the axis *ci*. Therefore if theſe axes *ai* and *ci* have not croſſed between the object and the image, that point which is the higheſt or the loweſt in one will appear the higheſt or the loweſt at the other. But they croſs no where except in the lens at *i*; and conſequently if the object is at *abc*, and the laſt image at *def* or any where elſe on the ſame ſide of the lens with the object, ſo that the lens where the axes croſs is not between them, they will both of them be in the ſame poſition.

But if the object *abc*, Plat. XII. fig. 11, is more remote than the principal focus of the convex lens *gsl*, there would be a diſtinct picture of the object formed ſomewhere on the other ſide of the lens, by propoſition 84. Suppoſe then that, if a paper was held at *fed*, this diſtinct picture would be painted there. Now if there had been no paper, though the rays of the ſeveral beams would ſtill be collected into their reſpective focuſes at that place, yet they would not ſtop there, but would proceed ſtrait forwards: the rays, that had converged to *d* in the lines *gd*, *ld*, after they had met there, would diverge from thence in the lines *do*, *dx*. And the rays, that had converged to any other points *f* or *c*, would proceed on and ſo diverge from thence, by propoſition 15. And thus every focus in the picture becomes a radiant: for which reaſon, if the eye is at *o*, the points from whence the rays laſt diverge, when they fall upon it, are the ſeveral focuſes in the diſtinct picture. Therefore the diſtinct picture is in this caſe the laſt image. And we have already proved, in propoſition 85, that the diſtinct picture is inverted in reſpect of the object. Therefore when the object is on one ſide of the lens and the laſt image is on the contrary ſide, the laſt image will be inverted.

144. *In all lenſes, the diameter of the object is to the diameter of the laſt image as the diſtance of the object from the lens to the diſtance of the laſt image from the lens.*

This has been proved already of the diſtinct picture, in propoſition 88, therefore the propoſition is true, when the object and laſt image are on contrary ſides of the lens, as they are, Plat. XII. fig. 11. But in the concave lens fig. 8, or the convex one fig. 7, where the object and image are on the ſame ſide, the higheſt point of both will be ſomewhere in the ſtrait line *dai*, and the loweſt point of both will be ſomewhere in the ſtrait line *fci*; by what has been proved in propoſition 143. Now if *ac* and *df* or the object and image are parallel to one another, the triangles *aic* and *dif* are ſimilar: for the angle at *i* is the ſame in both of them; and becauſe *df* and *ac* are parallel, the external angle will be equal to the internal oppoſite angle on the ſame ſide; Euc, b. I. prop. 29; that is, *iac* is equal to *idf*, and *ica* to *ifd*. Therefore the ſides of theſe two equiangular triangles are proportional, Euc. b. VI. prop. 4, *ac* to *df* as *ai* to *di*, or the height of the object is to the height of the image, as the diſtance of the object from the lens to the diſtance of the image from the lens. The ſame might be proved in the ſame manner of their reſpective breadths or of any other correſponding diameters.

145. *The object itſelf is the object of plane viſion, the laſt image is the object of refracted viſion.* This follows from propoſition 138.

146. *The apparent magnitude of an object ſeen through any lens is proportional to the apparent magnitude of its laſt image.* This likewiſe follows from propoſition 138.

147. *The apparent magnitude of an object ſeen through a lens of any ſort is equal to its apparent magnitude ſeen without the lens, if the lens is cloſe either to the eye or to the object.*

Firſt; ſuppoſe the eye to be cloſe to the vertex *i* either of the convex lens *gil*, Plat. XII. fig. 7, or of the concave lens *gil*, fig. 8, the object ſeen through the lens will appear of the ſame ſize that it would do to the naked eye in the ſame ſtation, or to the eye, if the lens was out of the way. For the diameter of the object of refracted viſion *df* is to the diameter of the object of plane viſion *ac*, as *ei* to *bi*; or their diameters are as their reſpective diſtances from *i* the vertex of the lens, by propoſition 143. Therefore their apparent diameters are equal, when they are ſeen from *i*, by propoſition 128. Now the laſt image or object of refracted viſion is what we ſee through the lens, and the object itſelf or object of plane viſion is what we ſee when the lens is taken away from the eye: and conſequently the apparent diameter of what we ſee, when we make uſe of the lens, upon ſuppoſition that the eye is placed cloſe to the lens, is equal to what the naked eye in the ſame ſtation would ſee, if the lens was taken from before it.

Secondly; the apparent magnitude of an object is not altered by being ſeen through a lens, if the lens is cloſe to the object. For when the real radiant is cloſe either to a convex or a concave lens, the imaginary radiant will likewiſe be cloſe to it, by propoſitions 73; that is, if the object, which conſiſts of real radiants, touches the lens, the laſt image, which conſiſts of imaginary ones, will touch it. And when both of them are cloſe to the lens their diameters will be equal, by propoſition 144. Therefore whether we uſe the lens and ſo ſee the laſt image, or from the ſame diſtance look at the object itſelf without the lens, their apparent magnitudes will be the ſame.

148. *If an object, when it is ſeen through a convex or plano-convex lens, is nearer to the lens than its principal focus, it will appear brighter than to the naked eye, and will be diſtinct and erect.*

When any object *ac*, Plat. XII. fig. 7, is nearer to the lens *gil* than its principal focus, the rays, that come from *b* or any other point in it, will diverge the leſs or will be the cloſer together for being refracted, by propoſition 67, and conſequently more of them will enter the eye; ſo that the object upon account of this refraction will appear the brighter, by propoſitions 104, 113. The rays of each pencil diverge after refraction, and therefore come to the eye in the ſame manner that they would if the object was at a moderate diſtance and no lens was uſed. But when the naked eye ſees an object at a moderate diſtance, it appears diſtinct. Therefore, when an object, that is nearer to a convex lens than its principal focus, is ſeen through the lens, it will appear diſtinct. The refracted rays of each beam diverge leſs than the incident ones, by propoſition 67; ſo that all the imaginary radiants are more remote than the real ones, by propoſition 28, or the laſt image is farther from the lens than the object; and conſequently, ſince the object is beyond the lens, the laſt image muſt neceſſarily be ſo too; or the laſt image is on the ſame ſide of the lens with the object. Therefore the laſt image, which is the object of refracted viſion will be erect, by propoſition 143.

149. *If an object, when it is ſeen through a convex or plano-convex lens, is farther from the lens than its principal focus, it will appear brighter than to the naked eye, and will be confuſed and erect, provided the eye is nearer to the lens than the place of the diſtinct picture.*

If *gsl*, Plat. XII. fig. 11, is a convex lens and *abc* is an object placed at a greater diſtance than the lens's principal focus, there is a place ſomewhere on the other ſide of the lens, ſuppoſe at *fed*, where the picture of this object would be diſtinctly painted upon a paper. This, whether a paper is held there or not, is called the place of the diſtinct picture. Now if the eye is any where between *e* and *s*, the object will appear brighter through the lens than it would to the naked eye. For the rays of each beam would diverge, if there was no lens; but the lens makes them converge, by propoſition 63: upon that account they are kept cloſer together, and conſequently more of them will enter the eye. The object ſeen in this manner will be confuſed, for when an object is ſeen by the naked eye, the rays of each beam diverge, if it is near, or are parallel, if it is very remote; no rays that come from one and the ſame point in an object naturally converge. Therefore ſince the rays of each beam by the refraction of the lens are made to converge, they come to the eye in ſuch a ſtate as the eye is not uſed to; and it has not a power of changing its figure ſo as to have diſtinct viſion by rays that are in ſuch a direction as no rays are naturally. But ſince there is no picture formed till the rays come to the retina, the picture there, though it is a confuſed one, will be inverted, and conſequently the object will appear erect, by propoſition 115.

150. *If an object, when it is ſeen through a convex or plano-convex lens, is in the lens's principal focus, it will appear brighter than to the naked eye, and will be diſtinct and erect.*

If the object *abc*, Plat. XII. fig. 7, was in the principal focus of the lens *gil*, the rays of each particular beam, ſuch as *eg*, *ei*, *el*, will be parallel after refraction, by propoſition 62: this makes them cloſer together than if they diverged, as they would do if the lens was taken away; and conſequently more of them will enter the eye, which will make the object appear the brighter. The rays from very remote objects are nearly parallel, by propoſition 30. Therefore an object placed in the lens's principal focus, becauſe it is ſeen by parallel rays, will be as diſtinct as remote objects uſually are to the naked eye. The laſt image *def*, becauſe the refracted rays, which diverge from the ſeveral points of it, are parallel, will be very remote; and conſequently it will be on the ſame ſide of the lens with the object. Therefore, by propoſition 143, the object will appear erect.

152. *When an object ſeen through a convex or plano-convex lens is nearer than the principal focus it is magnifyed; unleſs the eye touches the lens, or the lens touches the object; and as the eye departs from the lens, its apparent magnitude will decreaſe.*

The apparent magnitude of *df*, Plat. XII. fig. 7, is greater, than that of *ac*, if the eye is at *o*, or if it is any where elſe, except at *i*; that is, unleſs

leſs the eye touches the lens, the object of refracted viſion will appear bigger than the object of plane viſion. If the object *abc* continues in the ſame place or does not change its diſtance *bi* from the lens, then the laſt image will likewiſe be always at the ſame diſtance *ei*, by propoſition 68. Therefore, by propoſition 144, the real diameter of the laſt image will be invariable, wherever the eye is placed. And if the ſtation of the eye is at the vertex *i* of the lens, the apparent diameter of the laſt image ſeen through the lens is equal to the apparent diameter of the object ſeen by the naked eye in the ſame ſtation, by propoſition 147. But if the eye withdraws from the vertex to any other ſtation *o*, the apparent diameter of the laſt image, which is the object of refracted viſion, will decreaſe, by propoſition 126, becauſe the eye is at a greater diſtance from it. By the eyes withdrawing itſelf from the vertex to any other remoter ſtation *o* the apparent diameter of *ac*, when ſeen by the naked eye, will likewiſe decreaſe. Since then either *df* or *ac* have the ſame apparent diameter, when the eye is at *i* cloſe to the lens, and ſince the apparent diameter of both decreaſes, when the eye withdraws from thence to *o*; why ſhould *df* appear larger than *ac* to the eye at *o*; why ſhould the apparent diameter of the object of refracted viſion be greater than the apparent diameter of the object of plane viſion, or why ſhould the object, when ſeen from the ſtation *o* through the lens, ſeem bigger than when ſeen from the ſame ſtation by the naked eye? The reaſon is, that though the apparent diameter both of the laſt image and of the object are diminiſhed by the departure of the eye from them, yet they are not diminiſhed in the ſame proportion; the apparent diameter of the laſt image decreaſes in a leſs proportion than the apparent diameter of the object. For when the eye has withdrawn from *i* to *o* the apparent diameter of the laſt image decreaſes in the inverſe proportion of *oe* to *ie*; and the apparent diameter of the object in the inverſe proportion of *ob* to *ib*. But *oe* bears a leſs proportion to *ie*, than *ob* bears to *ib*. Therefore the apparent diameter of the laſt image decreaſes in a leſs proportion than the apparent diameter of the object. That *oe* bears a leſs proportion to *ie* than *ob* to *ib*, may be eaſily proved, whatever is the length of *io* or whatever is the diſtance of the eye from the lens. For *ie* and *ib* are two unequal quantities of which *ie* the diſtance of the image is always the greater, by propoſitions 67, 28. And when the eye departs from the lens *io* is a given quantity added to both of them. Therefore both of them are equally encreaſed, but not proportionally: for a given addition when made to a greater quantity will encreaſe it leſs in proportion than when made to a leſs quantity. Thus particularly, let *ie* be to *ib* as 2 to 1, and let $io=1$, then $ie+io=2+1=3$;

and

and $ib+io=1+1=2$. Therefore, by the addition of *io*, *ie* is encreaſed only in the proportion of 3 to 2, and *ib* by the ſame addition is encreaſed in the proportion of 2 to 1. Since then the apparent diameter of the image is equal to that of the object, when the eye is cloſe to the lens, and ſince though they both decreaſe as the eye departs from the lens, yet the apparent diameter of the image decreaſes the leaſt of the two, it follows that, when the eye is at any diſtance from the lens, the object of refracted viſion will be bigger than the object of plane viſion; or the apparent magnitude of the object ſeen through the lens will be greater than its apparent magnitude ſeen by the naked eye.

The latter part of the propoſition, that the apparent diameter of the object ſeen through the lens will decreaſe as the eye withdraws itſelf, has been proved already in proving the firſt part. All that we need obſerve upon this head is, that when the eye departs to an infinite diſtance from the lens, the apparent diameter of the object ſeen through it will be infinitely ſmall. The uſe of this obſervation will be ſeen hereafter.

There is one thing more that will be worth our notice before we take leave of this propoſition, which is, that though we ſay an object ſeen through the lens, when the eye is at *o*, appears to be magnifyed, yet by that we mean only that the apparent ſize, if ſeen through the lens, when the eye is there, is greater than it would be, if ſeen by the naked eye at the ſame ſtation *o*. For the apparent magnitude of it ſeen through the lens when the eye is at *o*, is not greater than the ſame object might appear to the naked eye, if the naked eye was nearer to it. Nay the eye may be ſo near as to ſee the object bigger without the help of a lens, than it ſees it at the ſtation *o* with this help. Thus particularly, by the ſecond part of this propoſition, the object ſeen through the lens, when the eye is at *o* appears leſs than it would, if the eye was at *i*. But if the eye was at *i*, the object would appear no bigger with the lens than without it, by propoſition 147. Therefore the object with the help of the lens does not ſeem ſo big, when the eye is at *o*, as it would without the lens, if the eye was at *i* or in the ſtation of the lens.

Therefore the chief uſes of a convex lens in viſion may be reduced to theſe two. Firſt, if we want to look at an object, and the eye muſt neceſſarily be placed at a certain diſtance from it; we ſhall ſee it larger at that diſtance by the help of a lens than we could without one. Secondly, if we want to ſee an object of a certain ſize, and the naked eye, in order to ſee it of that ſize, muſt come ſo near to the object as to render it confuſed; by the help of a lens we may ſee it of that ſize and yet ſee it diſtinctly.

153. *If an object ſeen through a convex or plano-convex lens is more remote than the principal focus, and the eye on the other ſide is nearer than the place of the diſtinct picture, the object appears magnifyed, unleſs the eye touches the lens, and the apparent magnitude of it will be inverſely as the eyes diſtance from the diſtinct picture; ſo that as the eye withdraws from the lens towards the picture the apparent magnitude of the object will encreaſe.*

Let *gsl* be the lens, *abc* an object farther off from the lens than its principal focus, and *fed* the diſtinct picture received upon a paper. In order to prove the propoſition, we will ſhew that wherever the eye is between *s* and *e* or between the lens and the picture, if it turns itſelf one way and looks through the lens at the object *abc* or turns itſelf the other way and looks at the picture upon the paper, the apparent diameter of the object will always be equal to the apparent diameter of the picture. Suppoſe the eye was at the ſide of the lens *g*, then it would ſee both the object and the picture without looking through the lens, and ſince the real diameter of the object is to the real diameter of the picture, as their reſpective diſtances from the lens, or from the eye when it is at *g*, by propoſition 88, conſequently their apparent diameters will be equal, by propoſition 128. Now if the eye, inſtead of being at *g*, was cloſe to the lens at *s*, and was to look through it, the apparent diameter of the object would not be altered, by propoſition 147; and ſince its diſtance from the picture *fed* is the ſame whether it is at *g* or at *s*, it follows that if the eye is cloſe to the lens, whether it turns itſelf one way and looks through it at the object, or turns itſelf the other way and looks at the picture upon the paper, the apparent diameters of the object and picture are equal; and either of them appears of the ſame ſize that the naked eye in the ſame ſtation would ſee the object. But ſuppoſe the eye was cloſe to the paper at *f* or *e* or *d*, the apparent diameter of the picture would then be infinite, by propoſition 126. And if the eye when cloſe to the paper was to turn itſelf and look at the object, the apparent diameter of that would likewiſe be infinite. For ſuppoſe it was at *f*; all the rays, which enter it, are thoſe collected there; and they are only ſuch as come from the point *c*: therefore nothing but the point *c* will appear diffuſed over the whole ſurface of the lens though that ſurface was ever ſo large. The ſame would be true of the point *b*, if the eye was at *e*; and of the point *a*, if it was at *d*. Therefore in this ſtation of the eye every point in the object is infinitely magnifyed, and conſequently its apparent diameter ſeen through the lens is infinite. Since then the object ſeen through the lens, and the picture ſeen upon the paper appear equal, when the eye is cloſe to the lens, and appear equal again, and are both of them infinite, when the eye is at the paper; it follows that they muſt both of them have apparently encreaſed, whilſt the eye departed from *s* to *e*, or from the lens to the paper: and they muſt both of them have encreaſed equally; becauſe they were equal at firſt and are equal at laſt. But ſince they have encreaſed equally, wherever the eye was between *s* and *e*, the apparent diameter of the object ſeen through the lens muſt always have been equal to the apparent diameter of the picture ſeen upon the paper. From hence we may prove the three parts of the propoſition.

Firſt, the object will be magnifyed, if the eye does not touch the lens, for it appears of the ſame ſize either with the lens or without it, if the eye is at *s*, and as the eye withdraws towards *a* the apparent magnitude of the object encreaſes.

Secondly; the apparent diameter of the object ſeen through the lens is every where equal to the apparent diameter of the diſtinct picture; and the apparent diameter of the diſtinct picture is every where inverſely as the diſtance of the eye from the picture, by propoſition 126. Therefore the apparent diameter of the object ſeen through the lens is inverſely as the diſtance of the eye from the diſtinct picture.

Thirdly, that as the eye withdraws from the lens towards the diſtinct picture, the apparent magnitude of the object ſeen through the lens will keep encreaſing has been proved as we went along.

154. *If an object ſeen through a convex or plano-convex lens is placed in the principal focus, it will be magnifyed; unleſs the eye touches the lens; and the apparent magnitude of it will not be altered by the eyes withdrawing from the lens.*

If the object *abc*, Plat. XII. fig. 7, was in the lens's principal focus, the refracted rays of each beam would neither diverge, as they are there repreſented, nor would they converge, as in fig. 11, but would be parallel, by propoſition 62. But ſince the refracted rays are parallel, if we conſider them as diverging ones, the imaginary radiants or the laſt image muſt be at an infinite diſtance, and if we conſider them as converging ones the focuſes or the diſtinct picture muſt be at an infinite diſtance, by propoſitions 30, 31. Now when the eye departs from a lens, if the apparent magnitude of an object ſeen through the lens decreaſes, it is becauſe the eye departs to a greater diſtance from the laſt image, as in propoſition 152; or if the apparent magnitude of the object encreaſes, it is becauſe the eye by departing from the lens gets nearer to the place of the diſtinct picture, as in propoſition 153. But when the object is in the principal focus, there is neither laſt image nor diſtinct picture, except at an infinite diſtance. Therefore the apparent magnitude of the object, as the eye departs from the lens, will neither decreaſe nor encreaſe; ſince no finite departure of the eye can either remove it farther from the laſt image or bring it nearer to the diſtinct picture. From hence it follows, that when an object is placed in the lens's principal focus, its apparent magnitude ſeen through the lens is the ſame whether the eye touches the lens, or is withdrawn to any finite diſtance whatever. Now if the eye touches the lens, the object is not magnifyed, but appears of the ſame ſize through the lens that it would appear of without it to the eye in the ſame ſtation, by propoſition 147. But ſince, let the eye be where it will behind the lens, the object appears of the ſame ſize as when the eye touches the lens, why ſhould we ſay that the object is not magnifyed in this ſtation of the eye but will be magnifyed in any other? The anſwer is, that if in any place as *o* behind the lens, the object *abc* appears as big as the naked eye could ſee it at the ſtation *i*, then at the ſtation *o* the object ſeen through the lens is ſaid to be magnifyed, not becauſe it appears bigger than the naked eye could ſee it in any ſtation, but becauſe it appears bigger than the naked eye could ſee it in that particular ſtation *o*.

In the three laſt propoſitions we have conſidered what alterations are made in the apparent magnitude of an object ſeen through a convex or plano-convex lens, by the motion of the eye. In the two following ones we will conſider what alteration it will undergo by the motion either of the lens or of the object.

155. *When the eye and object are fixed in their ſtations, as a convex or plano-convex lens moves from the eye the apparent magnitude of the object keeps encreaſing, till it has got to the middle between them; and then as the lens moves farther, the apparent magnitude keeps decreaſing, till it has got cloſe to the object: provided that in this whole motion of the lens the eye is never more remote from it than the place of the diſtinct picture.*

When the lens is at either extreme, either cloſe to the eye or cloſe to the object, by propoſition 147, the object is not magnifyed. But when the lens is any where between the two extremes, by propoſitions 152, 153, 154, the object is magnifyed. Therefore it will be moſt magnifyed, when the lens is equally removed from the two extremes, or when it is in the middle between the eye and the object. Now ſince the object is not magnifyed, when the lens is cloſe to the eye; but is moſt magnifyed when the lens is half way between the eye and the object, it follows that, as the lens moves from the eye towards that middle ſtation, the apparent magnitude muſt keep encreaſing. And ſince the object is magnifyed, when the lens is in that middle ſtation, but is not magnifyed at all, when it gets to the other extreme and is cloſe to the object, it follows that, as the lens moves from that middle ſtation towards the object, the apparent magnitude muſt keep decreaſing.

156. *When an object ſeen through a convex or plano-convex lens departs from the lens; if the eye is in the principal focus, the objects apparent magnitude is not altered; if the eye is more remote than the principal focus, the apparent magnitude is encreaſed; if the eye is nearer than the principal focus the apparent magnitude is diminiſhed.*

When an object departs from a convex lens, its laſt image on the ſame ſide of the lens with itſelf departs at the ſame time, by propoſition 72, and conſequently the apparent magnitude is diminiſhed, ſo that if the object was to depart to an infinite diſtance its apparent magnitude would be infinitely ſmall, by propoſition 152. But whilſt the object and its laſt image depart on one ſide, the diſtinct picture will approach on the other ſide where the eye is, by propoſition 71: and if the diſtinct picture was to approach till it came to the place where the eye is, the object would be infinitely magnifyed, by what was ſaid in proving propoſition 153. Now if the eye is ſuppoſed to be in the principal focus of the lens, the diſtinct picture cannot come to the place of the eye, unleſs the object departs to an infinite diſtance, as appears under propoſition 107. Therefore when the eye is in the principal focus, if the object departs to an infinite diſtance, the apparent magnitude is infinitely diminiſhed upon one account and infinitely encreaſed upon another; infinitely diminiſhed, becauſe the laſt image is at an infinite diſtance; and infinitely encreaſed, becauſe the diſtinct picture will be at the ſame place with the eye. Conſequently, ſince in an infinite departure of the object from the lens, the apparent magnitude is encreaſed and diminiſhed equally, it muſt have kept decreaſing juſt as much upon one account, whilſt the object was departing, as it has kept encreaſing upon the other; and therefore upon both accounts together, it muſt have been always the ſame.

If the eye was more remote than the principal focus; then the apparent magnitude, as before, would not be infinitely diminiſhed, but by an infinite departure of the object. But ſince, when the object is at an infinite diſtance, the diſtinct picture will be in the principal focus, and ſince the eye is farther from the lens than the principal focus, the diſtinct picture will be at the place where the eye is, before the object has departed to an infinite diſtance. Conſequently a finite departure of the object will magnify it infinitely. But if the apparent magnitude may upon one account become infinitely great by the objects departing only to a finite diſtance, but cannot upon the other account become infinitely ſmall but by departing to an infinite diſtance; it follows that in this ſtation of the eye whilſt the object is departing, its apparent magnitude muſt encreaſe faſter than it decreaſes, and therefore it will encreaſe upon the whole.

If the eye was nearer to the lens than the principal focus, the apparent magnitude would, as before, be infinitely diminiſhed by the objects departing to an infinite diſtance. But it cannot be infinitely encreaſed, unleſs the diſtinct picture is at the ſame place with the eye. Now when the object has departed to an infinite diſtance, the diſtinct picture is no nearer to the lens than its principal focus. Therefore, ſince the eye is ſuppoſed to be nearer, an infinite departure of the object will not bring the diſtinct picture to the place of the eye, and conſequently will not infinitely encreaſe its apparent magnitude. But ſince an infinite departure of the object will infinitely diminiſh its apparent magnitude upon one account, but will not infinitely encreaſe it upon the other account, if the eye is nearer than the principal focus, it follows that the apparent magnitude in the departure of the object muſt decreaſe faſter than it encreaſes, and conſequently muſt decreaſe upon the whole.

157. *We commonly make a wrong judgment of an objects apparent diſtance, when we ſee it erect, through a convex or plano-convex lens.*

The object of refracted viſion is always more remote from the lens than the object of plane viſion. For if the refracted rays diverge leſs than the incident ones, as in propoſition 148, then the imaginary radiants or the laſt image, will be more remote than the real ones, or than the object, by propoſition 67. Or if the refracted rays either are parallel or converge, as in propoſitions 149, 150, then the imaginary radiants are at an indefinite diſtance, and the place of the laſt image though indeterminate will be at a greater diſtance than the place of the object. And yet in any of theſe caſes, the object, when ſeen through the lens, ſeems to be nearer, than when ſeen by the naked eye. For in looking through a lens we ſeldom uſe more than one eye at once; and there is commonly nothing ſeen beſides the object, ſo that there is no viſible extenſion between the eye and the object by which we may eſtimate its apparent diſtance. In theſe circumſtances the apparent magnitude is the principle, by which we judge of its diſtance: and conſequently, by propoſition 135, ſince

the

the lens encreaſes its apparent magnitude, we ſhall judge the diſtance to be leſs than it is. But if the object is placed at the end of a ruler, which is parallel to the horizon, and the lens is large enough for us to ſee the ruler through it; there is then a viſible extenſion between the lens and the object. And by ſeeing the ruler through the lens its apparent length, which is the apparent diſtance of the object at the end of it, will be planely encreaſed. Or if the lens is large enough to uſe both eyes at once and the object is placed, as in propoſition 135, ſo that part of it appears through the lens and part of it may be ſeen by the naked eye at one ſide of the lens, we have then another principle to judge by; and the part which is ſeen through the lens will ſeem at a greater diſtance than the other part, which is ſeen by the naked eye.

158. *If an object is farther from a convex or plano-convex lens than its principal focus, and the eye on the other ſide is nearer than the diſtinct picture, the place of the object of refracted viſion is at a greater diſtance from the lens than the place where we look for it.*

In this ſituation the object ſeen through the lens appears double; and if the right eye is ſhut, the right-hand object vaniſhes; or if the left eye it ſhut, the left-hand object vaniſhes. Therefore the place of the object ſeen by refraction is beyond the concourſe of the optic axes, or at a greater diſtance than the place where we look for it, by propoſition 117.

159. *If an object is farther from a convex or plano-convex lens than its principal focus, and the eye is at a greater diſtance from it on the other ſide than the place of the diſtinct picture, the object appears fainter than to the naked eye, it is inverted, and may be ſeen diſtinctly.*

If the object *abc*, Plat. XII. fig. 11, is at a greater diſtance from the lens *gsl* than the lens's principal focus, and the eye at *o* is at a greater diſtance than *fed* the place of the diſtinct picture; then the laſt image or object of refracted viſion is the diſtinct picture itſelf, by propoſition 138. For after the rays that come from the ſeveral points in the object, have met in as many focuſes, at the place of the diſtinct picture, they will diverge laſt from the focuſes, when they come to the eye at *o*. Now as the diſtinct picture is nearer to the eye than the object, the rays coming from that picture will diverge ſo much, that many of them will be too far aſunder to enter the eye: and upon this account the object of refracted viſion will appear faint. The object and the diſtinct picture, which is now the object of refracted viſion, are on contrary ſides of the lens:

therefore the object will appear inverted, by propoſition 143. The rays diverge from the ſeveral imaginary radiants in the picture *fed* juſt as they would from the ſeveral real radiants of an object in the ſame place; and conſequently if the eye is too near it the appearance will be confuſed, juſt as it would be in too near an approach of the eye to a real object. But if the eye is removed to a moderate diſtance it will appear as diſtinct as any real object would at the ſame diſtance.

160. *When an object is more remote from a lens than its principal focus, and the eye is more remote than the diſtinct picture, the apparent diameter of the object is directly as the pictures diſtance from the lens, and inverſely as its diſtance from the eye.*

In this poſition of the object and eye the diſtinct picture is the object of refracted viſion, by propoſition 138. And the apparent diameter of the object ſeen in this manner is proportional to the apparent diameter of that picture. Now the apparent diameter of the picture, like that of any real object is directly as its real diameter and inverſely as its diſtance from the eye, by propoſitions 127, 126. But the real diameter of the picture is directly as its diſtance from the lens, by propoſition 88. Therefore the apparent diameter of the picture, and conſequently of the object, is directly as its diſtance from the lens, and inverſely as its diſtance from the eye.

161. *If an object is farther from a convex or plano-convex lens than its principal focus, and the eye is farther from it than the diſtinct picture; the place of the object of refracted viſion is nearer to us than the place where we look for it.*

We look at the lens, and the object of refracted viſion is nearer to us; for it is at the place of the diſtinct picture, which is between the lens and the eye. And accordingly we find that the object thus ſeen appears double; and in this double appearance the right-hand one is ſeen by the left eye, and the left-hand one by the right eye. Therefore the place of the object ſeen by refraction is nearer to us than the concourſe of the optic axes or than the place where we look for it, by propoſition 118. But ſuppoſe the object that we are looking at to be a candle, and let a pencil or any other ſlender body be held at the place of the diſtinct picture as a guide to our eyes; if we look directly at the pencil we may ſee the candle at the ſame time; but then it will appear double no longer. For the optic axes concurring in the pencil, concur, by the ſuppoſition, in the place of the candle ſeen by refraction: therefore the candle ought to appear ſingle, by propoſition 116.

162. *When an object is farther from a convex or plano-convex lens than its principal focus, and the eye is farther from it than the diſtinct picture; if the lens is moved towards the right hand or towards the left, the object will appear to move in the ſame direction; but if the eye is moved towards the right hand or towards the left, the object will appear to move in the contrary direction.*

Firſt; from any point as *b* in the object *abc* let a line *bse* be conceived to be drawn through the middle of the lens to the object of refracted viſion *fed.* This line is the axis of the rays that come from *b.* Now, ſince the object is ſuppoſed immoveable, one end of this line is fixed at *b*, and the other end will always be at that point *e* in the diſtinct picture, where the rays that come from *b* are united in their focus. But if the lens *gsl* is moved towards the left hand the middle point *s* moves the ſame way with the whole lens. And ſince the line *bse* always paſſes through the middle of the lens; this line whilſt the lens is moved towards the left-hand will turn round upon *b* as upon a center and will move towards the left hand with the point *s*. But the other end of it is always at *e* in the diſtinct picture. Therefore as this line turns round upon *b* and moves towards the left hand, the point *e* in the diſtinct picture will move the ſame way. This might be proved of every other point in the picture; and conſequently the whole picture or the object of refracted viſion moves in the ſame direction with the lens; towards the left hand, if the lens moves towards the left; or towards the right hand, if the lens moves towards the right. So that, when the lens is moved, the apparent motion of the object ſeen through it ariſes from the real motion of the picture, which is the object of refracted viſion.

Secondly; ſuppoſe the eye to be at *o* directly before *s* the middle of the lens; and from *s* through *e* the middle of the diſtinct picture to the eye at *o* conceive the line *seo* to be drawn. If the object and lens are both fixed, the diſtinct picture *fed* and the middle point of it *e* will be immoveable. Now the line *seo* paſſes through this middle point. Therefore if the eye *o* or one end of the line *seo* is moved towards the left hand, it will turn round upon *e* as a center, and the other end of it next the lens will move the contrary way or towards the right; that is, towards *l.* The eye refers the diſtinct picture to the lens and determines the apparent place of it by the part of the lens upon which it appears, by propoſition 161, and conſequently the end of the line *seo* which is at the lens determines the apparent place of the diſtinct picture. But if the eye moves towards the left hand, that end of the line *seo* which is next the lens moves towards the right; and upon that account the picture or object of refracted viſion appears to move upon the lens towards the right: or if the eye was to move towards the right hand, the picture would ſeem upon the lens to move towards the left. So that, when the eye is moved, the apparent motion of the object ſeen through the lens ariſes from an apparent motion of the picture or laſt image in reference to the lens.

The many caſes of viſion through convex or plano-convex lenſes is owing to the many different directions, in which the refracted rays may come to the eye. When the object is nearer than the principal focus, the refracted rays diverge, or if it is in the principal focus, they are parallel, when they come to the eye, wherever the eye is placed. When the object is more remote than the principal focus if the eye is nearer than the diſtinct picture, the rays are converging, when they fall upon it; if the eye is at the place of the diſtinct picture, they are met in their ſeveral focuſes; if it is more remote than this, they have paſſed their focuſes and have begun to diverge from thence. But the caſes of concave or plano-cave lenſes are very few; for wherever the eye is, the refracted rays, when they fall upon it, always diverge from an image on the ſame ſide of the lens with the object.

163. *When an object is ſeen through a concave or plano-concave lens, it ſeems fainter, than when ſeen by the naked eye; it appears erect and may be ſeen diſtinctly.*

If the object *abc*, Plat. XII. fig. 8. is ſeen through the concave lens *gil*, the rays that come from the point *b* or any other points in the object are made to diverge more by paſſing through the lens than they would have done, if the object had been viewed with the naked eye, by propoſition 76. But the more they diverge, the farther they are aſunder, when they come to the eye at *o*; and conſequently fewer of them will come within the compaſs of the pupill and enter the eye. Therefore the lens will make the object ſeem fainter. The refracted rays diverge from imaginary radiants *d*, *e*, *f*, &c. which are behind the lens, by propoſition 77. Therefore the object and the laſt image are on the ſame ſide of the lens, and the object ſeen through the lens will appear erect, by propoſition 143. If the eye is very near to the lens, ſo as to receive the extreme rays *em*, *en*, of the cone coming from the imaginary radiant *e*, and the extreme rays of all other refracted cones coming from the ſeveral imaginary radiants *d*, *f*, &c. which compoſe the laſt image; theſe extreme rays will diverge too much and the picture upon the retina of moſt eyes will be confuſed, by propoſition 99. But there are two ways of rendering it diſtinct. One way is to have a piece of paper with a pin-hole in it cloſe to the lens; for by this means the extreme rays will be ſtopped and only the middle rays of each cone, which are leſs diverging, will enter the eye. The other way requires no ſuch paper; for if the eye only withdraws itſelf to a moderate diſtance from the lens, the extreme rays will be ſo far aſunder, as not to enter the pūpill, only the middle or leſs diverging rays will enter it, and then the picture upon the retina and conſequently the viſion will be diſtinct, by propoſitions 99, 113.

164. *The apparent magnitude of an object ſeen through a concave or plano-concave lens is diminiſhed, unleſs the eye touches the lens or the lens touches the object; and as the eye departs from the lens, its apparent magnitude decreaſes.*

If the eye touches the vertex of the lens at *i*, the apparent diameter of the laſt image *df* is the ſame, that the apparent diameter of the object itſelf *ac* would be to the naked eye in the ſame ſtation, by propoſition 147. Now as the eye withdraws from the lens to any point *o*, its diſtance both from the laſt image and from the object encreaſes, and conſequently the apparent magnitude either of the laſt image ſeen through the lens from the ſtation *o* or of the object ſeen by the naked eye at the ſame ſtation will be leſs, than when ſeen from the ſtation *i*, by propoſition 126. But as the eye withdraws from the lens, its diſtance from the laſt image, which is the object of refracted viſion, encreaſes faſter in proportion than its diſtance from the object itſelf, which is the object of plane viſion. Therefore ſince, by propoſition 126, the apparent diameter decreaſes as the diſtance of the eye encreaſes, the apparent diameter of the object of refracted viſion decreaſes faſter than the apparent diameter of the object of plane viſion. Their apparent diameters were equal when the eye was at *i*. But the apparent diameter of that will be the leaſt, when the eye is at *o*, which in this departure of the eye has decreaſed the moſt; and this is the apparent diameter of the object of refracted viſion. Therefore when the eye is at *o* the object of refracted viſion will be apparently leſs than the object of plane viſion; or the object, when ſeen through the lens by the eye at *o*, will ſeem leſs, than when ſeen by the naked eye at the ſame ſtation. That the diſtance from *e* the laſt image encreaſes faſter than the diſtance from *b* the object, whilſt the eye departs from *i* to *o*, is evident from what has been ſhewn under propoſition 152. For the diſtance of the image from the lens is always leſs than the diſtance of the object: *ie* is leſs than *ib*: and as the eye departs from *i* to *o*, both theſe quantities encreaſe by the addition of one and the ſame quantity *io* made to each of them. But if the ſame quantity is added reſpectively to two unequal quantities, the leſſer of the two will be encreaſed more in proportion by this addition than the greater is; ſo that *oe* will bear a greater proportion to *ie*, than *ob* bears to *ib*.

The ſecond part of this propoſition, that the apparent magnitude of the object ſeen through the lens will decreaſe, as the eye departs from the lens, has been ſufficiently proved in proving the firſt part.

165. *When the eye and object are fixed in their places, as a concave or plano-concave lens moves from the eye the apparent magnitude of the object keeps decreaſing, till it has got to the middle between them, and then, as the lens moves farther, the apparent magnitude keeps encreaſing, till it has got cloſe to the object.*

When the lens is at each extreme, when it is either cloſe to the eye or cloſe to the object, the apparent magnitude of the object is the ſame, when ſeen through the lens, as when ſeen with the naked eye, by propoſition 147. In all other ſtations of the lens, the object ſeen through it appears to be diminiſhed, by propoſition 164. But ſince the object is not at all diminiſhed, when the lens is at either extreme, and is diminiſhed, when the lens is in any intermediate ſtation; it will be moſt of all diminiſhed, when the lens is equally removed from each extreme or is in the middle between the eye and the object. And ſince the object is not diminiſhed, when the lens is cloſe to the eye, and is moſt diminiſhed, when it is in the middle ſtation, it follows, that, as the lens moves from the eye to this ſtation, the apparent magnitude of the object muſt keep decreaſing: ſo likewiſe ſince the object is moſt diminiſhed, when the lens is in the middle ſtation and is not diminiſhed at all, when it is cloſe to the object; it follows, that, as the lens moves from this ſtation to the object, the apparent magnitude muſt keep encreaſing.

166. *The apparent magnitude of an object ſeen through a concave or plano-concave lens decreaſes as the object departs from the lens.*

For as the object departs, the laſt image will depart with it, but ſlower, by propoſitions 78, 138. And conſequently as the object departs the diameter of the laſt image, which is the object of refracted viſion, will decreaſe on account of the objects diſtance from the lens being encreaſed, by propoſition 144.

167. *We commonly make a wrong judgment of an objects apparent diſtance, when we ſee it through a concave or plano-concave lens.*

We ſeldom uſe both eyes in looking through a concave lens, and there is ſeldom any thing elſe to be ſeen through it beſides the object. Therefore we judge of the objects diſtance by its apparent magnitude: and as the lens diminiſhes the object, we conclude it to be farther off, when ſeen through the lens, than when ſeen without it. This is certainly a wrong judgment: for the object of refracted viſion is nearer to the eye than the object of plane viſion, by prop. 76, 28. And this wrong judgment of the apparent diſtance, which is owing to our want of all other principles to go upon, except the apparent magnitude, may be corrected, as in propoſition 157. For if the object is placed upon a ruler, which is parallel to the horizon, ſo that we may ſee the ruler through the lens, at the ſame time that we ſee the object; then if we judge of the objects diſtance by the viſible extenſion of the ruler, we ſhall ſee that it is nearer to us, when we look at it through the lens, than when we look at it with the naked eye. Or if we place the object in ſuch a manner as to ſee one part of it through the lens, and another part of it by the naked eye, and the lens is broad enough to ſee through it with both eyes together; upon comparing the apparent diſtances of the two parts, we ſhall find, that the part, which we ſee through the lens, appears nearer than the other.

168. *Short-ſighted people ſee objects diſtinctly at moderate diſtances by the help of concave lenſes.*

The defect of ſuch perſons eyes as are ſhort-ſighted has already been explaned, in propoſition 121. Their eyes are too convex and will bring the ſeveral pencils of rays to their reſpective focuſes, or will form a diſtinct picture, before the rays come to the retina, unleſs the object is brought very near to the eye. Now all that is done by bringing the object near to the eye may be done by looking through a concave lens, whilſt the object remains at a moderate diſtance. Bringing the object very near makes the rays diverge ſo much the more, by propoſition 28. And the refraction of a concave lens will do the ſame, by propoſition 76. So that when we look through a concave lens at objects, which are not near us, the rays come to the eye, as they do when we look at very near objects without a lens. Therefore ſuch perſons, as ſee very near objects diſtinctly with the naked eye, will be able to ſee remoter objects as diſtinctly by the help of a concave lens.

Though concave glaſſes help ſhort-ſighted people for the preſent, yet the conſtant uſe of them will make their eyes worſe. For when an object is ſeen through a concave lens, the rays, that come from it, diverge as much as they would if that object was very near. So that the conſequence of uſing a concave lens conſtantly is the ſame as if a man was only to look at near objects. The eye has no occaſion to ſtrain itſelf; but is always kept and by cuſtom is confirmed in that very figure, which is fit only to ſee near objects, and which is the occaſion of ſhort-ſightedneſs. And though the perſons, who have this defect in their eyes, find that it ſtrains their eyes to look at diſtant objects without their glaſs, yet this ſtraining would be of ſervice to them; it might gradually change the figure of their eyes, and either remedy the defect, or at leaſt prevent it from encreaſing.

169. *Old people ſee objects diſtinctly at moderate diſtances by the help of convex lenſes.*

The defect of old peoples eyes has been explaned, in propoſition 122. Their eyes are too flat and would not refract the rays enough to bring them to their reſpective focuſes, or to form a diſtinct picture upon the retina, unleſs the object was removed to a conſiderable diſtance from the eye. Now removing an object far from the eye only makes the rays of each beam diverge the leſs, by propoſition 28. And though the object is near, yet the rays will be made to diverge leſs, if they paſs through a convex lens, by propoſition 67. Therefore they, who can ſee only diſtant objects diſtinctly with the naked eye, will be able to ſee near ones as diſtinctly, if they make uſe of convex ſpectacles. For the rays come to the eye from near objects, when we look through a convex lens, juſt as they do from remote ones, when we look at them with the naked eye.

Perſons, that have good eyes, and when they are employed in fine work, frequently uſe convex ſpectacles only for the ſake of magnifying their work, will ſoon be unable to ſee diſtinctly any thing, that is near them, without ſuch ſpectacles. For the rays come to the eye through convex lenſes in the ſame manner that they would do from remote objects: and conſequently the eye, when we look through ſuch lenſes, puts on the figure that fits it for ſeeing remote objects diſtinctly. Therefore if by uſing ſpectacles we keep it conſtantly of this figure, it will by degrees fix itſelf too much to be altered; and then we ſhall be unable without ſpectacles to ſee any objects diſtinctly but remote ones. The work might be magnifyed, as much as there is occaſion for, by looking very near to it without ſpectacles: and though this may be found to ſtrain the eyes, yet ſuch ſtraining will ſeldom do them any great harm: the only harm it can do them is to make them ſhort-ſighted; and the good it may do them is to prevent them from ever having occaſion for convex ſpectacles.

170. *Objects ſeen through a plane glaſs are erect, nearer, and ſeem brighter and larger, than when they are ſeen by the naked eye.*

If GL, Plat. XII. fig. 6, is a plane glaſs the object PR, when ſeen through it will appear erect. The axes of the ſeveral beams, that come from any points P, Q, R, in the object, are the rays, which paſs through the glaſs without being refracted: and theſe rays, by propoſition 41, muſt be, PA, QC, RB, perpendicular to the ſurface of the glaſs. But lines perpendicular to a plane ſurface are parallel to one another. Euc. b. XI. prop. 6. Therefore the axes PA, QC, RB are parallel. Now all the rays of each beam diverge from one and the ſame point before refraction, and they diverge likewiſe from one and the ſame point after the two refractions, which they ſuffer in paſſing through the glaſs, by propoſition 45. But the axis of any beam is one of the rays. Therefore the axis of any beam both before and after refraction diverges from the ſame point with the other rays of the ſame beam. But the axis itſelf is not refracted at all; and conſequently the point from whence the other rays diverge both before and after refraction muſt be ſomewhere in the axis. Thus for inſtance, the rays of the higheſt beam, and PA amongſt the reſt, diverge before refraction from the point P. Suppoſe that all the other rays diverge after refraction from the point *p*, then this point *p* muſt be ſomewhere in the axis PA; for ſince PA paſſes ſtrait forwards and yet after all the other rays are refracted is to diverge from the ſame point with them, this point muſt be ſomewhere in the line PA, or elſe the ray, whoſe direction is in the line PA could not diverge from it. The ſame may be ſaid of the rays that come from the loweſt point R; they will diverge after refraction from ſome point in the axis RB. From hence it follows that to the eye on the other ſide of the glaſs, the higheſt point P of the object will appear in the line P*a*, and the loweſt point R will appear in the line R*b*, by propoſition 137. Now P*a* and R*b* are always parallel, and conſequently cannot croſs one another between the eye and the object. Therefore that axis, which is higheſt at the object, will be higheſt at the eye, and that, which is loweſt at the object, will be loweſt at the eye. And ſince the higheſt point of the object appears in the higheſt axis or line *a*P, and the loweſt point appears in the loweſt axis or line *b*R, and the middle point Q appears for the ſame reaſon in the middle axis or line *c*Q; every point appears in its proper place, or the object is erect.

The laſt image is nearer to the glaſs than the object: for every imaginary radiant is nearer than every real one. If *abkl*, Plat. IX. fig. 9, is a plane glaſs and *c* is a radiant, the rays *cd*, *ce*, which diverge from it will be refracted, when they enter the glaſs, into the lines *dg*, *en*, and again, when they come out of it, into the lines *gi*, *no*, parallel to *cd*, *ce*, by propoſition 47. But if from *g* and *n* lines as *gf*, *nf*, are drawn parallel to *cd*, *ce*, they will meet at *f*. Therefore the rays after their laſt refraction appear to diverge from the imaginary radiant *f*, which is nearer than the real one *c*. In like manner every imaginary radiant will be nearer than every real one in the object PQR, Plat. XII. fig. 6, or the laſt image *pqr*, which is the object of refracted viſion, will be nearer than the object itſelf.

If *gn*, Plat. IX. fig. 9, is the diameter of the pupill the rays *gi* and *no* will enter it, which, if there had been no glaſs between the radiant *c* and the eye, would have paſſed ſtrait on in the lines *cd*, *ce* continued, and when they came to the place of the eye would have been too far aſunder to enter it. Therefore ſince more rays from each radiant enter the eye, when it looks through the glaſs, than would have entered, if there had been no glaſs, each point of the object will appear the brighter.

The laſt image is equal to the object. For the object PQR is contained between the extreme axes PA, RB; and ſo is the laſt image *pqr*. And theſe axes are parallel, as has been proved already. Therefore they are juſt as far aſunder at the place of the image as they are at the place of the object; and conſequently the height of the image will be equal to the height of the object. But then the image is nearer to the eye than the object; and for that reaſon will appear larger, by propoſition 126.

Common window-glaſs is too thin to make any alteration in the apparent diſtance or apparent magnitude of objects, which are ſeen through it. For it is plane that if the glaſs *abkl*, Plat. IX. fig. 9, had no thickneſs, or if the ſides *ak* and *bl* were cloſe together, then *dg* and *en* would have no length, ſo that *g* would coincide with *d* and *n* with *e*. But *gf* and *dc* are parallel. Therefore if *g* and *d* are cloſe together *gf* and *dc* would coincide, and ſo likewiſe would *nf* and *ec*, that is, *f* and *c* would be at the ſame place. Now if the imaginary radiant is at the ſame place with the real one, the laſt image will be at the ſame place with the object and conſequently will appear at the ſame diſtance and of the ſame ſize. This would be the caſe if the glaſs had no thickneſs at all: from whence it is evident that the leſs the thickneſs of the glaſs is the leſs alteration will be made in the apparent diſtance and apparent magnitude of the object. Therefore in ſuch thin glaſs as common window-glaſs the alteration is too ſmall to be perceived.

CHAP.

CHAP. IX.

Of teleſcopes and microſcopes.

171. *An aſtronomical teleſcope conſiſts of two convex lenſes; whoſe diſtance from each other is equal to the ſum of their principal focuſes: that lens, which is towards the object, is called the object-glaſs, and that, which is next the eye, is called the eye-glaſs.*

IF *nl*, Plat. XIII. fig. 1, is one convex lens, whoſe principal focal diſtance is *mf*, and *bd* is another convex lens, whoſe principal focal diſtance is *cf*; let theſe two lenſes be ſo placed that the diſtance between them may be equal to *mf* added to *cf*, that is, let their diſtance from each other be *mc*, and in this ſituation they make an aſtronomical teleſcope.

172. *Very remote objects ſeen through an aſtronomical teleſcope appear diſtinct and inverted.*

If *zxy*, Plat. XIII. fig. 1, is a very remote object, the rays that come from one and the ſame point in it will be parallel to one another, by propoſition 30. Therefore let the parallel lines *pn*, *pm*, *pl*, be the rays, which come from *x* the middle point in the object, let *an*, *am*, *al*, be thoſe, which come from *z* the higheſt point, and *gn*, *gm*, *gl*, thoſe, which come from *y* the loweſt point. Theſe rays, and likewiſe the rays which come from all the other points, becauſe they are parallel to one another, will be collected into their ſeveral focuſes at *g*, *f*, *e*, the principal focus, by propoſition 60, and will there paint the diſtinct picture of the object. This diſtinct picture is the laſt image, and therefore, by propoſition 145, is the object of refracted viſion. But by the conſtruction of the teleſcope *gfe* is in the principal focus of the eye-glaſs, for *fc* is the eye-glaſs's principal focal diſtance. Conſequently the rays which diverge from any point *g* in this picture, will become parallel to their axis, after they have paſſed through the eye-glaſs, by propoſition 61. So that if *gc* is the axis of the beam that proceeds forwards and diverges from the point *g*, this axis goes ſtrait through the lens, but any other ray, as *gd*, will be refracted, and will come out of the eye-glaſs in the direction *do* parallel to *gc*. The ſame may be proved of the rays of any other beam. Therefore if the eye is at *o* or any where on the other ſide of the eye-glaſs, the laſt image *fed* or object of refracted viſion may be ſeen as diſtinctly as any object is by the naked eye, when that object is ſo remote that the rays

PLATE XII.

Fig. 1.

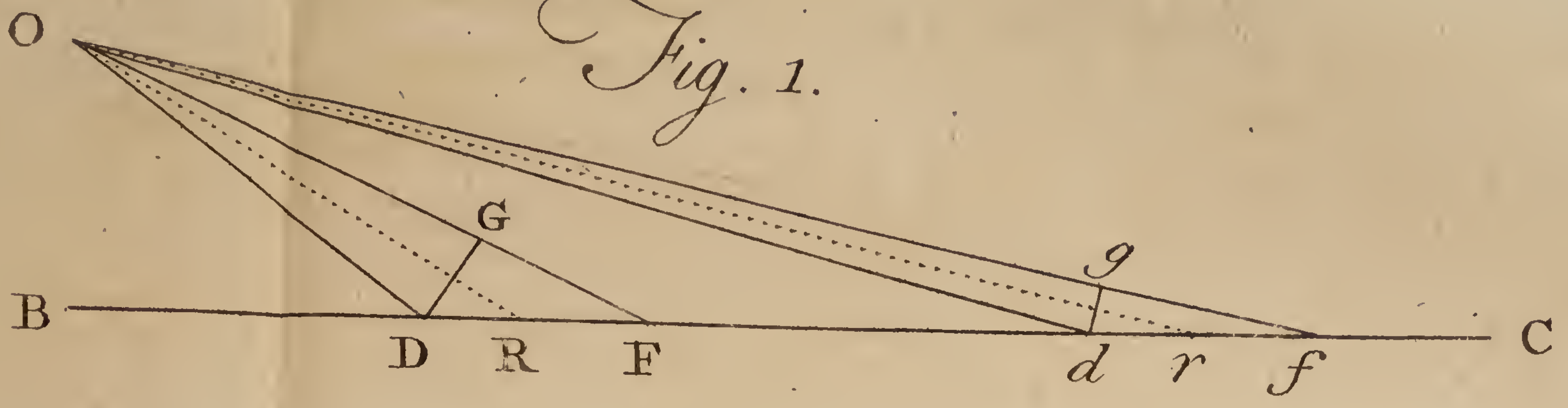


Fig. 2.

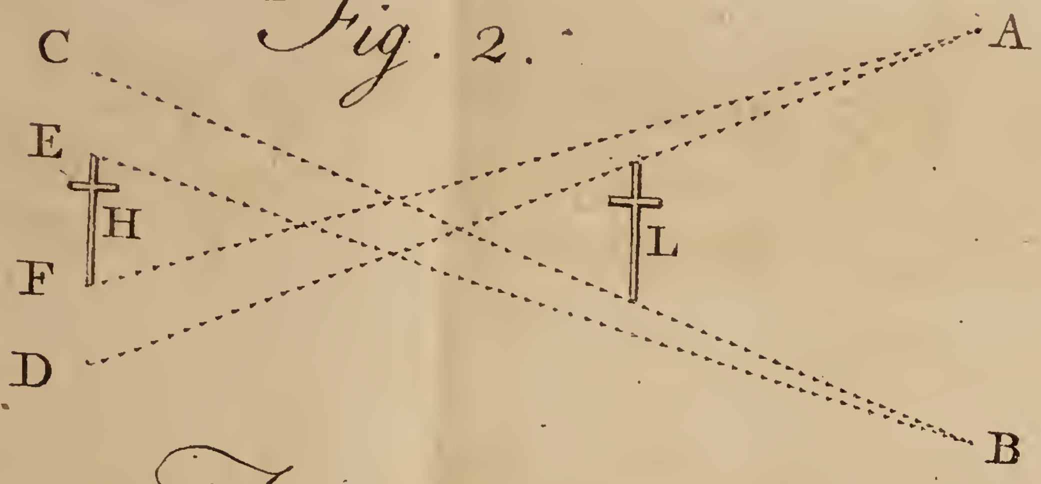


Fig. 3.

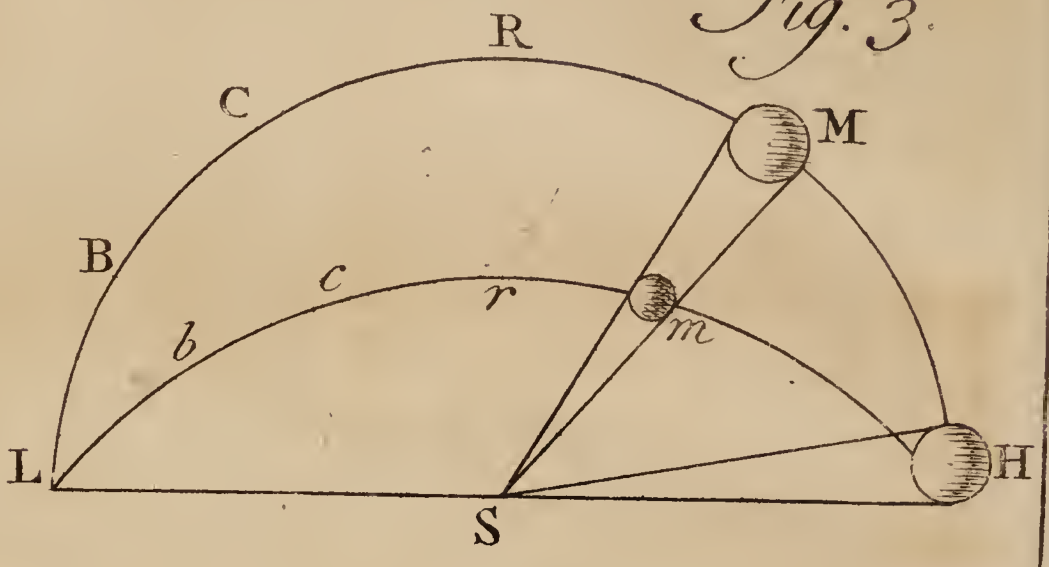


Fig. 4.

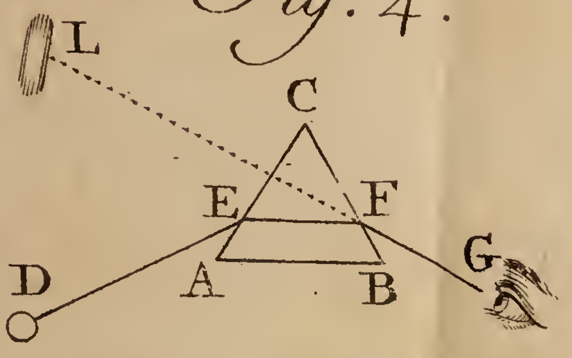


Fig. 5.

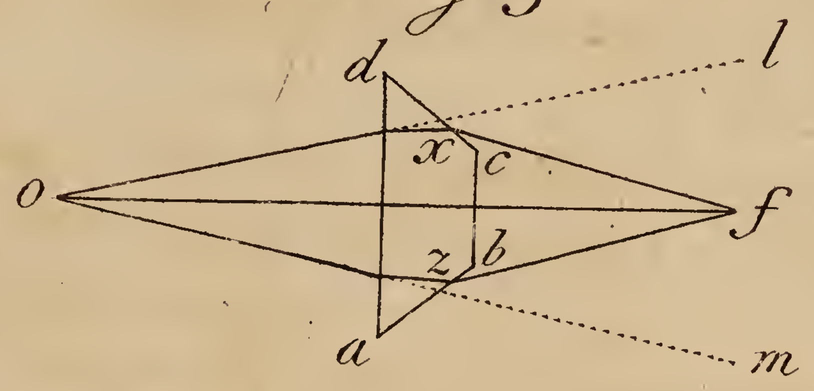


Fig. 6.

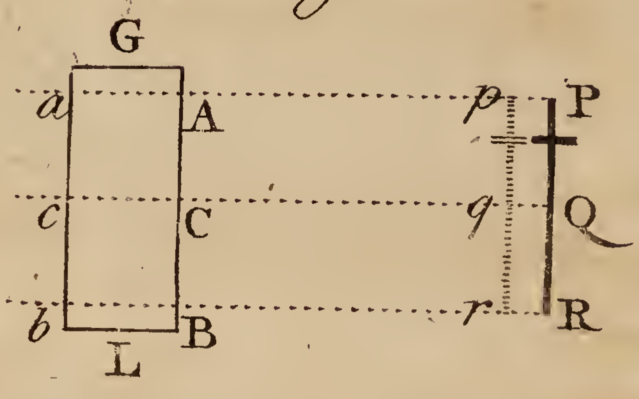


Fig. 7.

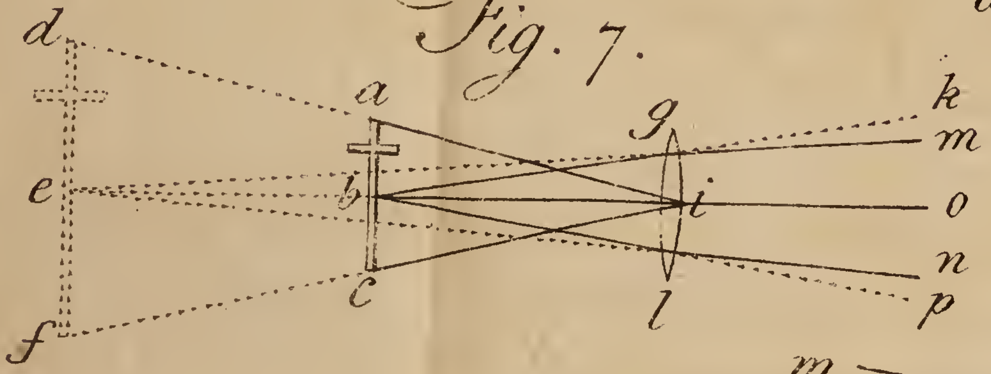


Fig. 8.

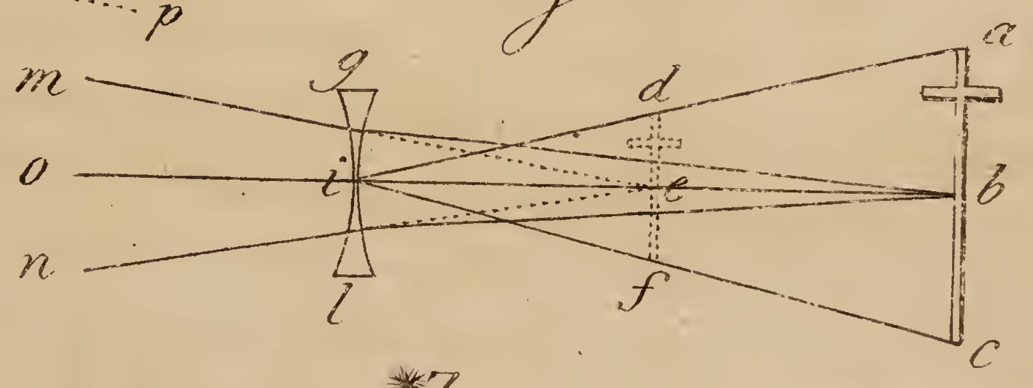


Fig. 9.

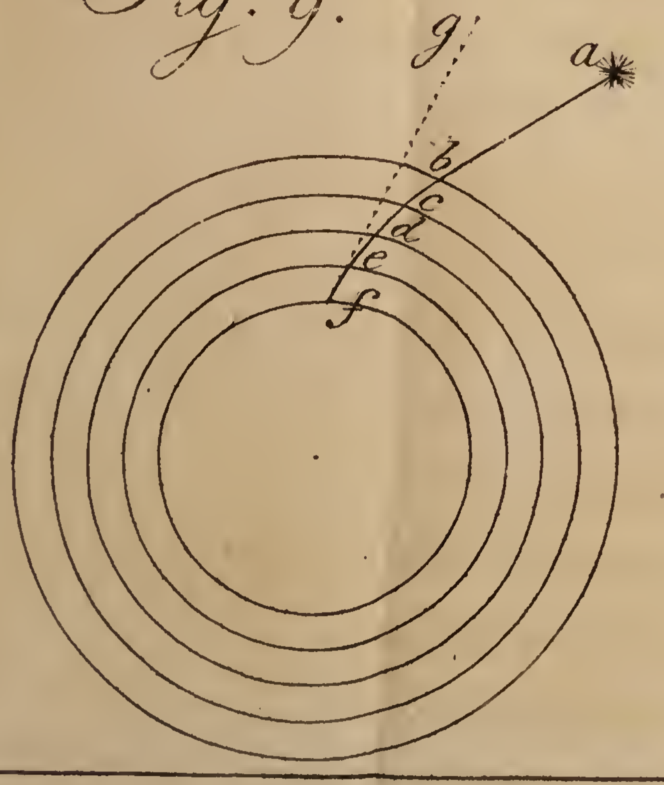


Fig. 10.

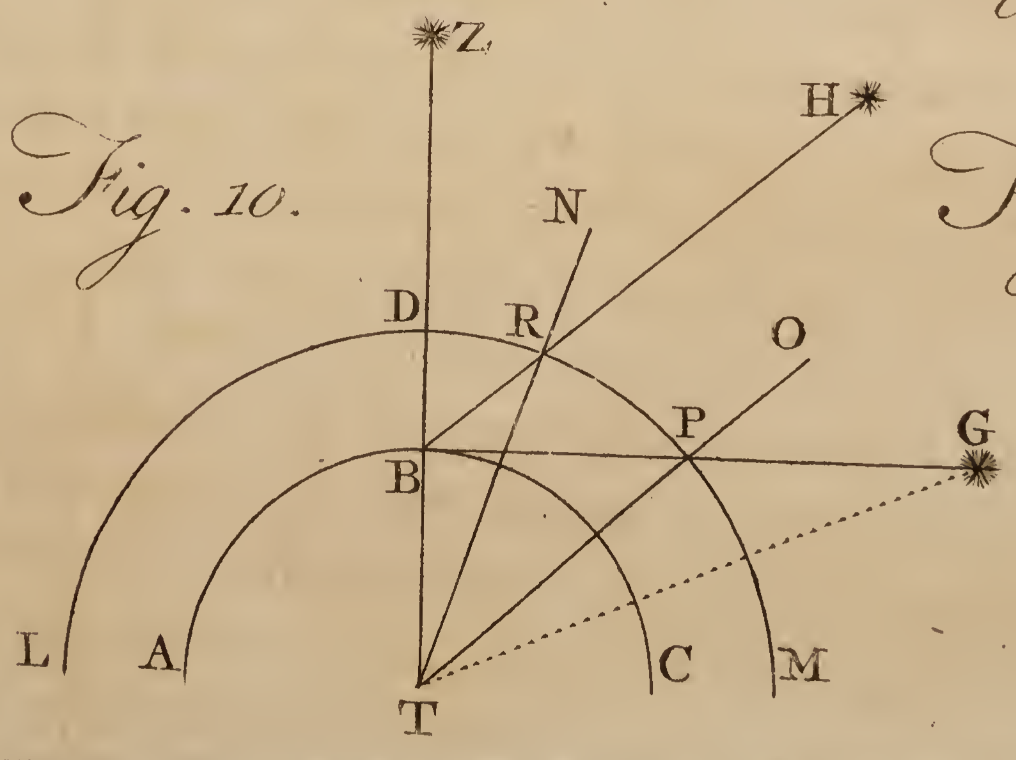
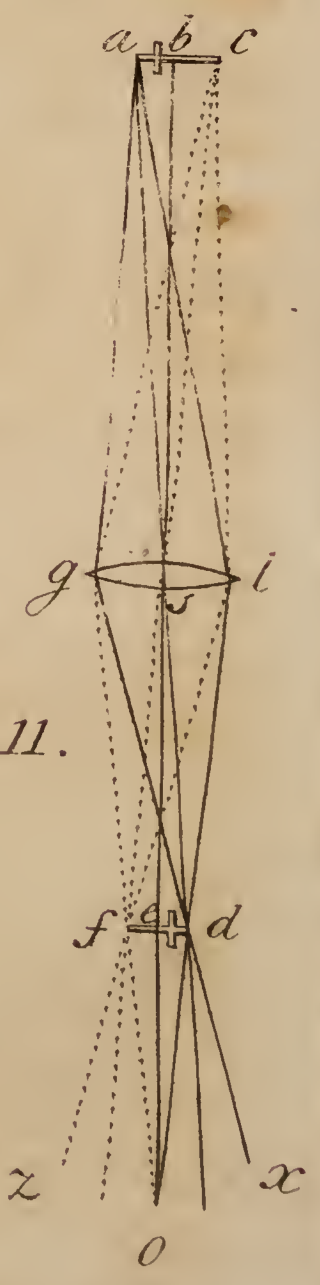


Fig. 11.



of each beam are parallel to one another. It is plane from what has been ſaid that the object thus ſeen will appear inverted; for the diſtinct picture is the object of refracted viſion, and the diſtinct picture is inverted, by propoſition 85.

173. *The apparent diameter of an object ſeen through an aſtronomical teleſcope, is to the apparent diameter of the ſame object ſeen by the naked eye at the ſtation of the object-glaſs, as the diſtance of the diſtinct picture from the object-glaſs to its diſtance from the eye-glaſs.*

In the teleſcope *mc*, Plat. XIII. fig. 1, the diſtance of the picture from the object-glaſs is *mf*, and its diſtance from the eye-glaſs is *fc*. The diameter of any remote object *zy* will appear longer through the teleſcope when the eye is at *o*, than it would appear to the naked eye placed at *m*, which is the ſtation of the object-glaſs. This propoſition determines how much longer the diameter of the object will appear; it is magnifyed in the proportion of *mf* to *fc*; or the diameter ſeems as much longer with the teleſcope than it would without it, as *mf* is longer than *fc*. Thus ſuppoſe that *mf* is to *fc* as 40 to 1, then the apparent diameter of the object *zy* will be magnifyed in the proportion of 40 to 1, or the object will appear 40 times longer and likewiſe 40 times broader through the teleſcope than it would appear to the naked eye at *m*.

If the diſtinct picture was received upon a paper at *efg*, the apparent diameter of the object ſeen by the naked eye at *m* the ſtation of the object-glaſs is equal to the apparent diameter of the picture ſeen from the ſame ſtation, as was proved under propoſition 153. Now the real diameter of the picture is given, by propoſition 88, becauſe its diſtance *mf* from the lens is given: conſequently the apparent diameter of the picture will be inverſely as the eyes diſtance from it; by propoſition 126, or 160. So that if the eye is any where at a leſs diſtance from the picture than *mf*, its apparent diameter will be greater in proportion as that diſtance is leſs. Suppoſe then that the paper was taken away and that there was no eye-glaſs, and let the eye be at *c* the ſtation of the eye-glaſs; its diſtance from the place of the diſtinct picture is then *fc*. Therefore the picture to the eye in that ſtation will appear bigger than it did at the ſtation *m* in the proportion of the diſtances *fc* and *mf* inverted, or in the proportion of *mf* to *fc*. But the apparent diameter of the picture ſeen from the ſtation *m* is equal to the apparent diameter of the object ſeen by the naked eye. Therefore the picture ſeen from the ſtation *c* appears bigger than the object in the proportion of *mf* to *fc*. This would be the caſe, if there was no eye-glaſs and the eye was placed in that ſtation where the eye-glaſs ſhould be. But if the eye-glaſs was ſlipped between the eye and the picture, ſo that the eye ſhould touch the eye-glaſs; this would make no alteration in the apparent diameter of the picture ſeen through it, by propoſition 147. Therefore if the eye is cloſe to the eye-glaſs of the teleſcope, the object ſeen through it will he magnifyed in the proportion of *mf* to *fc* But ſince the picture is in the eye-glaſs's principal focus, by the conſtruction, it appears of the ſame ſize whether the eye is cloſe to this lens or at any diſtance from it, by propoſition 154. Therefore, wherever the eye is, the apparent diameter of the object ſeen with the teleſcope is to the apparent diameter of the ſame object ſeen by the naked eye at the ſtation of the object-glaſs, as *mf* to *fc*, or as the diſtance of the diſtinct picture from the object-glaſs to its diſtance from the eye-glaſs. What is the moſt convenient ſituation for the eye to be placed in, ſhall be ſhewn hereafter.

174. *An aſtronomical teleſcope will not magnify an object unleſs the principal focal diſtance of the object-glaſs is greater than the principal focal diſtance of the eye-glaſs.*

Becauſe the objects, at which we look through teleſcopes, are remote ones, ſo that, by propoſition 30, the rays of each beam, that come from them, are nearly parallel to one another; it follows, by propoſition 60, that the diſtinct picture *gfe*, Plat. XIII. fig. 1, will be in the principal focus of the object-glaſs; and this picture, by the conſtruction of the teleſcope is likewiſe in the principal focus of the eye-glaſs. Therefore the diſtance of the diſtinct picture from the object-glaſs is the principal focal diſtance of the object-glaſs; and the diſtance of it from the eye-glaſs is the principal focal diſtance of the eye-glaſs. But, the apparent diameter of an object ſeen through a teleſcope is to the apparent diameter of the ſame object ſeen with the naked eye, as the diſtance of the picture from the object-glaſs to its diſtance from the eye-glaſs, by propoſition 173, that is, by what has juſt been proved, as the principal focal diſtance of the object-glaſs to the principal focal diſtance of the eye-glaſs.

From hence it follows, that, if *mf* the principal focal diſtance of the object-glaſs is greater than *fc* the principal focal diſtance of the eye-glaſs, the teleſcope will magnify the object. But, if *mf* was equal to *fc*, the object will appear no larger with the teleſcope than it does to the naked eye. Or laſtly, if *mf* was leſs than *fc*, the object would be apparently diminiſhed. Now the leſs convex a lens is the greater is its principal focal diſtance, by propoſition 53; and conſequently ſince the principal focal diſtance of the object-glaſs muſt be greater than that of the eye-glaſs for a teleſcope to magnify an object, the object-glaſs muſt be leſs convex or muſt be a ſegment of a larger ſphere than the eye-glaſs.

175. *An object may be equally magnifyed by two teleſcopes of very different lengths.*

If the principal focal diſtance of the object-glaſs is 20 feet, and that of the eye-glaſs is ½ foot; one is to the other as 40 to 1; and ſuch a teleſcope 20½ feet long will magnify an object 40 times; ſo that its apparent diameter, when ſeen through the teleſcope, will be 40 times longer, than when ſeen with the naked eye, by propoſition 173, 174. Another teleſcope 20½ inches long would magnify in the ſame proportion, if the principal focal diſtance of its object-glaſs was 20 inches and that of its eye-glaſs ½ inch: for in this teleſcope as well as in the former the principal focal diſtance of the object-glaſs is to the principal focal diſtance of the eye-glaſs as 40 to 1. And univerſally, ſince a teleſcope magnifyes an object in the proportion of the focal diſtance of the object-glaſs to the focal diſtance of the eye-glaſs; all teleſcopes, in which this proportion is the ſame, will magnify equally, whatever may be the lengths of the teleſcopes. The reaſon why ſhort teleſcopes that magnify an object very much are not ſo uſeful as long ones that magnify no more ſhall be ſhewn in its proper place.

176. *If an aſtronomical teleſcope is inverted, objects, that are ſeen through it, will appear to be diminiſhed.*

The principal focal diſtance of the object-glaſs is greater than the principal focal diſtance of the eye-glaſs, when the teleſcope is uſed in its proper poſition, by propoſition 174. But when it is inverted the eye-glaſs is turned towards the object, and therefore becomes the object-glaſs; and the object-glaſs is next the eye, and therefore becomes the eye-glaſs. So that in this poſition of the teleſcope the principal focal diſtance of the object-glaſs is leſs than the principal focal diſtance of the eye-glaſs; and conſequently, by propoſition 173, 174. the apparent diameter of the object will be diminiſhed.

177. *The viſible area, or ſpace that may be ſeen at one view through a teleſcope, is proportional to the area of the eye-glaſs.*

Since the diſtance of the picture from the object-glaſs is equal to the lens's principal focal diſtance; the area of the picture is a given quantity, by propoſition 96; but how much of that picture we can ſee at a time depends upon the breadth of the eye-glaſs. If a picture of any determinate

terminate ſize drawn upon canvas was to hang up againſt a wall, and we were to ſtand at a certain diſtance from it and to look at it through a hole in a piece of board, it is evident that we ſhould ſee more of the picture as the area of the hole was greater, and leſs of it as the area of the hole was leſs; ſo that the quantity of the picture, which might be ſeen at one view, would be proportional to the area of the hole. This is the caſe of the diſtinct picture in a teleſcope: the ſize of it is determinate; it is at a certain diſtance from the eye-glaſs, for it is always in the eye-glaſs's principal focus; and the eye-glaſs is the hole through which we look at it. Therefore the quantity of the picture that we can ſee at one view, or the viſible area, will be greater as this hole is greater, or leſs as this hole is leſs; that is, it will be proportional to the area of the eye-glaſs. Thus if *gfe* Plat. XIII. fig. 1. is the picture of the moon, the area of the eye-glaſs may be large enough to ſee the whole moon at once, or it may be ſo ſmall that we can only be able to ſee a part of it at a time. Or if there is at *f* the diſtinct picture of any planet that we are looking at through the teleſcope, and *ef*, *gf* is the diſtinct picture of the blue ſky round the planet; how much of this blue ſky we ſhall ſee depends upon the area of the hole or eye-glaſs, through which we look.

There is a great inconvenience in uſing a teleſcope where the eye-glaſs is ſmall, which is, that it will be difficult to find an object. If the area of the eye-glaſs is large the teleſcope will take in a large compaſs at one view; ſo that we may ſee a planet that we are looking for, though the teleſcope is not directed exactly towards it. For though it is neceſſary that it ſhould be ſo directed in order to ſee the planet in the very middle of the viſible ſpace; yet without ſuch exactneſs, where the ſpace taken in at one view is large, the planet may be ſeen ſomewhere in it, though it is not in the middle. But if the area of the eye-glaſs, and, in conſequence of that, the viſible area is very ſmall, unleſs the teleſcope is directed exactly towards the planet, we ſhall not be able to ſee it.

178. *The brightneſs of an object ſeen through a teleſcope depends upon the area of the object-glaſs: but whether the area of the object-glaſs is great or ſmall the ſpace that may be ſeen at one view will be the ſame, if the area of the eye-glaſs is given.*

The brightneſs of an object ſeen through a teleſcope depends upon the brightneſs of the diſtinct picture, and this, by propoſition 101, is proportional to the area of the lens, through which the rays paſs to form that picture. But this lens is the object-glaſs. Therefore an object ſeen through a teleſcope will appear the brighter for uſing an object-glaſs of a large area, and the fainter for uſing one of a ſmall area.

But, by propoſition 100, the ſize of the diſtinct picture is the ſame whether the area of the object-glaſs is great or ſmall, and conſequently, by propoſition 177, if we look at it through a given hole, or through an eye-glaſs of a given area, the quantity that we can ſee at once, or the ſpace that we ſee at one view, will not be altered by any alteration in the area of the object-glaſs.

179. *The diſtance of the eye from the eye-glaſs in looking through a teleſcope ſhould be equal to the eye-glaſs's principal focal diſtance.*

Since the diſtinct picture *gfe* is in the principal focus of the glaſs *dcb*, wherever the eye is placed on the other ſide of the glaſs, it will ſee that picture equally magnifyed, by propoſition 154. Therefore it is not for the ſake of magnifying the object more, that the eye is to be placed at *o*, ſo that *co* the diſtance of it from the eye-glaſs may be equal to this glaſs's principal focal diſtance. The advantage gained by this poſition of the eye is, that we may ſee more of the object at one view, or that the viſible area will be greater, when the eye is placed there, than when it is placed any where elſe. This will be beſt underſtood by conſidering the two glaſſes *nl* and *db* fixed, as they commonly are, in a tube whoſe ſides are parallel to *mc*, which *mc* is the common axis of the two lenſes continued, and is called the axis of the teleſcope. Now no rays can make any ſurface viſible, but ſuch as come from that ſurface, by propoſitions 110. 113. But no rays, that are parallel to the inner ſurfaces of the ſides of the tube, come from thoſe ſurfaces. Therefore no rays, that are parallel to the axis of the teleſcope, can make the ſides of the tube viſible. At *o*, which is the principal focus of the eye-glaſs *bd*, no rays meet but ſuch as before refraction were parallel to the axis of the lens *db*, or to *mc*, which as it is the axis of the teleſcope, is the axis of the lens *db* continued, by propoſition 60; conſequently no rays that are collected at *o* will make the ſides of the tube viſible. And upon this account if the eye is placed at *o*, ſince it receives no other rays but ſuch as are collected at *o*, the ſides of the tube will not be ſeen through the eye-glaſs. But if the eye was placed any where elſe, either nearer to the eye-glaſs or farther from it than *o* its principal focus; ſome rays would enter the eye, which were not parallel to one another before refraction: ſo that in any other poſition of the eye, the ſides of the tube will be ſeen through the eye-glaſs as well as the object that we are looking at. Now the area taken in at one view is proportional to the area of the eye-glaſs, by propoſition 177. Therefore a greater area of the object will be taken in at one view, when nothing elſe is ſeen through the eye-glaſs: but if we can ſee the ſides of the tube, ſome part of the area of the eye-glaſs will be taken up by thoſe ſides of the tube; and the object will be ſeen only through the remaining part. Therefore more of the object will be taken in at one view, if the eye is at *o*, where the ſides of the tube cannot be ſeen, than if it was placed in any other ſtation.

The caſe would be the ſame though the glaſſes were uſed without a tube. Then indeed the eye placed any where beſides at *o* could not ſee the ſides of a tube, but it would ſee the open ſpace *nd*, *lb*; and when any part of the eye-glaſs is taken up with ſeeing that open ſpace, the remaining area, through which the picture is to be ſeen, will be ſo much the leſs; and conſequently ſo much the leſs of the picture will be ſeen at one view.

180. *A teleſcope conſiſting of four convex lenſes is a double aſtronomical teleſcope.*

The two lenſes *nml*, *dcb*, Plat. XIII. fig. 2. if they are placed at the diſtance *mc* equal to the ſum of their principal focuſes, make one aſtronomical teleſcope; and the two lenſes *ptq*, *rus* placed at the diſtance *tu*, equal to the ſum of their principal focuſes is another, by propoſition 171. If theſe two teleſcopes are fixed at the diſtance *ct* from each other, ſo as to be both of them uſed together, they make one teleſcope conſiſting of four convex lenſes. In this compound teleſcope the lens, that is towards the object, is called the object-glaſs, and all the others are called eye-glaſſes. Of theſe eye-glaſſes, *dcb*, which is next the object-glaſs is called the firſt, *ptq* is the ſecond, and *rus*, which is next the eye, is the third.

181. *An object appears diſtinct and erect through a teleſcope conſiſting of four convex lenſes.*

Such a teleſcope is in effect two aſtronomical teleſcopes, by propoſition 180. Therefore as the object would appear diſtinct through each of them, by propoſition 172, one will not change the diſtinctneſs produced by the other; and conſequently it will appear diſtinct through both of them, when they are uſed together. But ſince, by propoſition 172, each of them uſed alone would invert the object, it follows that when they are both uſed together, the firſt will invert it and the ſecond by inverting it again will ſhew it upright. Therefore through ſuch a compound teleſcope the object appears erect.

It will be no great trouble to prove this propoſition by tracing the rays through all the four lenſes. The rays of each beam are parallel to one another when they fall upon the object-glaſs *nml*, by propoſition 30, becauſe the object is very remote: therefore a diſtinct inverted picture will be formed at *gfe* the principal focus of the object-glaſs, by propoſitions 60. 84. This picture, by the conſtruction is in the principal focus of *dcb* the firſt eye-glaſs. Therefore the rays of each beam which diverge from the ſeveral points in the diſtinct picture *gfe*, will be made parallel to one another by paſſing through the firſt eye-glaſs, by propoſition 61. Conſequently the rays of each beam, when they fall upon the ſecond eye-glaſs *ptq*, are parallel to one another, juſt as they were when they fell upon the object-glaſs *nml*. Therefore a diſtinct inverted picture will be formed again at *kih*, the principal focus of this ſecond eye-glaſs. And this picture, by the conſtruction, is in the principal focus of the third eye-glaſs *rus*. Therefore the rays of each beam, which diverge from the ſeveral points in this picture will be parallel to one another, when they have paſſed through the lens *rus* and come to the eye at *o*. Conſequently the object ſeen by theſe rays will be ſeen as diſtinctly through the teleſcope as the naked eye can ſee any object where the rays of each beam are parallel.

From what has been ſaid we may obſerve the reaſon why the object appears erect. The firſt picture *gfe* is inverted and ſo is the ſecond *kih*. But the firſt is inverted in reſpect of the object, and the ſecond is inverted in reſpect of the firſt. Therefore the ſecond picture, which is the laſt image, or object of refracted viſion, is in the ſame ſituation with the object itſelf.

182. *A teleſcope conſiſting of four lenſes commonly magnifyes an object in the proportion of the object-glaſs's principal focal diſtance to the firſt eye-glaſs's principal focal diſtance.*

The teleſcope *mu*, Plat. XIII. fig. 2. conſiſts of two aſtronomical teleſcopes, by propoſition 180. The firſt of them *mc* magnifyes the object in the proportion of *mf* to *fc*, or in the proportion of the principal focal diſtance of the object-glaſs to the principal focal diſtance of the firſt eye-glaſs, by propoſition 173, 174. But the ſecond teleſcope *tu* is commonly made up of two convex lenſes of equal convexities. Therefore this ſecond teleſcope does not alter the apparent magnitude of the object, by propoſition 174. From whence it follows, that, when both teleſcopes are uſed together, the object is magnifyed only by the firſt of them: ſo that the apparent diameter of the object ſeen through the teleſcope is to its apparent diameter ſeen by the naked eye, as *mf* the principal focal diſtance of the object-glaſs *nml* to *fc* the principal focal diſtance of the firſt eye-glaſs *dcb*.

183. *Galileo's teleſcope conſiſts of a convex object-glaſs and a concave eye-glaſs; and the diſtance between them is the difference of their principal focuſes.*

If *yf*, Plat. XIII. fig. 3. is the principal focal diſtance of the convex lens *zyx*, and *cf* is the principal focal diſtance of the concave lens *bca*; then *yf*—*cf*=*yc* is the difference of their principal focal diſtances. And if theſe two lenſes are placed at this diſtance *yc* from each other, they make Galileo's teleſcope.

184. *An object ſeen through Galileo's teleſcope appears erect and diſtinct.*

The rays that come from each point in a very remote object are parallel to one another, by propoſition 30: and if theſe rays paſs through the convex lens *zyx*, Plat. XIII: fig. 3. they would converge to as many points at the lens's principal focus, by propoſition 60, 83, and at *gfe* would paint the picture of the object, if the concave eye-glaſs was out of the way. Suppoſe therefore the eye was at *c* the ſtation of the eye-glaſs, then the object ſeen through the object-glaſs alone would appear erect and confuſed, by propoſition 149. And if the concave eye-glaſs was to be ſlipped between the eye and the object-glaſs, this would not change the apparent ſituation of the object, by propoſition 163, ſo that as the object would appear erect through the convex lens alone, it will likewiſe appear erect when ſeen through both the lenſes. But the concave eye-glaſs will make the object diſtinct, which would be confuſed without it. For ſince the ſeveral beams of rays, when they fall upon the concave glaſs are converging to ſeveral focuſes *g*, *f*, *e*, &c. and all theſe focuſes are at the diſtance *cf* from the lens, or, by the conſtruction of the teleſcope, at the diſtance of the eye-glaſs's principal focus, it follows, from propoſition 62, that the rays of each beam, when they are refracted, will become parallel, and the eye will ſee the object as diſtinctly through the teleſcope, as the naked eye ſees an object when it is far enough off for the rays of each beam of rays, that come from it, to be parallel to one another.

185. *Galileo's teleſcope magnifyes an object in the proportion of the object-glaſs's principal focal diſtance to the eye-glaſs's principal focal diſtance.*

The apparent diameter of an object ſeen through the teleſcope *yc*, Plat. XIII. fig. 3. is as much longer than its apparent diameter ſeen with the naked eye, as *yf* the principal focal diſtance of the object-glaſs is greater than *cf* the principal focal diſtance of the eye-glaſs.

If the concave lens *bca* was taken away and the diſtinct picture *gfe* was received upon a paper, the apparent diameter of the object either through the lens *zyx* by the eye cloſe to it at *y*, or by the naked eye at the ſame place is equal to the apparent diameter of this picture ſeen from *y* at the diſtance *yf*, by propoſitions 147, 153. And ſince the apparent diameter of the object ſeen through the lens *zyx* is inverſely as the eyes diſtance from the diſtinct picture, by propoſition 153, if the eye was to withdraw from *y* towards *c* the ſtation of the eye-glaſs, the object ſeen through the lens *zyx* would keep encreaſing as the diſtance *yf* decreaſed, ſo that when the eye was arrived at *c* the object would ſeem as much bigger than it did when the eye was at *y* as *cf* is leſs than *yf*, or as *yf* is greater than *cf*. But when the eye was at *y* its apparent diameter ſeen through the lens was equal to its apparent diameter ſeen with the naked eye. Therefore if the eye is at *c* the ſtation of the eye-glaſs the object will be magnifyed by the lens *zyx* in the proportion of *yf* to *cf*, or in the proportion of the principal focal diſtance of the object-glaſs to the principal focal diſtance of the eye-glaſs. Now ſuppoſe, whilſt the eye continues at *c*, the concave lens *bca* to be placed juſt before it ſo as to touch it; this lens, by propoſition 147, in ſuch a ſituation will make no alteration in the apparent magnitude of the object. Therefore the object ſeen through the teleſcope, or through both the lenſes, will continue to be magnifyed in the ſame proportion as before, if the eye is cloſe to the eye-glaſs: the only difference will be that the object ſo magnifyed is ſeen confuſedly without the eye-glaſs, by propoſition 149, but is ſeen diſtinctly with it, by propoſition 184. And having thus ſhewn that the propoſition is true, if the eye is cloſe to the eye-glaſs, we may ſhew it to be equally true in any other ſtation of the eye. For ſince the rays of each beam, when they come out of the eye-glaſs are parallel to one another, by propoſition 184, the apparent magnitude of the object will not be altered by the departure of the eye from *c*, for a reaſon aſſigned in propoſition 154; ſince, as the rays of each beam are parallel, the eye by departing from the eye-glaſs neither gets farther from the laſt image nor nearer to the diſtinct picture.

A teleſcope of this ſort cannot be ſo contrived as not to magnify the objects, that are ſeen through it. For if the teleſcope does not magnify, the principal focal diſtances of the convex object-glaſs and the concave eye-glaſs muſt be equal. Now the diſtance of the two glaſſes from each other muſt be the difference between their principal focal diſtances, by propoſition 189. Conſequently where the principal focal diſtances are equal, that is, where the difference between them is nothing, the diſtance between the two glaſſes muſt be nothing, or the two glaſſes muſt touch each other; and then, inſtead of making a teleſcope, they will rather make a meniſcus having its convexity on one ſide equal to its concavity on the other.

186. *The viſible area in Galileo's teleſcope depends upon the breadth of the pupill.*

The rays of any beam, when they fall upon the eye-glaſs, are not far from their focus: therefore they do not ſpread over the whole ſurface of the lens; but extend only over a ſmall part of it. Thus the beam, which is converging to the extreme focus *g*, Plat. XIII. fig. 3, covers but a ſmall part of the lens at *b* the edge of it; and the beam, which is converging to the other extreme focus *e*, covers but a ſmall part at *a* the other edge of the lens. Now if the pupill, when it is cloſe to the lens, is wide enough to reach from *b* to *a*, the rays from the extreme parts of the object will enter it, and conſequently thoſe parts may be ſeen through the teleſcope. But if the pupill was narrower, if for inſtance it was no wider than *do*, the rays of the extreme beams would not enter it, ſo that the extreme parts of the object could not be ſeen. Therefore the viſible area or ſpace that may be taken in at one view will be greater as the breadth of the pupill is greater, or leſs as that breadth is leſs.

187. *The eye is in the moſt convenient poſition for looking through Galileo's teleſcope, when it is cloſe to the eye-glaſs.*

The apparent magnitude of the object ſeen through the teleſcope would be the ſame, wherever the eye was placed, by what was ſhewn under propoſition 185. But when the eye is cloſe to the concave eye-glaſs, more of the object is ſeen at one view, than could be ſeen in any other ſtation of the eye. The rays of each beam are indeed parallel to one another: the rays of the beam *bk*, for inſtance, are parallel amongſt themſelves; and ſo likewiſe are thoſe of the beam *al*. But then the beams or rays of different beams are not parallel to one another: the beam *bk* is not parallel to the beam *al*. For ſince the beam *bk* before refraction was tending to *g* and the beam *al* to *e*, ſo that one of them was nearly parallel to the other; it follows that after refraction they will diverge from one another, by propoſition 54. Therefore the farther theſe beams are from the eye-glaſs the farther they will be aſunder, by propoſition

10: conſequently though, when the eye is cloſe to the glaſs, theſe extreme beams may be near enough to each other to enter the pupill, yet, when the eye is placed at any diſtance from the glaſs, they will be ſo far aſunder that they cannot enter it: therefore the extreme parts of the object, which might be ſeen, if the eye was cloſe to the eye-glaſs, will not be ſeen, if the eye is removed to any diſtance from it.

188. *A ſingle microſcope is only a very convex lens.*

A lens of a great convexity is a ſegment of a ſmall ſphere; and therefore will have a ſhort principal focus, by propoſition 53. Suppoſe therefore ſuch a lens to have its principal focus only $\frac{1}{10}$ inch: if any ſmall object is placed in the principal focus or at the diſtance of $\frac{1}{10}$ inch of the lens, the eye on the other ſide will ſee it diſtinctly through the lens, and as big as the naked eye could ſee it, if the eye was at the ſtation of the lens or within $\frac{1}{10}$ inch of the object, by propoſitions 147, 151, 154. But the naked eye does not ſee an object diſtinctly unleſs it is 5 inches from it. Therefore the apparent diameter of the object ſeen through the lens is as great as it would be, if the eye was 50 times nearer than it ever is, when we have diſtinct viſion, and conſequently is 50 times as great as we are ever uſed to ſee it.

189. *A double microſcope conſiſts of two convex lenſes, of which the object-glaſs is more convex than the eye-glaſs: and the diſtance between them is equal to the diſtance of the diſtinct picture from the object-glaſs added to the principal focal diſtance of the eye-glaſs.*

The microſcope, Plat. XIII. fig. 4, conſiſts of a very convex object-glaſs *nml* and a leſs convex eye-glaſs *dcb*. The ſmall object *zxy* is nearer to the object-glaſs than its principal focus, ſo that ſomewhere behind the glaſs there will be a diſtinct picture of it *gfe*, by propoſition 84. The diſtance of this picture from the object-glaſs is *mf*, the diſtance of it from the eye-glaſs, or *cf*, muſt be equal to the eye-glaſs's principal focus. And the diſtance of the two glaſſes from each other muſt be *mc*, or *mf*+*fc*, which is the diſtance of the diſtinct picture from the object-glaſs added to the principal focal diſtance of the eye-glaſs.

From this conſtruction it is evident that the diſtinct picture is always in the eye-glaſs's principal focus. And by this deſcription of a compound microſcope we may ſee what is the eſſential difference between ſuch a microſcope and an aſtronomical teleſcope. The object-glaſs of a compound microſcope is more convex than the eye-glaſs: but, if an aſtronomical teleſcope magnifyes, the object-glaſs of it is leſs convex than the eye-glaſs. How each of theſe inſtruments is fitted for their reſpective purpoſes by this conſtruction will appear as we go along.

Some compound microſcopes are made with three glaſſes; ſo that the rays, after they have been refracted by paſſing through the lens *nml*, are received upon another convex lens, and are refracted again before they form the diſtinct picture. In this caſe the rays are made to converge by the refraction of both theſe lenſes, and they are both of them called object-glaſſes. The diſtinct picture ſo formed is in the principal focus of the third glaſs, which being next the eye is in this as in other microſcopes called the eye-glaſs.

190. *An object ſeen through a double microſcope appears diſtinct and inverted.*

The demonſtration of this propoſition is the ſame with that of propoſition 172: and the letters of reference are made the ſame, in Plat. XIII. fig. 1, and 4, that the reader, if he pleaſes, may the more readily apply it.

191. *The apparent diameter of an object ſeen through a double microſcope is to the apparent diameter of the ſame object ſeen by the naked eye in the ſtation of the object-glaſs, as the diſtance of the diſtinct picture from the object-glaſs to its diſtance from the eye-glaſs.*

This propoſition is demonſtrated on Plat. XIII. fig. 4, in the ſame manner as propoſition 173, is on fig. 1.

Now an aſtronomical teleſcope will not magnify an object unleſs the principal focal diſtance of the object-glaſs is greater than the principal focal diſtance of the eye-glaſs, or unleſs the object-glaſs is leſs convex than the eye-glaſs. For when the object is very remote, as objects are which we look at through a teleſcope, the diſtinct picture is in the principal focus of the object-glaſs; and conſequently it cannot be farther from the object-glaſs than it is from the eye-glaſs unleſs the principal focal diſtance of the object-glaſs is greater than the principal focal diſtance of the eye-glaſs. But when the object is very near the object-glaſs, as objects are which we look at through a microſcope, the diſtinct picture will be more remote from the glaſs than its principal focus, by propoſition 61: and conſequently the diſtinct picture *gfe*, Plat. XIII. fig. 4, may be more remote from the object-glaſs *nl* than it is from the eye-glaſs *df*, though the object-glaſs is more convex or has a ſhorter principal focus than the eye-glaſs.

Here we may obſerve, firſt; that an aſtronomical teleſcope will not anſwer the purpoſes of a microſcope or cannot be uſed for magnifying near objects. For if the object *zxy*, Plat. XIII. fig. 1. was near the lens *nml*, the picture *g f e* would be ſo far from it as not to be within the compaſs of the inſtrument or between the two lenſes. Secondly, a double microſcope cannot be uſed for magnifying diſtant objects: becauſe the object-glaſs has a ſhorter principal focus than the eye-glaſs, and conſequently is like an aſtronomical teleſcope inverted, and inſtead of magnifying a remote object will diminiſh it, by propoſition 176. Thirdly, an aſtronomical teleſcope inverted will not anſwer the purpoſe of a microſcope in magnifying very near objects: for though if *bd*, Plat. XIII. fig. 1. is towards the object and *nl* is towards the eye, the object-glaſs will then be more convex than the eye-glaſs, as in a double microſcope; yet *db* the eye-glaſs of an aſtronomical teleſcope is not convex enough to ſerve for the object-glaſs of a microſcope: ſo that when the object is very near *d b*, the picture *g f e* will be very far from this glaſs *db*, and conſequently, unleſs the teleſcope is made much longer than it uſually is, this picture will either not be between the lenſes, or however will not be, as it ought, in the principal focus of the lens *n l*, which is next to the eye.

192. *An object ſeen through a microſcope appears magnifyed in the proportion of the limit of diſtinct viſion to the diſtance of the object-glaſs from it.*

The magnifying power of a microſcope, as determined in propoſition 191, is in reference to the eye at the ſtation of the object-glaſs *m*, Plat. XIII. fig. 4. Thus if *mf* is to *cf* as 10 to 1, the apparent diameter of the object ſeen through the microſcope will be 10 times greater than the naked eye would ſee it at *m*. But if *mx* the diſtance of the object-glaſs from the object is leſs than the diſtance at which the naked eye can ſee an object diſtinctly, or leſs than the limit of diſtinct viſion, the apparent diameter of the object ſeen by the naked eye at *m* will be as much greater than we are ever uſed to ſee it, as *mx* is leſs than that limit, by propoſition 126. Thus for inſtance, the limit of diſtinct viſion being ſuppoſed 5 inches, we are never uſed to look at an object nearer than 5 inches, becauſe if we were to hold it nearer to our eye, we ſhould ſee it confuſedly. Therefore if *mx* is only 1 inch, the object would appear to the naked eye at *m* 5 times longer than we uſually ſee it. Therefore the object ſeen through the microſcope appears larger than uſual upon two accounts. Firſt it is 10 times larger than the naked eye at *m* would ſee it, becauſe *fm* is 10 times *fc*. And ſecondly, the naked eye at *m* would ſee it 5 times larger than it has ever been uſed to ſee it, becauſe *mx* is 5 times leſs than the limit of diſtinct viſion, or 5 times leſs than the diſtance at which the naked eye has been uſed to look at it. But if the apparent diameter is 10 times greater than uſual upon one account, and 5 times greater upon the other account, it follows, that upon both accounts together it will be 50 times greater. For the microſcope magnifyes in the proportion of 10 to 1, and of 5 to 1, that is in the proportion of 50 to 1. From whence we may deduce the following general propoſition for determining how much larger an object will appear through a microſcope than the naked eye has been uſed to ſee it.

193. *A microſcope magnifyes the apparent diameter of an object in a proportion compounded of the diſtance of the diſtinct picture from the object-glaſs to its diſtance from the eye-glaſs, and of the limit of diſtinct viſion to the diſtance of the object-glaſs from the object.*

It magnifyes in the firſt proportion, by propoſition 191, and in the ſecond, by propoſition 192.

194. *When the ſame eye-glaſs is made uſe of, the magnifying power of a microſcope is encreaſed, by encreaſing the convexity of the object-glaſs.*

If the microſcope *cm*, Plat. XIII. fig. 4. magnifyes an object in the proportion of 50 to 1 with the object-glaſs *m*, it would magnify the ſame object in a greater proportion, if this object-glaſs was changed for a more convex one.

The more convex an object-glaſs is, the nearer will be the diſtinct picture, if the object was to continue at the ſame diſtance, by propoſition 95. But by bringing the glaſs nearer to the object the picture will depart from it, by propoſition 98. And thus, whatever is the convexity of the glaſs, by altering its diſtance from the object the picture is always kept, where it ought to be, at *gfe* the principal focus of the eye-glaſs. Now when the place of the diſtinct picture, and conſequently its diſtance from each of the glaſſes, is given; the microſcope magnifyes in the proportion of the limit of diſtinct viſion to the diſtance of the object-glaſs from the object, by propoſition 192. Therefore the nearer this glaſs is to the object the more the microſcope will magnify: and ſince a more convex object-glaſs muſt be brought nearer to it than a leſs convex one, the more convex one will magnify the moſt.

If *mf* is to *cf* as 10 to 1, and *mx* is 1 inch or $\frac{1}{5}$ of the limit of diſtinct viſion, we found, by propoſition 192, that the objects apparent diameter ſeen through the microſcope will be 50 times greater than uſual. But ſuppoſe a more convex object-glaſs was made uſe of; ſo that to keep the picture at *gfe*, the lens muſt be brought within $\frac{1}{2}$ inch of the object. Then *mf* is ſtill to *cf* as 10 to 1. But *mx*, being now only $\frac{1}{2}$ inch or $\frac{1}{10}$ 5 inches or $\frac{1}{10}$ the limit of diſtinct viſion, the microſcope will magnify the object again in the proportion of 10 to 1, by propoſition 192. Since therefore the object is magnifyed in the proportion of 10 to 1 and of 10 to 1, upon both accounts together it will be magnifyed by this more convex lens in the proportion of 100 to 1, or twice as much as it was by the leſs convex one. And univerſally, ſince in the ſame microſcope the diſtinct picture is always kept in the ſame place, whatever object-glaſs we make uſe of, the magnifying power of it will always be inverſely as the diſtance of the object-glaſs from the object.

We need ſay nothing here about the viſible area through a double microſcope, or about the proper poſition for the eye; ſince the ſimilitude between this inſtrument and an aſtronomical teleſcope planely ſhews that theſe particulars are to be determined by the ſame rules in both.

195. *The aperture of the object-glaſs in a microſcope ſhould be very ſmall.*

For as the object-glaſs is very near to the object, the outermoſt rays of every beam will diverge very much, and would hinder the diſtinctneſs of the picture, by propoſition 99, unleſs they were excluded by making the aperture ſmall. But the ſmallneſs of the aperture will make the object appear faint, by propoſition 178. Therefore to make amends for this, many ways are contrived to illuminate the object as much as poſſible. For this purpoſe it is placed in the ſun-ſhine, or the light is collected by a convex lens, or a concave mirrour, and is thrown upon it.

196. *Short-ſighted perſons ſee beſt through a teleſcope or microſcope if it is ſhortened a little.*

When an aſtronomical teleſcope or double microſcope is of the uſual length the diſtinct picture is in the eye-glaſs's principal focus, and the rays of each beam come out parallel, by propoſitions 172. 190. But if either inſtrument is ſhortened a little, the eye-glaſs is brought nearer to the picture, and then, as the picture is nearer to it than the principal focus, the rays of each beam, that diverge from thence, will continue to diverge after they have paſſed through it, by propoſition 67. So likewiſe in Galileo's teleſcope the rays of each beam, when they fall upon the concave eye-glaſs converge to its principal focus, and therefore when they have paſſed through it they become parallel, by propoſition 184. But by ſhortening the teleſcope the eye-glaſs is moved farther from the place of the diſtinct picture, for which reaſon the rays of each beam converge to points more remote than its principal focus; and therefore, after refraction they will diverge, by propoſition 65. Since therefore ſhortening any of theſe inſtruments a little will make the refracted rays diverge, and ſince, by propoſition 168, ſhort-ſighted perſons ſee better by diverging rays than by parallel ones, it follows that ſuch perſons will ſee more diſtinctly through any of theſe inſtruments, when they are ſhortened, than when they are of the uſual length.

197. *Old perſons ſee beſt through a teleſcope or microſcope, if it is lengthened a little.*

When a teleſcope or double microſcope is of the uſual length the rays of each beam are parallel when they come to the eye, by propoſitions 172, 190, 184. But if an aſtronomical teleſcope or double microſcope is lengthened by drawing the eye-glaſs farther from the object-glaſs, the diſtinct picture will be more remote from the eye-glaſs than its principal focus, and conſequently the rays of the ſeveral beams, which diverge from the picture will converge, after they have paſſed through the glaſs, by propoſition 63. And if Galileo's teleſcope is lengthened, the rays, which fall converging on the eye-glaſs, will converge to points nearer than the glaſs's principal focus, and will therefore continue to converge after refraction, by propoſition 69. But old perſons ſee more diſtinctly, when the rays converge, than when they are parallel, by propoſition 169. Therefore ſuch perſons will ſee the better through any of theſe inſtruments, if they are lengthened a little.

CHAP. XII.

Of the cauſe and laws of the reflection of light.

198. *The reflection of light from tranſparent bodies, is either partial or total. The partial reflection happens either at the firſt or ſecond ſurface; the total one at the ſecond ſurface only.*

WHEN a beam of light falls upon a thick piece of poliſhed glaſs; all the rays will not paſs through it; but ſome of them will be reflected from the firſt ſurface of the glaſs, where the beam enters. For if a candle is placed before the glaſs, when the beams that come from the candle fall upon it, though the greateſt number of rays paſs into it; yet a faint image of the candle may be ſeen by reflection from the firſt ſurface, which ſhews that ſome few of the rays are reflected there, whilſt the reſt enter the glaſs. Secondly, when as much of the beam as entered the glaſs is come to the ſecond ſurface, all the rays, that come thither, will not paſs out of it into the air, the greateſt part may paſs out, but ſome will be reflected from thence. For a ſecond faint image of the candle may be ſeen there. Theſe are the two partial reflections; and theſe happen, whatever is the obliquity of the rays, when they fall upon the firſt or ſecond ſurface. The total reflection at the ſecond ſurface does not happen, unleſs the rays fall very obliquely upon it. If the angle of incidence at the ſecond ſurface is under 40 degrees, greateſt part of the beam will paſs out of the glaſs into the air, and only ſome few rays will be reflected, as we ſaid before. But if the angle of incidence or obliquity is greater than 40 or 41 degrees, none of the rays will come out of the glaſs, but the whole beam, that is, as much of the beam as came to the ſecond ſurface, will be reflected.

By ſhewing the cauſe of theſe reflections from tranſparent ſubſtances we ſhall ſee to what cauſe the reflection of light from other ſubſtances, ſuch as poliſhed metals, is owing.

199. *The rays of light are not reflected by ſtriking upon the ſolid parts of bodies.*

When rays of light fall upon a thick piece of poliſhed glaſs, ſome of them are reflected from the firſt and ſome from the ſecond ſurface, by propoſition 198, and as many or more rays are reflected at the ſecond ſurface, where they are to paſs out of the glaſs into the air, as at the firſt ſurface, where they are to paſs out of air into glaſs. Now this could ſcarce be the caſe, if theſe reflections were occaſioned by the ſtriking of the rays upon the ſolid parts of bodies: for upon this ſuppoſition, ſince glaſs is denſer, or in a given ſpace contains more ſolid parts, than air, a greater quantity of rays ought to be reflected, when they are paſſing out of air into glaſs, than when they are paſſing out of glaſs into air.

When a beam of light is to paſs out of glaſs into air, if its obliquity is leſs than 40 degrees, greateſt part of it will paſs out, but if the obliquity is greater than 41 degrees, it will all be reflected, by propoſition 198. But it is ſcarce probable that the beam at one obliquity ſhould meet with nothing but pores or interſtices in the air, and ſo paſs freely into it, and at a little greater obliquity ſhould meet with nothing but ſolid parts and ſo be totally reflected. Or if the reader imagines, that what is ſo unlikely ſhould however be poſſible; we may obſerve farther that if there was water behind the glaſs inſtead of air, the rays would not be reflected from the water at the ſecond ſurface, though their obliquity was greater than 41 degrees. But ſince water is denſer than air, and the rays which would have been reflected back into the glaſs, if there had been air behind it, will paſs freely out of it, if there is water behind it, we cannot imagine that the reflection of them from the air could be occaſioned by their ſtriking upon its ſolid parts: for if air reflects them by this means, then certainly water, which is more denſe than air, could not fail of reflecting them.

Nor can any one ſuppoſe that the total reflection from the ſecond ſurface, though it is not occaſioned by the rays ſtriking againſt the ſolid parts of the air, may yet be owing to their ſtriking againſt the ſolid parts in the ſecond ſurface of the glaſs itſelf. This ſuppoſition is attended with the ſame difficulty as the other. For it is ſcarce poſſible that the rays at one degree of obliquity ſhould meet with nothing but pores in that ſurface, and at a greater obliquity ſhould meet with nothing but ſolid parts. Nay indeed this ſuppoſition is attended with a greater difficulty. For if the rays, falling at an obliquity greater than 41 degrees, meet with nothing but ſolid parts in the ſecond ſurface of the glaſs, and are reflected by ſtriking againſt thoſe parts; then this reflection would happen, whether there was air or water behind the glaſs; whereas at the obliquity of 43 degrees, they will be reflected, if there is air behind it, but would be tranſmitted if there was water behind it.

200, *Bodies reflect and refract light by one and the ſame power differently exerciſed in different circumſtances.*

When rays of light are to paſs out of one medium into another of a different denſity, they are refracted; and this refraction will be ſo much the greater as the mediums differ more from one another in denſity, by propoſition 37. Rays are more refracted, when they paſs out of air into a diamond, than when they paſs out of air into glaſs, and more ſtill than when they paſs out of water into glaſs. And in theſe or any other inſtances, where the refraction is the ſtrongeſt, the reflection is likewiſe the ſtrongeſt. There is a ſtronger reflexion, when light is paſſing out of air into a diamond, than when it is paſſing out of air into chryſtal, or into common glaſs: and if a diamond, or a piece of chryſtal or of common glaſs is immerged in water, as they differ leſs in denſity from water than from air, the refraction is the weaker, and the reflexion from their ſurface becomes the fainter.

We are therefore to explane the reflection of light upon the ſame principles, that we made uſe of in explaning its refraction. Bodies act upon the rays of light two ways, by propoſition 5 of mechanics, they attract all thoſe rays that are very near them, and repell thoſe that are a little farther from them. Any ray, if it is at a certain diſtance from the ſurface of a piece of glaſs, is attracted by the glaſs, but is repelled, if it is at a diſtance ſomewhat greater. The ſpace contiguous to the ſurface of the glaſs, where the rays are attracted, we may call the attracting ſurface; and the ſpace beyond this or farther from the ſurface of the glaſs, where they are repelled, we may call the repelling ſurface. Farther we muſt obſerve of the rays themſelves that they have fits of eaſy reflection and eaſy tranſmiſſion. That they have ſuch fits ſhall be proved hereafter. What theſe fits are owing to is not eaſy to determine; perhaps the rays are put into them at their firſt emiſſion from luminous bodies, and continue in them during all their progreſs. And it is not unlikely that thoſe rays, which are in a fit of eaſy tranſmiſſion, move the faſteſt.

Now if GGMM, Plat. XIII. fig. 5. is the repelling ſurface of a piece of glaſs and any ray R*a* falls upon it; this ray when it enters the ſurface will be retarded by the repulſion, and conſequently will be refracted from a perpendicular, ſo that inſtead of going ſtrait forwards in the line R*n*, it will be bent into *bo*, by propoſition 38: for though the repelling ſurface may not be a rarer medium than the air out of which the ray paſſes; yet the effect is the ſame; the ray will be retarded as much by paſſing into the repelling ſurface as it would be by paſſing out of a denſer medium into a rarer. Now this repulſion encreaſes as the ray approaches the glaſs, till it gets into the ſurface of attraction. And conſequently, as the ray moves in the repelling ſurface, it is conſtantly retarded, by which means it will be conſtantly refracted from a perpendicular. The ray therefore will not go ſtrait forwards in the direction *bo*, but will be bent, as it paſſes along, into the directions *ct*, *dv*, and at laſt into *fw*, ſo as to be parallel to MM the limit of the repelling ſurface. But the ray cannot proceed ſtrait in this direction *fw*, for being repelled by the glaſs it will be inflected from it, into the line *gp*; and whilſt it is thus moving from the glaſs, and is ſtill in the ſurface where it is repelled from the glaſs, the repulſion will accelerate the ray, and conſequently, by propoſition 37, it will be conſtantly refracted towards a perpendicular, and will be bent, as it goes along, into the ſeveral directions *hp*, *iq*, *kl*, and will emerge from the ſurface at *l* in the line *l*S. Thus a conſtant retardation refracted the ray from a perpendicular, till it became parallel to the limit of repulſion at *f*, it was there inflected from the ſurface of the glaſs, and then a conſtant acceleration refracted it towards a perpendicular, till it was reflected or came out of the repelling ſurface on the ſame ſide where it entered it at firſt. And in the courſe of its motion, it deſcribed the curve *abcdefghikl*. Thus the reflexion is occaſioned at the firſt ſurface, or when the rays are about to enter the glaſs. But then this reflexion will be only a partial one: thoſe rays only will be reflected, which are moving the ſloweſt, the moment of the reſt which have a greater velocity, will be ſufficient to carry them through the repelling ſurface; and when they have paſſed that, they will then be attracted by the glaſs, and will be tranſmitted to its ſecond ſurface.

The partial reflection at the ſecond ſurface is produced in the ſame manner; only as it was produced, by the repelling ſurface, when the rays were entering the glaſs, it will be produced by the attracting ſurface when they are coming out of it. Let GG therefore now be conſidered as the ſecond ſurface of the glaſs, MM the limit between the attracting and repelling ſurface, and conſequently the ſpace between GG and MM is the attracting ſurface. As the ray R*a* comes out of the glaſs, it will continue to be attracted towards it, ſo long as it continues within the ſpace GGMM: this attraction will retard the ray and bend it from a perpendicular, till at laſt the direction of the ray becomes parallel at *f* to MM the limit of attraction. At this place the attraction will inflect the ray, and will make it begin to move towards the glaſs; ſo that afterwards, when the ray is moving towards the glaſs, the ſame attraction will accelerate it and bend it towards a perpendicular, till it enters the glaſs again at *l*, and then paſſing back again through the ſubſtance of the glaſs it will come out of it at its firſt ſurface, or will be reflected. This likewiſe will be a partial reflection: thoſe rays, which are in a fit of eaſy tranſmiſſion, or which are moving with the greateſt velocity, will make their way through the attracting ſurface, and thoſe only will be reflected, which were moving with the leaſt velocity.

The total reflection at the ſecond ſurface, which happens when the rays fall very obliquely is planely nothing but a refraction. Let *fo*, Plat. VIII. fig. 5. be a ray of light paſſing out of the glaſs Z into the air X through the plane ſurface *ab*, this ray will be ſo refracted from a perpendicular, by propoſition 38, as to deſcribe the line *or*, when it comes into the air: and *rc* the ſine of the refracted angle is to *fd* the ſine of incidence as 3 to 2. Now if the ray *fo* falls more obliquely, or if the ſine of incidence *fd* encreaſes, the refracted ſine *rc* will encreaſe with it, by propoſition 40. Suppoſe therefore the obliquity to be ſo great, or by encreaſing the angle *fod* make the ſine of incidence *fd* ſo great, that *oa* the radius may be to *fd* as 3 to 2, then the radius *oa* will be the refracted ſine, that is, *coa* will be the refracted angle, or the ray will be refracted along the ſurface of the glaſs. Here therefore the reflection will begin: for if the obliquity of the incident ray is ever ſo little encreaſed, the refracted angle will be greater than *coa* or greater than a right one; and then the refracted ray will not deſcribe *or* or any other line on that ſide of *oa*, for any ſuch line would make a leſs angle than a right one with *co*; nor will it deſcribe *oa*, for then the refracted angle would be a right one; but it will deſcribe ſome line on the other ſide *oa* within the glaſs; that is, it will return into the ſubſtance of the glaſs and ſo paſſing back again through it, will come out at the ſame ſurface, where it firſt entered, or will be reflected. And univerſally, when a ray is to paſs out of a denſer medium into a rarer, if the obliquity of the ray is ſo great that the ſine of incidence is to the refracted ſine, as the ſine of incidence to the radius, the refracted ſine will be the radius, and the total reflection will begin at that obliquity. Therefore the more the ſine of incidence differs from the refracted ſine, ſo much leſs the ſine of incidence may be, and yet the refracted ſine will be equal to the radius. Now the ſine of incidence differs moſt from the refracted ſine, when the mediums differ moſt from each other in denſity. Conſequently the total reflection will begin at a leſs obliquity, when rays are paſſing out of glaſs into air, than when they are paſſing out of water into air: becauſe glaſs and air differ more in denſity, than water and air.

Now here we muſt obſerve, that, when a beam of light is totally reflected at the ſecond ſurface of the glaſs, the retardation, which occaſions this total reflection is not made all at once. For when the rays come ſo near to the ſurface as to make the attraction towards the glaſs greater than that towards the air, the retardation, and conſequently the refraction begins. Thus if MM, Plat. XIII. fig. 5. is the ſecond ſurface of the glaſs, ſuppoſing the diſtance between MM and any imaginary plane GG in in the glaſs to be equal to the ſemidiameter of the ſphere of attraction of any ſingle particle of the glaſs; then till the ray R*a* comes to *a* it is equally attracted all ways; but as ſoon as it is at *b*, or nearer to MM than GG it will be more attracted by the particles, that are behind it than by thoſe that are before it; for there will be fewer particles attracting it between *b* and MM than there are attracting it the contrary way. Conſequently, ſince the rays are gradually retarded till they are reflected, they will deſcribe ſuch a curve as *abcdef*, and after reflection being accelerated in the ſame manner as they return in the glaſs's ſubſtance they will deſcribe a ſimilar curve *fghikl*.

201. *In all caſes of reflection, the complement of reflection is equal to the complement of incidence, and the angle of reflection is equal to the angle of incidence.*

In all caſes of reflection the rays deſcribe ſuch a curve as *abcdefghikl*, as has been proved under propoſition 200: and ſince they deſcribe one half of this curve *abcdef* by being retarded, and the other half *fghikl* by being ſimilarly accelerated, one half will be ſimilar to the other: and conſequently, ſince they are ſimilarly placed, one half will make the ſame angle with any plane ſurface GG that the other half makes with it. Therefore the angle R*a*G, which is the complement of incidence, will be equal to the angle S*l*G, which is the complement of reflection. But ſince the complements are equal, the angles themſelves are likewiſe equal, the angle of incidence equal to the angle of reflection.

In the reflection of light whether it is partial or total, the bending of the rays is made within a very ſmall compaſs: ſo that the curve *abcdefghikl* will be ſo ſmall that we may neglect it, and may look upon the point of incidence *a* as coinciding with the point of reflection *l*: and upon this account we may conſider the reflection of light in the ſame manner, as in mechanics, we have conſidered the reflection of elaſtic bodies, where the body is always reflected from the point of incidence.

202. *All reflection is reciprocal*

If any reflected ray is turned back again to the reflecting ſurface, it will be reflected into the line of incidence. If the ray AC, Plat.VII. fig. 3. after it has been reflected in the line CB, is turned back again in the direction BC, it will be reflected into the line CA, which it had before deſcribed at its incidence. For the angle of reflection is always equal to the angle of incidence, by propoſition 201. Therefore if, when the angle of incidence is ACO, the angle OCB will be the angle of reflection, becauſe it is equal to it; for the ſame reaſon, when BCO is the angle of incidence, OCA will be the angle of reflection.

CHAP. XIII.

Of the reflection of light from plane mirrours.

203. *The cathetus of incidence is the axis of any beam, or is a perpendicular drawn from any radiant to the plane of a mirrour, on which rays fall that come from the radiant.*

IF A Plat.VII. fig. 7. is a radiant and rays diverging from it fall on the mirrour C*e*, the line AC, or A*a* which is AC continued behind the mirrour, is the cathetus of incidence. Or if *be* is the whole length of the mirrour ſo that no line can be drawn from the radiant A perpendicular to the mirrour itſelf, yet ſtill the line AC is the cathetus of incidence, though the mirrour reaches no farther than *b*: for AC is perpendicular to the plane of the mirrour, ſince *b*C, though no part of the mirrour, is in the ſame plane with it.

This explanation of the term in plane mirrours will ſufficiently ſhew the reader what is meant by it, when we come to apply it to ſuch mirrours as are not plane.

204. *Rays that are parallel, when they fall upon a plane mirrour, will be parallel, when they are reflected from it.*

If ED, Plat. XIII. fig. 6, is a plane reflecting ſurface, or is a line drawn along the ſurface of a common looking-glaſs, and any rays, FA, LC, GB, which are parallel to one another, are reflected from thence, the reflected rays, AH, CM, BI, will likewiſe be parallel. FA and LC are parallel at their incidence, by the ſuppoſition, therefore they are equally inclined to the plane ſurface, or to the right line ED drawn upon it; that is, the angle FAE, which one ray makes with it, is equal to the angle LCE which the other makes with it. For, ſince FA and LC are parallel, the right line ED, which croſſes them, makes the external angle FAE equal to LCE the internal oppoſite angle on the ſame ſide. Euc. b. I. prop. 29. But FAE and LCE are the complements of incidence of the two rays FA and LC. And ſince the complements of incidence are equal, the complements of reflection will likewiſe be equal, by propoſition 201, or the angles HAD, MCD, which the reflected rays AH and CM make with the plane ED, will likewiſe be equal. But MCD is the external angle, and HAD is the internal oppoſite one on the ſame ſide. Therefore, ſince theſe are equal, AH and CM, or the reflected rays are parallel to one another. Euc. b. I. prop. 28. In the ſame manner we might prove, that if GB is parallel to FA, when they fall upon the plane ED, BI will likewiſe be parallel to AH, when they are reflected. And that all rays, which are parallel before reflection, will likewiſe be parallel after it.

205. *If diverging rays are reflected from a plane mirrour, the diſtance of the imaginary radiant behind the mirrour, is equal to the diſtance of the real one before it; and the divergency of the rays is not altered by the reflection.*

Let A, Plat. VII. fig. 7, be the radiant, from which the rays A*b*, A*d*, A*e* diverge, when they fall upon the ſurface of a looking-glaſs or the ſurface of ſmooth water, they will be reflected from thence in the directions *bf*, *dg*, *eh*, ſo as to make the complements of reflection equal to the complements of incidence, by propoſition 201. In this caſe *ab*, the diſtance of the imaginary radiant *a*, from which the reflected rays appear to diverge, will be equal to A*b*, the diſtance of the real radiant. And the angle *b*A*d* or the divergency of any two rays at their incidence will be equal to the angle *bad* or *fag*, which is the divergency of the ſame rays after reflection.

The angle A*b*C is equal to the angle *fbe*, by propoſition 201; for one is the complement of incidence of the ray A*b* and the other is the complement of reflection of the ſame ray. But C*ba* is likewiſe equal to *fbe*, becauſe they are oppoſite angles made at the vertex, where the two right lines C*e* and *af* croſs. Euc. b. I. prop. 15. Therefore ſince A*b*C and *ab*C are each of them equal to *fbe*, they are likewiſe equal to each other. In the ſame manner we may prove that A*d*C is equal to *ad*C. But ſince A*b*C is equal to *ab*C, it follows that A*be* is likewiſe equal to *abe*; for A*be* is the complement of A*b*C and *abe* is the complement of *ab*C to two right angles; and where the angles themſelves are equal, their complements will be ſo too. Thus we have ſhewn that in the two triangles A*bd* and *abd* the angles at *b* and at *d* in each of them are equal; and the ſide *bd* is common to both. Conſequently the triangle A*bd* is in all reſpects equal to the triangle *abd*. Euc. b. I. prop. 26. The ſide A*b* in one is equal to the ſide *ab* in the other, and the ſide A*d* is likewiſe equal to the ſide *ad*; that is, the diſtance of the real radiant A either from the point *b* or from the point *d* is equal to the diſtance of the imaginary radiant *a* from the ſame point. And the angle *b*A*d* contained between the incident rays A*b* and *ad* is likewiſe equal to the angle *bad* contained between the reflected rays *bf* and *dg*; that is, the divergency of the incident rays is the ſame with the divergency of the reflected ones.

206. *If converging rays are reflected from a plane mirrour, the diſtance of the imaginary focus behind the mirrour is equal to the diſtance of the real focus before it; and the convergency of the rays is not altered by the reflection.*

If *fb*, *gd*, *he*, Plat. VII. fig. 7, rays converging to the focus *a*, fall upon the mirrour *be*, and are reflected from it, before they come to *a*, then *a* is the imaginary focus; and the point A, where the rays meet after reflection is the real one. Now *ab* the diſtance of the imaginary focus from the point *b*, or *ad* its diſtance from the point *d*, will be equal reſpectively to A*b* and A*d* or to the diſtances of the real focus from thoſe points; and the angle *bad* or *fag*, which is the convergency of the incident rays, will be equal to the angle *bAd*, which is the convergency of the ſame rays when they are reflected.

This is proved exactly in the ſame manner as propoſition 205.

CHAP. XIV.

Of the reflection of parallel rays from convex and concave mirrours.

207. *When parallel rays fall upon a concave ſpherical mirrour, they will be made to converge by being reflected from it; and the diſtance of the focus will be half a ſemidiameter of the mirrours concavity.*

IF a concave mirrour MBN, Plat. XIII. fig. 7, is part of a ſphere whoſe ſemidiameter is CB; any rays CB, LD, that are parallel, when they fall upon the reflecting ſurface, will converge after reflection to a focus F, and BF the diſtance of this focus from the mirrour will be equal to half CB.

Any ray CB, which paſſes through C the center of the concavity, is perpendicular to the ſurface of the mirrour, and conſequently will be reflected back again in BC the line of incidence, by propoſition 201, and propoſition 88, of mechanics. This ray, as the line it deſcribes is not altered by the reflection, may be conſidered as the axis of the beam. But any collateral ray, LD, will fall obliquely upon the mirrour; for, at the point of incidence D, DC drawn from that point to the center is the perpendicular. The angle of incidence here is LDC; and the ray will be reflected in the line DF, ſo as to make the angle of reflection CDF equal to LDC, by propoſition 201. Now all the perpendiculars drawn from the ſurface to the center C, as BC, DC, HC, converge to the axis BC. Therefore any collateral ray LD, which was parallel to the axis when they fell on one ſide of the perpendicular, will converge by being reflected in the line DF on the other ſide. BF the diſtance of the focus from the ſurface is half BC. The angle of incidence of the ray LD is LDC. LD and CB are parallel rays, by the ſuppoſition, and conſequently CD, which croſſes them, makes the alternate angles LDC and DCF equal. Euc. b. I. prop 29. Thus DCF is equal to the angle of incidence; becauſe it is the alternate angle. And CDF is equal to the angle of incidence; becauſe it is the angle of reflection. Therefore the two angles at the baſe of the triangle CFD are equal to one another, becauſe each of them is equal to the angle of incidence; and conſequently the ſides FC and FD, which ſubtend thoſe angles, are equal. Euc. b. I. prop. 6. But when D and B are very near together FB is very nearly equal to FD. Therefore FC and FB are very nearly equal, or the point F, to which the ray DF converges, divides the ſemidiameter CB in halves, ſo that BF is half the ſemidiameter.

208. *When parallel rays fall upon a convex ſpherical mirrour, they will be made to diverge by being reflected from it; and the diſtance of the virtual or negative focus will be half a ſemidiameter of the convexity.*

If the concave mirrour MBN, Plat. XIII. fig. 7. was to unbend itſelf, the ſemidiameter BC would lengthen and the focus F, which is always at the diſtance of half a ſemidiameter, would depart from the mirrour. When the ſurface was quite unbent, or was become plane, the ſemidiameter BC would be infinite, and the reflected rays would converge to an infinitely diſtant focus, or would be parallel, by propoſition 30. 204. But if MBN was to unbend itſelf ſtill farther, or to become leſs concave than when it was plane; its concavity would become leſs than nothing, or it muſt bend itſelf the contrary way and ſo become convex, or the concave mirrour MBN would then be changed into the convex one *mbn*, fig. 8. So that a convex mirrour is only a negative concave one. When the concavity is affirmative, the ſemidiameter BC, and the focus F, are on the ſame ſide with the incident rays. When the concavity is nothing, or when the mirrour is plane, the ſemediameter is infinite, and the focus at an infinite diſtance. But when the concavity is negative, or when the mirrour is convex the ſemidiameter will be greater than infinite, and the focus will be at more than an infinite diſtance; and conſequently they will be negative, by propoſition 49, that is, they will change ſides, and will be on the contrary ſide of the mirrour to the incident rays. Since therefore the focus is affirmative, when the mirrour is concave, but negative when it is convex; the convergency, which is affirmative, when the mirrour is concave, will become negative when it is convex, or the reflected rays will diverge. Thus if *mbn* is the mirrour, and *xb*, *ld*, two parallel rays that fall upon it; *xb*, becauſe it is perpendicular to the ſurface, will be reflected back again in the line *bx*; but *ld* which is oblique to the mirrour, will be reflected in the line *dz* on the other ſide of the perpendicular *cy*, and will diverge from the virtual or negative focus *f*. Now when the ſemidiameter is affirmative the focus F is before the mirrour and divides the ſemidiameter BC into two equal parts; and conſequently, when the ſemidiameter is negative or lyes behind the mirrour, the ne-

negative focus will be juſt as far in proportion from the mirrour as the affirmative one, and will likewiſe divide the ſemidiameter *bc* into two equal parts; ſo that *fb* the diſtance of this virtual or negative focus behind the mirrour will be half the ſemidiameter *bc*.

209. *When rays, that were parallel at their incidence, are reflected from a concave mirrour, they do not all of them converge exactly to one and the ſame focus; and when they are reflected from a convex mirrour, they do not all of them diverge exactly from one and the ſame virtual focus.*

LD, and GH, are two parallel rays falling upon the concave mirrour MN. LD, which is neareſt to the axis, is reflected in the line DF to the focus F: but GH will not be reflected in the line HF to the ſame focus. For if it was reflected thither, then, by what has been proved in propoſition 207, CFH muſt be an iſoſceles triangle, or FC muſt be equal to FH. But FC was there proved to be equal to FD. And conſequently, if the ray GH was reflected to F, the ſame line FC muſt be equal both to FH and to FD. But it cannot be equal to each of them, becauſe they are not equal to one another; for FH is longer than FD. Therefore the ray GH is not reflected to the ſame point F with the ray LD. If it be asked therefore whether the focus, to which GH is reflected, is nearer to the mirrour than F or farther from it; we anſwer it will be nearer. For as the diſtance of F from C is equal to FD, ſo the diſtance of the focus to which GH is reflected will be equal to FH; and ſince FH is longer than FD, the focus, to which GH is reflected, will be at a greater diſtance from the center than CF, and conſequently will be at a leſs diſtance from the mirrour than FB.

That the rays which fall far from the axis CB, will be reflected to a leſs diſtant focus than thoſe which fall near to it, may be otherwiſe proved. For the line BC, in which the axis itſelf is reflected, is the ſame that the incident ray deſcribed; becauſe the axis is perpendicular to the ſurface. And conſequently, ſince any ray LD, which falls near to the axis, is nearer perpendicular, than any other ray GH, which falls far from it, the line, in which LD is reflected, will differ leſs from DL the line of incidence than the line in which GH is reflected will differ from HG. But the leſs the line of reflection differs from the line of incidence ſo much leſs will the reflected ray deviate from being parallel to its axis; for all the lines of incidence are parallel to it, by the ſuppoſition. But the leſs a converging ray deviates from being parallel to its axis the leſs it converges, and the leſs it converges ſo much more remote is the focus, by propoſition 29. Therefore thoſe rays, which fall neareſt the axis, will have a remoter focus than thoſe which fall at a greater diſtance from it. Now the focus of thoſe rays, which fall neareſt to the axis, is at the diſtance of half a ſemidiameter of the concavity, by propoſition 207, and conſequently the focus of thoſe, which fall farther from the axis, is at a diſtance ſomething leſs than this.

But ſince the propoſition is true, when the concavity of the mirrour, the convergency of the reflected rays, and the focus are all of them affirmative, it will likewiſe be true for the ſame reaſons, when they are all of them negative. Therefore when the mirrour is convex, when the reflected rays diverge and the focus is negative, all the rays after reflection will not diverge from exactly the ſame point; but thoſe, which fall neareſt to the axis, will have their virtual focus at a greater diſtance from the mirrour, than thoſe have, which fall farther from it.

The reader however may obſerve, that in rays which do not fall very remote from the axis, the difference of the focal diſtance is ſo ſmall that we may neglect it, and may conſider them after reflection as all converging to the ſame point, if the mirrour is concave, or as all diverging from the ſame point, if it is convex.

CHAP. XV.

Of the reflection of diverging rays from a concave mirrour, and of converging rays from a convex one.

210. *The principal focal diſtance of a concave mirrour is the diſtance, at which rays, that were parallel at their incidence, are united after reflection. And the principal focal diſtance of a convex mirrour is the diſtance, from which rays, that were parallel at their incidence, diverge after reflection.*

THE principal focus of either a concave or a convex mirrour is at the diſtance of half a ſemidiameter of the ſphere, of which the mirrour is a ſegment, by propoſition 207, 208.

From the ſimilitude, which the reader muſt have already obſerved, between the effects of a concave mirrour and a convex lens, and between the effects of a convex mirrour and a concave lens, it follows that, as in propoſition 60, there will be three caſes of diverging rays, when they fall upon a concave mirrour, and three caſes likewiſe of converging rays, when they fall upon a convex one. For, when the mirrour is concave and the rays are diverging, the radiant may be either in the principal focus, or more remote than the principal focus, or nearer than the principal focus. And, when the mirrour is convex and the rays are converging, the imaginary focus may be either in the principal focus, or more remote than the principal focus, or nearer than the principal focus. And indeed the reader will find the ſame ſimilitude between the reflection and the refraction of light in moſt inſtances.

211. *If the radiant is in the principal focus of a concave mirrour, the reflected rays will be parallel.* (See prop. 61.)

After parallel rays have been reflected by the concave mirrour MBN, they converge to F, Plat. XIII. fig. 7. which is the principal focus of the mirrour. Therefore if theſe rays were turned back in the lines of reflection, they would diverge as from a radiant in the principal focus. And when rays are ſo turned back, they will be parallel after reflection, by propoſition 202. conſequently, when a radiant is in the principal focus, the rays, which diverge from thence, will be parallel after reflection.

212. *If the imaginary focus of converging rays is in the principal focus of a convex mirrour, the reflected rays will be parallel.* (See prop. 62.)

If the incident rays *zd*, *xb*, Plat. XIII. fig. 8, converge to *f* the principal focus of the convex mirrour *mbn*; that is, if the imaginary focus of theſe rays is at *f*, the reflected rays *dl*, *bx* will be parallel. For parallel rays *ld*, *xb*, after they have been reflected by the mirrour diverge from *f*, in the lines *dz*, *bx*. Therefore theſe rays, if they were turned back again in the lines of reflection *zd*, *xb*, or any other rays converging in the ſame manner to *f*, will be reflected in the lines *dl*, *bx*, and will be parallel to one another after reflection, by propoſition 202.

213. *If the radiant is more remote from a concave mirrour, than the mirrours principal focus; the rays, that diverge from it, will converge after they are reflected.* (See prop. 63.)

The reflection from the mirrour deſtroys juſt as much divergency as rays have, which come from the mirrours principal focus, and the reflected rays then become parallel, by propoſition 211. But if the radiant is at E, Plat. XIII. fig. 9. ſo that the diſtance BE is greater than the mirrours principal focal diſtance, the rays EB, ED, that come from it, will have leſs divergency, than they would have if the radiant was in the principal focus, by propoſition 28. Conſequently the reflection will deſtroy more than all their divergency, or will make their divergency ne-

negative, or will make them converge: ſo that they will be reflected in the lines BF, DF and will meet in a focus at F.

214. *If the radiant is more remote from a concave mirrour than the principal focus; as the diſtance of the radiant from the mirrours ſurface to its diſtance from the center of the concavity, ſo is the diſtance of the focus from the ſurface to its diſtance from the center.*

The rays which diverge from E, Plat XIII. fig. 9. will be reflected to ſome focus F; where EB is the diſtance of the radiant from the ſurface, and EC its diſtance from the center: EB is a perpendicular ray and conſequently is reflected in the line BC: FB is the diſtance of the focus from the ſurface, and FC its diſtance from the center. Now in this caſe EB is always to EC, as FB to FC. CD drawn from the center to the ſurface is a perpendicular, ED is the incident ray, DF is the reflected ray, EDC is the angle of incidence, and CDF is the angle of reflection. But, by propoſition 201, EDC and CDF are equal angles. Conſequently the line DC divides the angle EDF into two equal parts, ſo that one half of EDF it the angle of incidence, and the other half is the angle of reflection. But if in the triangle EDF the angle at E is thus divided into two equal parts, the line DC, which ſo divides it, will divide the ſide FE into two parts EC and FC, which are in the ſame proportion with ED and FD the other two ſides of the ſame triangle, Euc. b.VI. prop. 3; that is, ED is to EC, as FD to FC. But if B and D are very near together, EB will be nearly equal to ED, and FC will be nearly equal to FD. Therefore EB is to EC, as FB is to FC.

215. *If the imaginary focus is more remote from a convex mirrour than the mirrours principal focus; the rays that converge to that imaginary focus at their incidence, will diverge after they are reflected.* (See prop. 65.)

If the rays *yd*, *xb*, Plat. XIII. fig. 10. converge to *e* a point more remote than the mirrours principal focus, they will be reflected in the lines *bx*, *dz*, ſo as to diverge from *f*. If theſe rays had converged to the mirrours principal focus, the reflection would have juſt deſtroyed all their convergency, and would have made them parallel, by propoſition 212. Therefore if their convergency is leſs, as it is, by propoſition 29, when they converge to a point more remote than the principal focus, the reflection will deſtroy more than all their convergency, that is, it will make their convergency negative or will make them diverge.

216. *If the imaginary focus is more remote from a convex mirrour than the principal focus; as the diſtance of the imaginary focus from the mirrours ſurface is to its diſtance from the center of the convexity, ſo is the diſtance of the imaginary radiant from the ſurface after reflection to its diſtance from the center.*

If the rays *yd*, *xb*, Plat. XIII. fig. 10. converge to the imaginary focus *e*, and are reflected in the lines *dz*, *bx* ſo as to diverge after reflection from the imaginary radiant *f*; then *eb* is the diſtance of the imaginary focus from the ſurface, and *ec* is its diſtance from the center; *fb* is the diſtance of the imaginary radiant from the ſurface, and *fc* is its diſtance from the center. And *eb* is always to *ec* as *fb* to *fc*. This propoſition is the ſame with propoſition 214, only every thing is taken on the contrary ſide, or fig. 10, is the negative of fig. 9. For, firſt E is a radiant before the mirrour, from whence the incident rays diverge, and *e* is an imaginary focus behind the mirrour, to which the incident rays converge. Secondly, MBN is a concave mirrour, and *mbn* is a convex one. Thirdly, the principal focus of a concave mirrour is before it, and that of a convex mirrour is behind it. Fourthly, the reflected rays BF, and DF, converge, and the reflected rays *bx*, *dz*, diverge. Therefore fifthly, the focus F is before one of the mirrours, and the imaginary radiant *f* is behind the other. Since then whether diverging rays fall upon a concave mirrour, or converging rays fall upon a convex one, every thing is the ſame only taken on the contrary ſides, provided the radiant in one caſe and the imaginary focus in the other are more remote than the principal focus; this propoſition is the ſame with propoſition 214, only it is in negative terms. But making the terms negative does not alter the proportion. Therefore ſince as EB to EC, ſo FB to FC, it follows that as *eb* to *ec*, ſo *fb* to *fc*.

217. *If a radiant is more remote from a concave mirrour than its principal focus, as the radiant approaches to the mirrour, the focus departs from it; and as the radiant departs, the focus approaches.* (See prop. 71.)

As the radiant E, Plat. XIII. fig. 9. moves towards the mirrour, the focus F moves from it. Now as the radiant E approaches to the mirrour EB decreaſes, and ſo likewiſe does EC. But then they decreaſe equally, becauſe juſt as much as is taken from the length of EB by E's approach, juſt ſo much is taken from the length of EC. And conſequently if theſe two lines decreaſe equally, the longer of the two EB will decreaſe ſlower in proportion than the ſhorter EC. But if in the approach of the radiant to the mirrour, EB decreaſes ſlower in proportion than EC, then the proportion of EB to EC is changed, ſo that EB, by having decreaſed ſlower than EC, will bear a greater proportion to EC, than it did before. But EB is always to EC as FB to FC. Conſequently, when in the approach of the radiant the proportion of EB to EC is encreaſed, the proportion of FB to FC muſt encreaſe likewiſe. And ſince the proportion of FB to FC cannot encreaſe unleſs FB encreaſes, it follows, that whilſt the radiant approaches, FB or the diſtance of the focus from the mirrour will encreaſe, that is, the focus will depart.

In like manner, when the radiant departs from the mirrour, EB encreaſes, and EC encreaſes equally at the ſame time. But EB is longer than EC, therefore when equal parts are added to both of them, EB ſo encreaſed will be encreaſed leſs in proportion than EC, or whilſt they encreaſe equally EB encreaſes ſlower in proportion than EC: and conſequently in the equal encreaſe of theſe two quantities, the proportion of EB to EC decreaſes. And ſince as EB to EC ſo FB to FC, when the proportion of EB to EC decreaſes, the proportion of FB to FC muſt decreaſe at the ſame time. But the proportion of FB to FC cannot decreaſe unleſs FB decreaſes. Therefore when EB encreaſes FB muſt decreaſe, or as the radiant departs from the mirrour, FB, which is the diſtance of the focus, decreaſes, that is, the focus approaches to it.

218. *If the radiant is in the center of the mirrours concavity, the focus will be in the ſame place.*

If the radiant is in the center, all the rays diverging from it will be perpendicular to the mirrours ſurface, ſince they deſcribe lines drawn from the center to the ſurface. But all perpendicular rays are reflected in the lines of their incidence, by propoſition 202. Therefore the rays, which diverged from the center before reflection, will converge to it afterwards, or the focus will be in the center of the concavity, if the radiant is there.

This propoſition may be proved in another manner. For when the radiant is at the center it is more remote than the principal focus of the mirrour, by propoſition 210. Therefore as EB, Plat. XIII. fig. 9. to EC ſo FB to FC, by propoſition 214. But when the radiant E is at the center C then EC is nothing; and conſequently becauſe EC, or the ſecond term in the proportion, is nothing, FC, or the fourth term, which is the diſtance of the focus from the center, will likewiſe be nothing; that is, the focus will be at the center.

219. *If the radiant is on one ſide of the center of a mirrours concavity, the focus will be on the other ſide.*

Whilſt the radiant E, Plat. XIII. fig. 9. approaches to the mirrour the focus F departs from it, by propoſition 217. Now both the radiant by approaching to the mirrour and the focus by departing from it approach to the center C; and when one of them arrives at the center, the other will be there too, by propoſition 218. And if the radiant ſtill approaches, the focus will ſtill depart as before, ſo that if the radiant was on one ſide of the center at E, the focus would be on the other ſide at F; or if the radiant was at F the focus would then be at E, by propoſition 202.

220. *If the radiant is nearer to a concave mirrour than its principal focus; the rays, which diverge from thence, will continue to diverge, but leſs than they did at their incidence.* (See prop. 67.)

As the radiant approaches to the mirrour the focus departs from it, by propoſition 217. And when the radiant has got into the principal focus, the reflected rays will be parallel, by propoſition 211, and the focus will be at an infinite diſtance, by propoſition 31. Conſequently if the radiant E, Plat. XIII. fig. 11, gets nearer to the mirrour, the focus ſtill departing from the mirrour will be at a diſtance greater than infinite, or its diſtance will be negative. Now whilſt the radiant was more remote than the principal focus of the mirrour, and the diſtance of the focus, to which the rays were reflected, was finite, the rays converged after reflection and this focus was before the mirrour, as at F, fig. 10. Therefore when the radiant is nearer than the principal focus, and the diſtance of the focus, to which the rays are reflected, becomes greater than infinite, this diſtance will be negative, ſo that the focus will be behind the mirrour, and the reflected rays will have a negative convergency or will diverge, by propoſition 49. Thus if the radiant E, fig. 11, is nearer to the mirrour than its principal focus, the rays EB, ED, which diverge from it, will be reflected in the lines BE, DS, ſo as to diverge after reflection from a negative focus or imaginary radiant F behind the mirrour. However theſe reflected rays will diverge leſs than the incident ones. For the reflection will deſtroy juſt as much divergency as rays have that come from the principal focus, by propoſition 211. Therefore, though theſe rays have a greater divergency becauſe the radiant is nearer, and conſequently the reflection cannot deſtroy it all and make the rays parallel, yet it will deſtroy a part of it, and will make them diverge leſs than they did at their incidence.

221. *If the radiant is nearer to a concave mirrour than the principal focus; as the diſtance of the real radiant from the ſurface of the mirrour to its diſtance from the center of the concavity, ſo is the diſtance of the imaginary radiant from the ſurface after reflection to its diſtance from the center.*

If E, Plat. XIII. fig. 11, is the radiant and F is the negative focus or imaginary radiant; then EB is to EC as FB to FC. This is the proportion when E, as in fig. 9, is more remote than the principal focus, by propoſition 114, and, from what has been ſaid under propoſition 220, it appears that there is no difference between that caſe and this, except only that the focus which is affirmative in one caſe is negative in the other. Therefore this proportion is likewiſe true, when the radiant is nearer than the principal focus.

222. *If rays converge at their incidence upon a convex mirrour to an imaginary focus which is nearer than the principal focus; the reflected rays will continue to converge, but leſs than they did at their incidence.* (See prop. 69.)

If *fb* and *sd*, Plat. XIII. fig. 12, converge to an imaginary focus *e*, which is nearer to the mirrour than its principal focus; they will be reflected in the lines *bf*, *df*, ſo as to converge ſtill, only leſs than they did before. If the imaginary focus had been in the principal focus, the reflection would have deſtroyed all the convergency of the rays, by propoſition 112. Therefore if the imaginary focus is nearer, and the rays, by propoſition 29, converge more, the reflection will deſtroy only a part of their convergency; and conſequently the reflected rays will continue to converge, but leſs than the incident ones.

223. *If rays converge, at their incidence, to an imaginary focus, which is nearer to a convex mirrour than its principal focus; as the diſtance of the imaginary focus from the ſurface of the mirrour is to its diſtance from the center of the convexity, ſo after reflection is the diſtance of the real focus from the ſurface to its diſtance from the center.*

The imaginary focus to which the incident rays *fb* and *sd*, Plat. XIII. fig. 12, tend is *e*, *c* is the center of the convexity, and *f* is the real focus, where the reflected rays meet. And *eb* is to *ec* as *fb* to *fc*. This propoſition is the ſame as propoſition 221, only every thing is taken on the contrary ſide, or fig. 12 is the negative of fig. 11. For firſt, in fig. 11, E is a radiant nearer than the principal focus, and fig. 12, *e* is an imagiginary focus on the contrary ſide of the mirrour. Secondly MBN is a concave mirrour, and *mbn* is a convex one. Thirdly, the principal focus of the concave mirrour is before it, and that of the convex one is behind it. Fourthly, the reflected rays BE and DS diverge, and the reflected rays *bf* and *df* converge. Therefore fifthly, the imaginary radiant F is behind one mirrour, and the real focus *f* is before the other. Since then, whether diverging rays fall upon a concave mirrour, or converging rays upon a convex one, every thing is the ſame, only taken on contrary ſides, provided the radiant in one caſe, and the imaginary focus in the other is nearer than the mirrours principal focus, the proportions will be the ſame in both caſes: for if the propoſition is true when the terms are affirmative, it will not be made leſs true by changing them from affirmative to negative. Therefore ſince EB is to EC as FB to FC, *eb* is likewiſe to *ec* as *fb* to *fc*.

224. *If the radiant is nearer to a concave mirrour than its principal focus, as the real radiant on one ſide approaches to the mirrour, the imaginary radiant on the other ſide will approach to it, and as the real radiant departs the imaginary one will depart.* (See prop. 72.)

If the radiant E, Plat. XIII. fig. 11, is nearer to the mirrour MBN than its principal focus, then as E approaches to the mirrour, F the imaginary radiant will likewiſe approach to it. EB is to EC as FB to FC, by propoſition 221. Now if E approaches to the mirrour, EB muſt decreaſe, and the proportion of EB to EC muſt decreaſe, conſequently the proportion of FB to FC muſt decreaſe or FB muſt decreaſe, that is, F muſt approach to the mirrour. If the reader is not aware how the proportion which FB bears to FC will decreaſe by ſhortening FB, ſince as F moves towards B, it likewiſe moves towards C and conſequently FC ſhortens at the ſame time; in order to underſtand this he need only recollect what we have often obſerved, that theſe two lines ſhorten equally, as F moves towards B, and conſequently the ſhorter of the two or FB decreaſes faſter in proportion than the longer of the two FC.

In like manner, if the radiant E departs from the mirrour the imaginary radiant F will depart too. For ſince we have ſeen that, whilſt EB decreaſes FB will likewiſe decreaſe, it follows that whilſt EB encreaſes by the radiants motion from the mirrour, FB will encreaſe or the imaginary radiant will move from the mirrour. This motion of the imaginary radiant will continue no longer than till the real radiant E has got to the mirrours principal focus: for then the diſtance of F is infinite, by propoſition 30; and if the radiant departs farther this diſtance will be greater than infinite, and the radiant will change ſides, or will be before the mirrour, by what is proved under propoſition 49; that is, when E is more remote than the principal focus of the mirrour, F will be a negative radiant, or a focus, before the mirrour, as in propoſitions 213, 214.

225. *If the real radiant is cloſe to the ſurface of a concave mirrour, the imaginary radiant will likewiſe be cloſe to it.*

The diſtances of the real radiant E, Plat. XIII. fig. 11, and of the imaginary radiant F is thus determined, by propoſition 221, as EB to EC, ſo is FB to FC. Now when the real radiant E is cloſe to the mirrour at B, then EB is nothing. But when the firſt term or EB is nothing, the third term or FB will be nothing, and conſequently F will be cloſe to B or the imaginary radiant will be cloſe to the mirrour.

We may now recapitulate what has been ſaid in the foregoing propoſitions concerning the motion of E and F or of the radiant and focus in reſpect of a concave mirrour. When E, Plat. XIII. fig. 9, is at an infinite diſtance, ſo that the incident rays are parallel, F will be before the mirrour in its principal focus, by propoſition 207. As E approaches to the mirrour F will depart from it, and unleſs E is at an infinite diſtance F will be farther from the mirrour than its principal focus, by propoſitions 213, 214. Whilſt E keeps approaching, F will keep departing, till they both meet at C in the center of the concavity, by propoſitions 217, 218. When E is between the center and the ſurface, F will be on the oppoſite ſide of the center, by propoſition 219. And in the nearer approach of E to the mirrour F will ſtill keep departing, till E comes to the principal focus, and then the diſtance of F is infinite, or the reflected rays are parallel, by propoſitions 217, 211. If the radiant E approaches nearer to the mirrour than the principal focus, as in fig. 11, then the diſtance of F is greater than infinite, or F will be negative and on the contrary ſide of the mirrour or behind it, by propoſitions 220, 221. After this if E approaches to the mirrour F will approach to it likewiſe, by propoſition 224, till E is cloſe to the mirrour and then F will meet it again, by propoſition 225.

CHAP. XVI.

Of the reflection of converging rays from a concave mirrour, and of diverging rays from a convex one.

226. *If rays converge, when they fall upon a concave mirrour, they will converge more after reflection.* (See prop. 74.)

IF the rays SD, EB, Plat. XIII. fig. 11, converge to the imaginary focus F, but, before they arrive there, are reflected by the concave mirrour MBN, they will converge after reflection more than they did before, and will meet at E. For, by propoſitions 220, 221, if the rays EB, ED, were to diverge from E, they would be reflected in the lines BE, DS, and would diverge leſs than they did at their incidence, juſt as if they had come ſtrait from F. Therefore if theſe rays BE and DS are turned back again in the ſame lines EB, SD, ſo as to converge to F, they will be reflected in the lines of their firſt incidence BE and DE, ſo as to be made to converge more and to meet at the point E, by propoſition 202. In like manner any other rays which diverged before reflection, and diverge likewiſe after it, will diverge leſs, by propoſition 220. But all converging rays may be conſidered as ſuch rays turned back again: and ſince all ſuch rays, when turned back again will be reflected in the lines of their incidence, by propoſition 202; it follows that all converging rays, when they have been reflected, will be made to converge more.

227. *When converging rays are reflected from a concave mirrour, as the diſtance of the imaginary focus from the ſurface of the mirrour to its diſtance from the center of the concavity, ſo is the diſtance of the real focus from the ſurface to its diſtance from the center.*

If the rays EB, SD, converge to the imaginary focus F before reflection, they will converge to the real focus E after reflection, by propoſitions 221, 202. And as FB is to FC, ſo is EB to EC. This follows from propoſition 221. For we have there proved that as EB to EC, ſo FB to FC, and conſequently as FB to FC, ſo is EB to EC.

228. *If rays diverge, when they fall upon a convex mirrour, they will diverge more after reflection.* (See prop. 76.)

If there is any radiant at *f*, Plat. XIII. fig. 12, from which the rays *fb*, *fd* diverge; when they are reflected from the mirrour *mbn*, they will go on in the lines *bf*, *ds*, ſo as to diverge more, and will have the ſame di-

direction in reſpect of each other, as if they had come ſtrait forwards from a radiant *e*. If the rays *fb*, *sd*, which converge to *e*, were reflected from the mirrour, they would converge more after reflection, and would meet at the focus *f*, by propoſitions 222, 223. And if theſe rays were to be turned back again from *f* in the lines *fb*, *fd*, ſo as to diverge from *f*, they would be reflected into the lines of their former directions *bf* and *ds*, by propoſition 202. And conſequently would be made to diverge more. In like manner all rays, which converge at their incidence, and converge likewiſe after reflection, will converge leſs after reflection than they did before it, by propoſition 222. And all rays, which diverge at their incidence, may be conſidered as ſuch rays turned back again. But ſince all ſuch rays, when thus turned back again, will be reflected into the lines of their incidence, and will therefore be made to diverge more; it follows that all rays, which diverge when they fall upon a convex mirrour, will diverge more after they are reflected from it.

229. *When diverging rays are reflected from a convex mirrour, as the diſtance of the real radiant from the ſurface of the mirrour to its diſtance from the center, ſo is the diſtance of the imaginary radiant from the ſurface after reflection to its diſtance from the center.*

If rays diverge before reflection from the focus *f*, Plat. XIII. fig. 12. they will diverge after reflection from the imaginary focus *e*, by propoſitions 222. 202. And as *fb* to *fc* ſo is *eb* to *ec*. For we have proved, in propoſition 223, that *eb* is to *ec* as *fb* to *fc*; from whence it follows, that *fb* is to *fc* as *eb* to *ec*.

230. *As the real radiant moves either towards or from a convex mirrour, the imaginary radiant will move the ſame way.* (See prop. 78.)

If the real radiant *f* approaches to the mirrour *mbn*, Plat. XIII. fig. 12, the imaginary radiant *e* will likewiſe approach to it: or if *f* departs from the mirrour, *e* will likewiſe depart. For when *f* approaches to the mirrour *fb* decreaſes faſter in proportion than *fc*; or the proportion of *fb* to *fc* decreaſes, conſequently, ſince as *fb* to *fc* ſo *eb* to *ec*, by propoſition 228, the proportion of *eb* to *ec* muſt decreaſe at the ſame time. But the proportion of *eb* to *ec* cannot decreaſe unleſs *eb* decreaſes. Therefore when the real radiant *f* approaches to the mirrour, or when *fb* decreaſes, *eb* will likewiſe decreaſe, or the imaginary radiant *e* will approach to it. In like manner if the real radiant *f* departs from the mirrour, or if *fb* encreaſes, it will encreaſe faſter in proportion than *fc*, and conſequently the proportion of *fb* to *fc* will encreaſe; for which reaſon the

PLATE XIII.

Fig. 1.

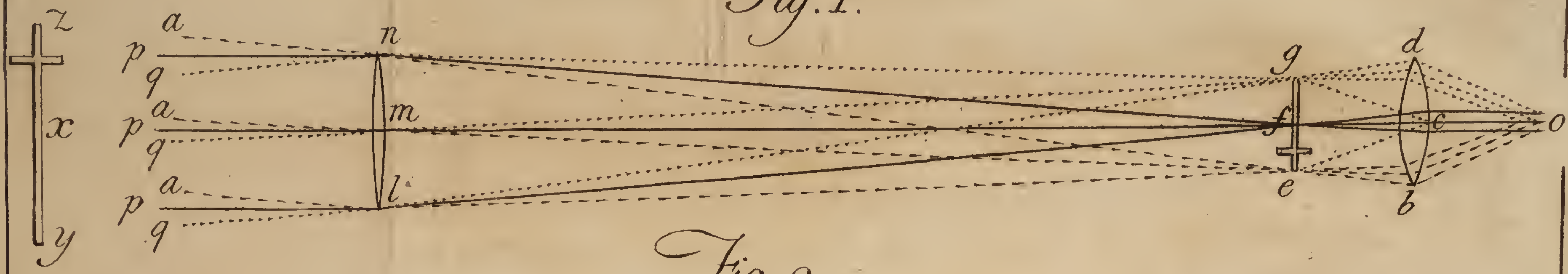


Fig. 2.

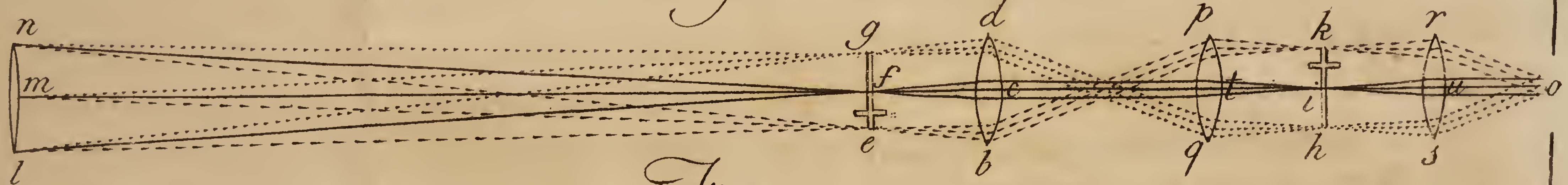


Fig. 3.

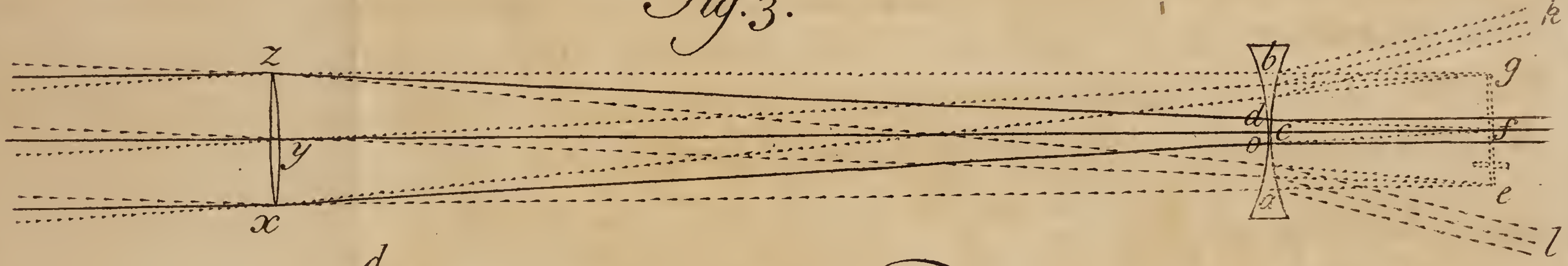


Fig. 4.

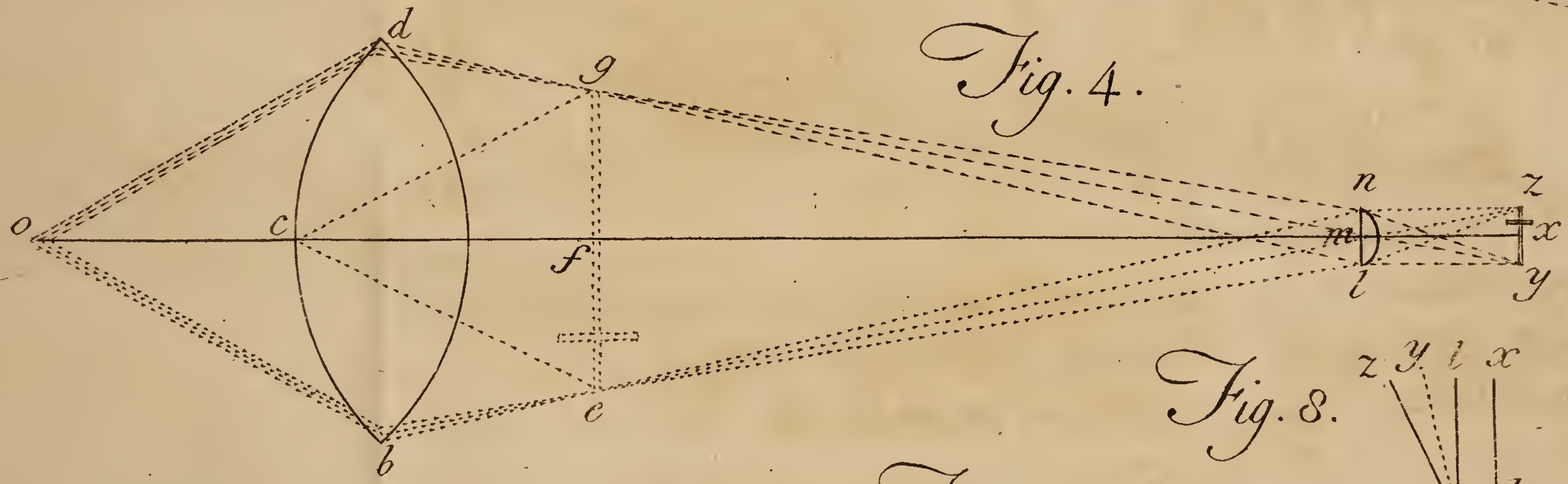


Fig. 5.

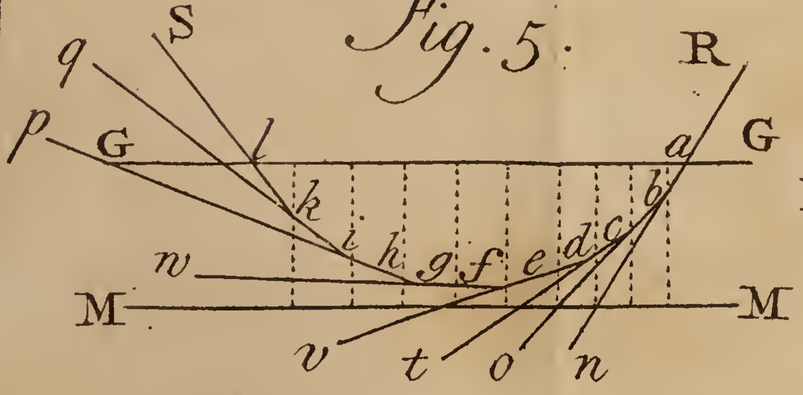


Fig. 6.

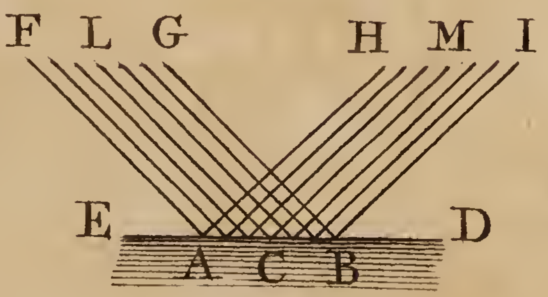


Fig. 7.

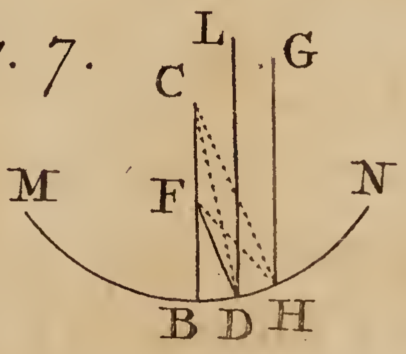


Fig. 8.

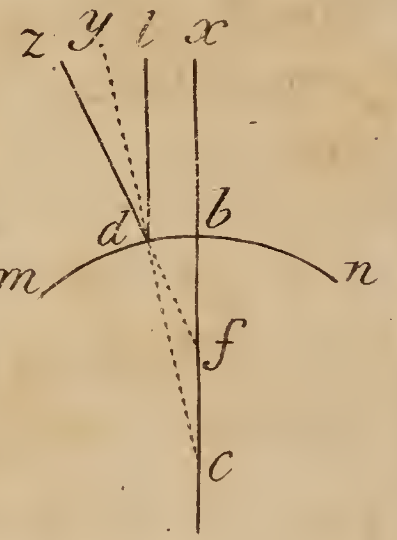


Fig. 9.

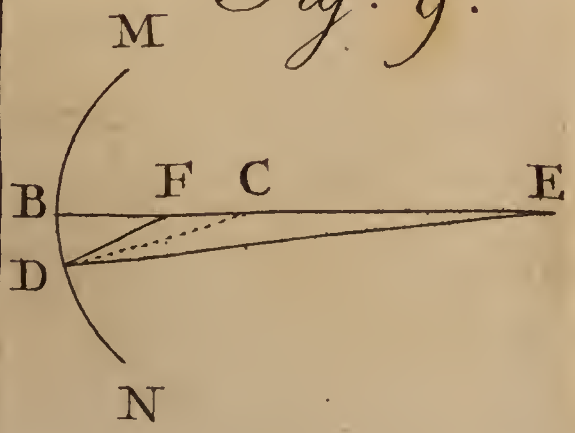


Fig. 10.

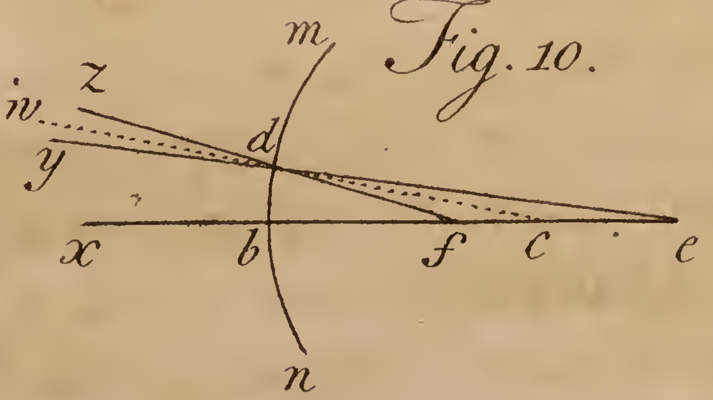


Fig. 11.

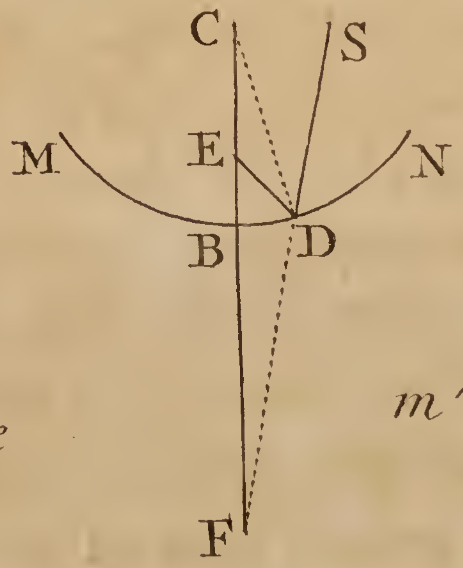
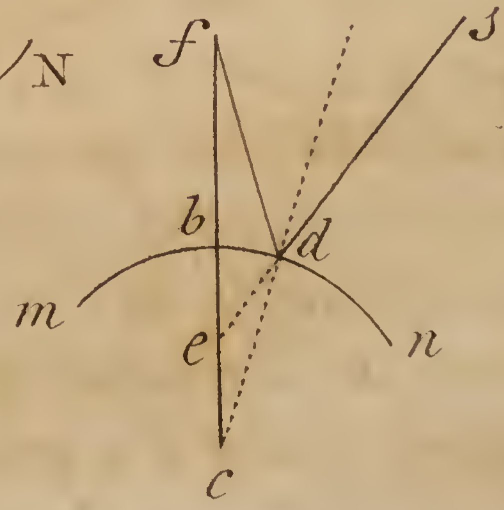


Fig. 12.



the proportion of *eb* to *ec* will encreaſe, that is, *eb* the diſtance of the imaginary radiant *e* will encreaſe, or the imaginary radiant will depart at the ſame time.

231. *If the real radiant is cloſe to a convex mirrour, the imaginary radiant will likewiſe be cloſe to it.* (See prop. 79.)

As *fb* to *fc*, Plat. XIII. fig. 12. ſo is *eb* to *ec*, by propoſition 228. But when the real radiant *f* is cloſe to the mirrour, *fb* or its diſtance from the ſurface is nothing. And when the firſt term of this proportion is nothing, the third term *ec*, or the diſtance of the imaginary radiant from the ſurface, will be nothing, that is, the imaginary radiant will be cloſe to the ſurface.

CHAP. XVII.

Of reflected viſion in plane mirrours.

232. *The paſſage of reflection is the incident ray added to the reflected ray.*

WE may ſufficiently explane this definition in an inſtance of reflected viſion from a plane mirrour *ab*, Plat. XIV. fig. 1. though the term defined is uſed in all other inſtances of reflected viſion. Let *de* be an object, and ſuppoſe the rays *df*, *dr*, which diverge from *d* the higheſt radiant in it, to be reflected in the lines *fc*, *rh*; here *df* is the incident ray and *fc* is the reflected one, and *df* added to *fc* is the paſſage of reflection; or *dr* is the incident ray and *rh* is the reflected one, and *dr* added to *rh* is the paſſage of reflection.

233. *When any ſmall object or any point of an object is ſeen by reflected light, it appears in the direction of that line, which the rays deſcribe after their laſt reflection.*

This is proved in the ſame manner as propoſition 137. For the picture of an object ſeen by reflection will be in the ſame part of the retina, that it would have been in, if we had looked with the naked eye at an object ſituated in the place from whence the reflected rays diverge; ſince the reflection gives the ſame direction to the rays as if they had originally come from that place: ſo that in reflected as well as in refracted viſion the laſt image conſiſts of imaginary radiants, from which the rays diverge when they come to the eye.

234. *In all mirrours, whether plane or ſpherical, the place of the imaginary radiant, when it can be determined, is the interſection of the cathetus of incidence and of any reflected ray.*

If *d*, Plat. XIV. fig. 1, is a radiant in any object *de*, and any ray *df*, which comes from this radiant, is reflected in the line *fc*, then draw the cathetus of incidence *di* perpendicular to *ab* the ſurface of the plane mirrour, and the plane of the imaginary radiant from whence *fc* appears to diverge is thus determined; continue the cathetus of incidence *di* behind the mirrour, and if the reflected ray *cf* is continued in the ſame manner till it croſſes the cathetus at *l*, this point of interſection *l* will be the place of the imaginary radiant. All the rays, which diverge from one and the ſame radiant *d* before reflection, diverge from one and the ſame imaginary radiant after reflection. Now amongſt the innumerable rays, which come from *d* in all directions there will be one that deſcribes *di* the cathetus of incidence. This ray will be reflected in the line of its incidence *id*, becauſe it falls perpendicularly upon the mirrour, by propoſition 88 of mechanics. Conſequently, ſince all the rays diverge after reflection from one and the ſame imaginary radiant, this radiant muſt be ſomewhere in the line *di* continued, for otherwiſe the reflected ray *id* could not diverge from it. The ray *fc* likewiſe diverges from the ſame radiant; and conſequently this radiant muſt be ſomewhere in the line *cf* continued. But it is impoſſible that the imaginary radiant ſhould be both in the line *cf* and in the line *di*, unleſs it is at ſome point which is in both theſe lines. And no point can be in both theſe lines, but *l*, which is the point of their interſection. Therefore *l* is the place of the imaginary radiant.

This is univerſal, whenever the ray, which deſcribes the cathetus of incidence, is reflected either from a plane or a ſpherical mirrour. For the imaginary radiant will always be ſomewhere in this ray; that is, it will be ſomewhere in the cathetus of incidence; and it will always be ſomewhere in any other reflected ray. Therefore it will always be in the common point of interſection of the cathetus of incidence and of any reflected ray. Thus in the concave mirrours *gil*, fig. 2, and 3, and in the convex one *gil*, fig. 4, if *a* is a radiant in the object *abc*, the place of the laſt imaginary radiant, when the rays, which come from *a*, are reflected by the mirrour, may be determined by this rule. The ray *ai* is reflected in ſome line *ic*; draw the cathetus of incidence *kad*, which in ſpherical mirrours always paſſes through *k* the center of the ſphere of which the mirrour is a ſegment; then continue the reflected ray *ic* till it croſſes the cathetus; and the point of interſection *d* is the place of the laſt imaginary radiant.

The place of the laſt radiant is determined in the ſame manner, though it may happen, either from the poſition of the object or from the ſhortneſs of the mirrour, that the ray, which deſcribes the cathetus of incidence, does not fall upon the mirrours ſurface, and conſequently is not reflected. Thus if the plane mirrour is not ſo long as *ab*, fig. 1. if, for inſtance, the length of it is only *fb*, ſo that the ray *di* does not fall upon it, the laſt imaginary radiant will ſtill be at *l*, or at the point where the cathetus *di* continued interſects any reflected ray *cf* or *hr* continued. This would be the place of the laſt radiant, if the mirrour was long enough to reflect the ray which deſcribes the cathetus, as has been proved already. But the length of the mirrour does not alter the place of the imaginary radiant, by propoſition 205. Therefore this will be the place of the laſt radiant, or the reflected rays *fc*, *rh*, will diverge from the ſame point; though the length of the mirrour is only *fb*, ſo that the ray *di*, which deſcribes the cathetus of incidence is not reflected.

235. *In plane mirrours, the diſtance of the laſt image from the mirrour is equal to the diſtance of the object from it: and the diſtance of any point in the laſt image from the eye is equal to the paſſage of reflection.*

The diſtances of each imaginary radiant *l*, *m*, Plat. XIV. fig. 1, behind the mirrour are reſpectively equal to the diſtance of the correſponding real radiants *d*, *e*, before it, by propoſition 205. Therefore the diſtance of the laſt image *lm*, which is made up of imaginary radiants, is equal to the diſtance of the object *de*, which is made up of real ones.

The diſtance of *l* or the higheſt point in the image from the eye at *c* is *lfc*, and *lfc* is equal to *dfc* the paſſage of reflection. For ſince *lf* is equal to *df*, by propoſition 205, $lf+fc=lfc$ muſt be equal to $df+fc=dfc$. So likewiſe *mnc*, which is the diſtance of the loweſt point in the image from the eye at *c* is equal to *enc*, which is the paſſage of reflection. For $mn=en$, by propoſition 205: and conſequently, $mn+nc=en+nc$.

236. *In plane mirrours the image is equal and ſimilar to the object.*

The image *lm*, Plat. XIV. fig. 1, is juſt as high as the object *de*. For if *d* is the higheſt point of the object, then the higheſt point of the image is in the cathetus of incidence *dil*, and if *e* is the loweſt point of the object, the loweſt point of the image is in the cathetus *ezm*, by propoſition 334. But *dil* and *ezm* are parallel to one another, becauſe, by propoſition 203, they are both of them perpendicular to the ſame plane ſurface *ab*. Euc. b. XI. prop. 6. Now the height of the object is *de* the diſtance between thoſe two lines at the object, and the height of the image is *lm* the diſtance between the ſame lines continued to the image. But theſe lines are parallel to one another, and conſequently the diſtance between them is the ſame at both places. Therefore the height of the laſt image *lm* is equal to the height of the object *de*. In the ſame manner we might ſhew that the breadth of the image is equal to the breadth of the object. And conſequently one is equal to the other in all reſpects. But ſurfaces that are every way equal to one another muſt be ſimilar. Therefore the image and object are ſimilar ſurfaces. What we mean by the ſurface of the object, when we ſay that the ſurface of the image is ſimilar to it, may be ſeen in propoſition 90, where we explaned the ſame thing in reference to the picture of an object in a dark room.

237. *If a plane mirrour and the object that is ſeen in it are both of them perpendicular to the horizon, the object ſeen in the mirrour appears erect.*

If the object *de*, Plat. XIV. fig. 1, and the mirrour *ab* are perpendicular to the horizon, the image *lm*, or the object of reflected viſion, is in the ſame poſition as the object itſelf. The higheſt point *d* of the object appears in the cathetus of incidence *di* drawn from thence, and the loweſt point *e* appears in the cathetus *ez*, by propoſition 234. But theſe cathetuſes being both perpendicular to the ſurface of the mirrour are parallel to each other; Euc. b. XI. prop. 6, and conſequently they do not croſs each other any where between the object *de* and the image *lm*. Therefore that cathetus, which is higheſt at the object, will be higheſt at the image, and that, which is loweſt at one, will be loweſt at the other: and conſequently the image, which is the object of reflected viſion, will be in the ſame poſition with the object itſelf.

238. *When the object is parallel to a plane mirrour, the length of that part of the mirrour, upon which the image appears, is to the length of the object, as any reflected ray is to the paſſage of reflection.*

If the object *de* is parallel to the mirrour *ab*, and the image *lm* is ſeen by the eye at *c*, then *fn* the length of that part of the mirrour, which is taken up by the image, ſubtends the optic angle *lcm*, or the angle under which the image appears. For ſince all the viſible length of the image is included within that angle *lcm*, by propoſition 125, no more of the mirrour can be taken up by that viſible length than what is likewiſe included within the ſame angle. Now the length of the image *lm* is equal to the length of the object *de*, by propoſition 236. And *fn* the length of that part of the mirrour, which is taken up by the image, is to *lm* the length of the image as any reflected ray *fc* is to *dfc* the paſſage of that rays reflection, or as the diſtance of the eye from the mirrour, which is the reflected ray *fc* to *lfc* the diſtance of the eye from the image, which is equal to the paſſage of reflection, by propoſition 235. For the apparent length of *fn* that part upon the mirrours ſurface, where the image is ſeen, is equal to the apparent length of the image; becauſe they both ſubtend the ſame angle; by propoſition 125. And from hence it follows, by propoſition 128, that their real lengths are to each other as their reſpective diſtances from the eye. But the eyes diſtance from that part of the mirrour, where the image appears is equal to the reflected ray *fc*, and the eyes diſtance from the image is *lfc* equal to the paſſage of the reflection. Therefore *fn* the length of that part of the mirrour where the image appears is to *lm* the length of the image, or to *de*=*lm* the length of the object, as *fc* a reflected ray is to *lfc*=*dfc* the paſſage of reflection.

Or if the eye was at *h*, then *lhm* is the optic angle, and *ro* is the length of that part in the mirrour where the image appears: and in this poſition of the eye *ro* is to *lm* or to *de* as *rh* any reflected ray is to *lrh*=*drh* the paſſage of reflection.

In the ſame manner we might ſhew that the breadth of that part in the mirrour, which is taken up by the image, is to the breadth of the object, or that any diameter of that part is to the correſponding diameter of the object, as a reflected ray is to the paſſage of that rays reflection.

239. *Though a common looking-glaſs ſhould be ſo ſhort that, at a certain diſtance of the eye and of a given object from it, we cannot ſee the whole length of the object in the glaſs by reflection; yet, by bringing the eye nearer to the glaſs or by removing the object farther from it, we may ſee the whole length of the object.*

The length of glaſs, which is requiſite to ſee the whole length of the object, bears the ſame proportion to the length of the object, that a reflected ray bears to the paſſage of reflection, by propoſition 238. From hence it follows that the leſs the reflected ray is in proportion to the paſſage of reflection, ſo much ſhorter a glaſs may be in proportion to any object, the entire length of which is ſeen by reflection in the glaſs. Now as the eye *c*, Plat. XIV. fig. 1. approaches to the glaſs, the length of a reflected ray *cf* decreaſes: at the ſame time indeed the paſſage of reflection *dfc*, or *lfc*=*dfc*, decreaſes too. But *cf* is ſhorter in itſelf than *lfc*, and by the approach of the eye to the glaſs theſe two unequal lines are equally diminiſhed; and conſequently this equal decreaſe bears a greater proportion to *cf* than to *lfc*; that is, though in the approach of the eye to the glaſs an equal part is taken away both from *cf* and from *lfc*, yet this equal part bears a greater proportion to *cf* than it does to *lfc*; upon which account by taking an equal part from *cf* and from *lfc*, the reflected ray *cf* will be more diminiſhed in proportion than the paſſage of reflection *lfc*. Therefore in the approach of the eye to the glaſs the proportion of the reflected ray to the paſſage of reflection decreaſes, ſince the former is more diminiſhed in proportion than the latter; and conſequently the length of glaſs, in which we ſee the object, is ſhorter in proportion to the objects length: ſo that if a glaſs was too ſhort to ſhew us the whole length of the object, at a certain diſtance of the eye, yet it may be long enough for this purpoſe, if the eye comes nearer to it.

The ſame purpoſe may be anſwered, by removing the object farther from the glaſs, whilſt the eye continues at the ſame diſtance. For this removal of the object will diminiſh the proportion of the reflected ray to the paſſage of reflection. If the eye continues at *c*, the reflected ray is *cf*, but as the object departs from the glaſs *df*, and conſequently *dfc*, or *lfc*=*dfc*, which is the paſſage of reflection, encreaſes. But the longer *lfc* is, ſo much leſs proportion the given line *cf* will bear to it. Therefore as the object departs from the glaſs the proportion of the reflected ray to the paſſage of reflection decreaſes.

Since the reader has been ſhewn that, as the eye approaches to the glaſs or as the object departs from it, the length of glaſs, upon which a given object is ſeen, decreaſes; he will eaſily infer that the contrary cauſe would produce a contrary effect, or that, as the eye departs from the glaſs or the object approaches to it, the length of glaſs, upon which a given object is ſeen, will encreaſe. For in the departure of the eye or in the approach of the object, the proportion of the reflected ray to the paſſage of reflection encreaſes.

240. *That part of a plane mirrours ſurface, upon which an object appears by reflection, is always ſimilar to the ſurface of the object.*

The length of the object is to the length of that part in the mirrours ſurface where the object appears, and the breadth of the object is to the breadth of this ſurface, as a reflected ray to the paſſage of reflection, by propoſition 238. Therefore the length of the object is to the length of this ſurface as the breadth of the object to the breadth of this ſurface; ſince each is to the other as a reflected ray to the paſſage of reflection. But two ſurfaces are ſimilar, when their lengths and breadths are reſpectively proportional to each other. Therefore that part of a mirrours ſurface, on which any object is ſeen, is ſimilar to the ſurface of the object itſelf.

241. *The area of that part in a plane mirrours ſurface, upon which an object appears, is to the area of the objects ſurface, as the ſquare of a reflected ray to the ſquare of the paſſage of reflection.*

That part of a mirrours ſurface, where any object appears is ſimilar to the objects ſurface, by propoſition 240. But the areas of ſimilar ſurfaces are to one another as the ſquares of their correſponding diameters. Euc. b. VI. prop. 20. corol. 1. Therefore the area, which the image covers on the mirrours ſurface, and the area of the objects ſurface, are to each other as the ſquares of their reſpective lengths. Now the length of that part in the mirrour, which the image covers, is to the length of the object, as a reflected ray to the paſſage of reflection, by propoſition 238. Therefore their areas are to each other as the ſquare of a reflected ray to the ſquare of the paſſage of reflection.

242. *If a perſon ſees himſelf entirely in a plane looking-glaſs, which is parallel to him; the glaſs muſt be half as long as he is, and half as broad as he is; or its ſurface muſt be equal to a quarter of his ſurface.*

The length of a mirrour muſt be to the length of a perſon, who ſees himſelf entirely in it, as a reflected ray to the paſſage of reflection, by propoſition 238. Now when a perſon is looking at himſelf, the incident ray is his diſtance from the glaſs, and the reflected ray is equal to it, for it is the glaſs's diſtance from him. The paſſage of reflection, is the incident ray added to the reflected ray, and conſequently is equal to twice his diſtance from the glaſs. Therefore the reflected ray is to the paſſage of reflection as 1 to 2. From whence it follows that the length of the glaſs, in which he can ſee himſelf from head to foot, muſt be to his length as 1 to 2; that is, the length of the glaſs muſt be half his length.

In a plane mirrour, that is parallel to a perſon who is looking at himſelf, he will not be able to ſee himſelf from head to foot, if the mirrour is at all ſhorter than this, whether he is near to the mirrour or far from

it,

it. If he was looking at any other object, then indeed the nearer he comes to the mirrour, the more of the object he might ſee, by propoſition 239: becauſe by coming near to the mirrour his eye approaches to it. But when he himſelf is the object that he is looking at, as he comes nearer to the mirrour his eye comes nearer to it, but then the object comes juſt as much nearer at the ſame time. And conſequently though in a ſhort mirrour he might ſee more of himſelf from the approach of the eye, yet he will ſee juſt as much leſs of himſelf from the equal approach of the object, by propoſition 239. Therefore upon both accounts together he will ſee juſt as much and no more of himſelf, when he is near to the mirrour, than he did, when he was farther from it; but if it was too ſhort to ſee himſelf entirely in it at any one diſtance, it will likewiſe be too ſhort for that purpoſe at any other leſs diſtance. In like manner if he was looking at any other object, the farther the object is from the mirrour the more he might ſee of it. But when he himſelf is the object that he is looking at, though by going farther from the mirrour, he might upon this account ſee more of himſelf; yet as he cannot go farther off, but his eye will depart equally from the mirrour, conſequently upon this ſecond account he will ſee juſt as much leſs of himſelf. Therefore upon both accounts together he will ſee neither more nor leſs of himſelf for encreaſing his diſtance: but if the mirrour was too ſhort to ſee himſelf from head to foot at any one diſtance, it will likewiſe be too ſhort for that purpoſe at any other greater diſtance.

In the ſame manner that we have proved a mirrour muſt be half a mans length for him to ſee his whole length in it, we might likewiſe prove that it muſt be half his breadth for him to ſee his whole breadth.

But the ſurface of the mirrour can only be a quarter of his ſurface, if it is but half as long and but half as broad as he is. This laſt part of the propoſition may be demonſtrated otherwiſe. The area of a mirrour, which is covered by the image of any object, is to the area of that objects ſurface, as the ſquare of a reflected ray to the ſquare of the paſſage of reflection, by propoſition 241. But when a man is looking at himſelf, the reflected ray is to the paſſage of reflection as 1 to 2, by what has been already ſhewn, and conſequently the ſquare of the reflected ray is to the ſquare of the paſſage of reflection as 1 to 4. Therefore the area of a mirrour, which will be juſt covered by a mans image, when he is looking at himſelf, is to the area of his ſurface, as 1 to 4; that is, the ſurface of the mirrour in which he can juſt ſee his whole length and his whole breadth is $\frac{1}{4}$ of his ſurface.

243. *In a plane mirrour what is on the right-hand ſide in the object, is on the left-hand ſide in the image, and what is on the left-hand ſide in the object will be on the right-hand ſide in the image.*

This will be beſt explaned and proved by the inſtance of a perſon looking at himſelf in a plane mirrour. His face is then towards the face of his image in the glaſs. Now if any other perſon ſtood with his face towards ours, it is evident that his right-hand would be towards our left, and his left-hand towards our right. And for the ſame reaſon it will be ſo in the caſe of our own image ſeen in the glaſs. If we hold a pen in our right-hand, the image with its face towards us will hold the pen in that hand which is on the ſame ſide: and if any perſon with his face towards us was to hold a pen in the hand which is on the ſame ſide with our right-hand, we ſhould ſay that he held it left-handed.

In reading a book we begin the lines from the left-hand, and read towards the right. But if we turn the book towards a looking glaſs we muſt then read the lines ſeen by reflection from the left-hand of our image in the glaſs towards the right-hand of that image. This we ſhall find is reading the contrary way from what we uſually do, for the lines will appear inverted and we muſt read them from our own right-hand towards our left.

Something like this may be ſeen in ſome prints that are copies from paintings; for all the figures in ſuch prints are frequently left-handed. If the figures engraved upon the copper-plate ſtand the ſame way that they do in the original painting and are all of them right-handed; then in the print taken off from that plate they will ſtand the contrary way and will be left-handed. For in taking off the print from the engraving, any figure upon the plate, as for inſtance, the figure of a man with a ſpear in his right-hand has its face towards the figure upon the paper; conſequently the right-hand of the man upon the plate will be towards the left-hand of the ſame man in the print: therefore as one of them holds the ſpear in his right-hand the other will hold it in his left.

244. *If a plane mirrour is parallel to the horizon ſuch objects, as are perpendicular to the horizon, when they are ſeen it, appear inverted.*

This is the caſe of trees, or buildings, or men, ſtanding on the bank of a river, when they are ſeen in the water, which is a plane mirrour. The bottom of any tree or other object is neareſt to the mirrour and the top of it is fartheſt from the mirrour. Now, by propoſition 205, each imaginary radiant is at the ſame diſtance behind the mirrour, that the real radiant is at before it. Therefore in the image the bottom of the tree will be neareſt to the ſurface of the water and the top will be fartheſt from it. But in this poſition of the image the top of the tree ſeen in the water will be nearer to the center of the earth than the bottom of it, or the tree will appear inverted.

If a man holds a looking-glaſs above his eye with the reflecting ſurface turned downwards and parallel to the horizon, he will ſee himſelf inverted in the glaſs. In this poſition his head is nearer to the glaſs than his feet, and conſequently, by propoſition 205, in his image on the other ſide the head will likewiſe be nearer to the glaſs than the feet. Therefore the feet of his image will be farther from the center of the earth than its head is, or the image will appear with its feet upwards.

245. *If a plane mirrour is inclined to the horizon ſo as to make with it half a right angle, any object, which is parallel to the horizon, will appear erect in the mirrour; and on the contrary any object, which is erect, will appear parallel to the horizon.*

If AB, Plat. XIV. fig. 1, is a plane looking-glaſs with its reflecting ſurface downwards, and CD is an object parallel to the horizon, then the mirrour AB, which makes half a right-angle with the horizon, will likewiſe make half a right-angle with the object, or ABC will be an angle of 45 degrees. Now at whatever diſtance any radiant C in the object is from the mirrour, the diſtance of the correſponding radiant *c* in the image will be the ſame, by propoſition 205; or CE will be equal to *c*E. For the ſame reaſon any other radiant D in the object will be juſt as far diſtant before the mirrour, as the imaginary radiant *d* is behind it; or DF will be equal to *d*F. And ſince every radiant in the image may in this manner be proved to be at the ſame diſtance behind the mirrour that the real radiant is at before it; it follows, that the image *cd* will make half a right angle *c*BA with the mirrour on one ſide as the object CD makes half a right angle CBA on the other. Conſequently $c\mathrm{BC} = c\mathrm{BA} + \mathrm{CBA}$ is a right angle, or the image *cd* is perpendicular to the object CD; and therefore, being perpendicular likewiſe to the horizon to which the object is parallel, it will appear erect.

If *cd* is the object and the mirrour AB has its reflecting ſurface upwards; then the object is erect; the mirrour is inclined to the horizon at half a right angle; and the image will be CD parallel to the horizon. If CB is an horizontal plane, the object *cd*, becauſe it is erect, by the ſuppoſition, will make a right angle *c*BC with the plane. But ABC the mirrours inclination is one half of this angle, therefore the other half AB*c* is likewiſe half a right angle; that is, the object makes half a right angle with the mirrour before it, and conſequently, from what has been ſaid in proving the other part of the propoſition, the image will make half a right angle with the mirrour behind it. Therefore the object *cd* and its image CD will make two half right angles or one whole right angle with each other. But the object *cd* makes likewiſe a right angle with the horizontal plane CB. Therefore the image CD will be in this plane or will be parallel to the horizon.

246. *If an object is placed between two plane mirrours inclined to one another at any angle, more images than one of that object may be ſeen.*

If two plane mirrours CB and CA, Plat. XIV. fig. 6. are inclined to one another ſo as to contain the angle BCA, and the object F is placed between them; then let FD be drawn from the object perpendicular to the mirrour CA, and in FD, which is the cathetus of incidence, let KD be taken equal to KF, and the image of this object will appear at D, by propoſition 235. In like manner, if FG is drawn perpendicular to the mirrour CB, and G is juſt as far behind the mirrour as F is before it, the image of the ſame object will appear at G, by propoſition 235. Thus an image of the object will appear in each mirrour.

There will be a ſmall difference between theſe two images. For as the object F has one ſide towards the mirrour CA, and the contrary ſide towards the mirrour CB; one of theſe images will repreſent one ſide of the object, and the other image will repreſent the other ſide of it. If F is the face of the ſpectator himſelf, and he looks ſtrait forwards the right ſide of his face will appear in the glaſs CA and the left ſide in the glaſs CB. If he ſtands with his face towards CA, then D will be the repreſentation of his face and G of his back. Indeed if both ſides of the object are alike, as ſuppoſe F to be a candle, then though this difference is ſtill the ſame as in any other object, yet the difference will not be perceivable, becauſe both images will be alike.

But beſides the images D and G there will be many more ſeen in each of the mirrours. Some of the rays, which are reflected from the mirrour CA and diverge from the image D, come to the eye of the ſpectator placed any where between the mirrours, and make that image viſible. But all the reflected rays will not come to the eye; for ſince they diverge from D in all directions, ſome of them will fall upon the oppoſite mirrour CB. So that the image D may be conſidered as an object placed before the mirrour CB: and conſequently when the rays, that apparently diverge from D, are reflected by this mirrour CB, if DHE is drawn perpendicular to CB, it will be the cathetus of incidence, and if EH is taken equal to DH, theſe reflected rays will repreſent the image of D at the point E, which is juſt as far behind the mirrour CB as D is before it, by propoſition 235. In like manner the rays which diverge from this ſecond image E will ſome of them fall upon the mirrour CA, and if a cathetus EM is drawn from E to this mirrour CA, the image of E or a third image of the object will appear in this cathetus at the point M, which is as far behind the mirrour as E is before it. For the ſame reaſon, ſince rays will diverge from M as from an object before the mirrour CB, the image of M or a fourth image of the object F will appear in this mirrour CB, in a cathetus drawn from M to a point as far behind the mirrour as M is before it. And thus as long as the image repreſented in one mirrour is before the other, ſo long an image of that image will be produced. Now all the images, that have been hitherto deſcribed, begin from the mirrour CA, and D is the firſt of them. Therefore whatever ſide of the object is towards CA, all theſe images will repreſent that ſide.

But there will be another ſet of images beginning from the mirrour CB, and of this ſet the image G is the firſt, and conſequently all this ſet will repreſent the ſame ſide of the object that is repreſented at G, or the ſide, which is towards the mirrour CB. For as the rays reflected from CB diverge from G in all directions ſome of them will fall upon the oppoſite mirrour CA; and if the cathetus CL is drawn theſe rays, when reflected from CA, will produce an image of G at the point L, which is as far behind the mirrour as G is before it. In like manner ſome of the reflected rays, which diverge in all directions from L, will be returned to the mirrour CB, and being reflected from thence will produce an image of L. And thus again as long as the image repreſented in one mirrour is before the other, ſo long an image of that image will be produced.

247. *All the images, that appear in two plane mirrours inclined to each other, are in the circumference of a circle, the ſemidiameter of which is the diſtance of the object from the vertex of the angle contained between the mirrours.*

The ſeveral images D, E, G, L, M, &c, Plat. XIV. fig. 6, are in the circumference of a circle, and CF the diſtance of F the object from C, the vertex of the angle made by the mirrours CA and CB, is the ſemidiameter of this circle.

CF the diſtance of the object from the vertex is equal to CD. For FK the diſtance of the object before the mirrour is equal to DK the diſtance

ſtance of the image behind it, by prop. 235; CK is common to both the triangles CFK and CDK; and the angle CKF is equal to the angle CKD, becauſe they are both right ones, ſince FD is the cathetus of incidence. But when two ſides in one triangle are reſpectively equal to two ſides in another triangle, and the angle contained between the equal ſides in one is equal to the angle contained between the equal ſides in the other, thoſe two triangles are equal to each other in all reſpects. Euc. b. I. prop. 4. Therefore the ſide CF in the triangle CFK is equal to the ſide CD in the triangle CDK.

CE is likewiſe equal to CD. For DE is the cathetus, and conſequently the angles CHD and CHE are equal, becauſe they are both right ones; D is the place of one image before the mirrour, and E is the place of the image of this image behind the mirrour, and conſequently DH is equal to EH; and laſtly, in the two triangles CDH, CEH, the ſide CH is common. Therefore all the ſides of one triangle are reſpectively equal to all the ſides of the other, ſo that CE is equal to CD.

Since then CD is equal both to CF and likewiſe to CE, it follows, that CF and CE are equal to one another. And in the ſame manner we might prove that any line drawn from C to G or to L or to M or to any other image, that appears in the mirrours will be equal to CF. Therefore if one foot of the compaſſes is placed at C, and a circle is ſtruck with the opening CF, this circle whoſe ſemidiameter is CF will paſs through D, E, G, L, M, and every other image, that appears in the mirrours. And conſequently all thoſe images are placed ſomewhere in the circumference of a circle, whoſe ſemidiameter is the diſtance of the object F from C the vertical point.

248. *When two plane mirrours are inclined to one another, the images of each ſet end, when each cathetus of incidence ends between the two mirrours continued.*

In propoſition 246, we obſerved that, if two plane mirrours, CA, CB, Plat. XIV. fig. 7, are inclined to one another and any object S is placed between them, there will be two ſets of images ſeen in the mirrours: one ſet begins from the mirrour CA, and *a*1 will be the firſt image in it; the other begins from the mirrour CB, and *b*1 will be the firſt image. Every image in theſe two ſets will be in the circumference of a circle, whoſe ſemidiameter is CS the diſtance of the object from the vertex C, by propoſition 247. And the place of each ſucceeding image is determined by drawing a cathetus of the image to that mirrour before which the preceeding image ſtands, by propoſition 235, and by what was proved in propoſition 246. Thus the image *a*1 ſtands before the mirrour CB, and if a cathetus is drawn from *a*1 to the mirrour CB and is continued to the circumference of the circle at *a*2, this will be the place of the next image. This image *a*2 ſtands before the mirrour CA: therefore if a cathetus is drawn from *a*2 to the mirrour CB and is continued to the circumference at *a*3, this will be the place of the third image in this ſet. This image *a*3 ſtands before the mirrour CB, and conſequently to determine the place of the ſucceeding one, a cathetus muſt be drawn from *a*3 to the mirrour CB and muſt be continued to the circumference of the circle, in which all the images are placed. Now as the angle *a*3CB, which *a*3 makes with the mirrour CB, is an obtuſe one or is greater than a right one, no cathetus or perpendicular can be drawn from *a*3 to the mirrour itſelf. But ſuch a cathetus may be drawn from *a*3 to CL, which is the plane of this mirrour continued from C to L. And the image ſeen by reflection will appear in the cathetus, whether it falls upon the mirrour itſelf or upon the plane of the mirrour continued, by propoſitions 234, 235. Therefore if a cathetus is drawn from *a*3 to CL and is continued to *a*4, this will be the place of the fourth image. The image *a*4 ſtands before the mirrour CA, and to determine the place of the fifth image a cathetus muſt be drawn from *a*4 to CR, which is CA continued, and when this cathetus is continued to the circumference *a*5, this point at the circumference will be the place of the fifth image. The image *a*5 ſtands before the mirrour CB, and conſequently if from *a*5 a cathetus is drawn to CL, and continued to *a*6, this will be the place of the ſixth image. The cathetus *a*5*a*6 ends between CR and CL, which are the two mirrours continued: and the propoſition affirms that the ſet of images which begins from CA, will end, where the cathetus of incidence drawn alternately from one mirrour to the other ends between the two mirrours continued: and conſequently *a*6 is the laſt image of this ſet. That this will be the laſt image is evident from the following conſideration. The reflecting ſurface of the mirrour CB is towards the object S, and ſo likewiſe is the reflecting ſurface of the mirrour CA. Conſequently the image *a*6, which is placed between the mirrours continued to R and L, will be behind both of them, it is behind BCL, and behind ACR. But no rays can be reflected by either mirrour, if they diverge from an image that is behind them. And unleſs there is ſuch a reflection no new image will be produced. Therefore *a*6 the image, which is placed between the two mirrours continued, will be the laſt of this ſet.

In like manner we might trace out the images of the other ſet, which begin from CB. The firſt of them is *b*1, and the laſt *b*6, where the cathetus ends between CR and CL the two mirrours continued. For the reaſon juſt aſſigned is univerſal. Whenever the cathetus ends between the two mirrours continued, the image is behind both of them, and conſequently, as no farther reflection can be made from either of them, this image will be the laſt.

249. *When two plane mirrours are inclined to one another, the angular diſtance between the two firſt images of each ſet is equal to double the angle of inclination.*

The angle of inclination is BCA, Plat. XIV. fig. 7, or is the the angle contained between the two plane mirrours AC and BC. The firſt image of one ſet is $a1$; and the firſt of the other ſet is $b1$. The angular diſtance between theſe firſt images is the angle $b1Ca1$. This angular diſtance is double the angle of inclination; or $b1Ca1$ is equal to twice BCA.

The angle of inclination BCA is equal to the angular diſtance of the object S from one mirrour BC added to its angular diſtance from the other mirrour AC, or $BCA=BCS+ACS$: this is ſelf-evident. Now BCS is equal to $BCb1$, and ACS is equal to $ACa1$, by propoſition 235. Therefore $BCb1+ACa1=BCS+ACS=BCA$. Conſequently $BCb1+ACa1+BCA=2BCA$. But the angular diſtance between $b1$ and $a1$, or between the two firſt images of each ſet is $BCb1+ACa1+BCA$. Therefore this angular diſtance is equal to 2BCA or to double the angle of inclination.

250. *When two plane mirrours are inclined to one another, the angular diſtance between any two images of each ſet, that are produced by the ſame number of reflections, is greater than the angular diſtance between the two preceeding ones by double the angle of inclination.*

The image $a1$, Plat. XIV. fig. 7, is the firſt of one ſet, and the image $b1$ is the firſt of the other: the angular diſtance between them is $b1Ca1$. The image $a2$ belongs to one ſet, and the image $b2$ belongs to the other: each of theſe images is produced by the ſame number of reflections, for each of them is produced by two reflections: the angular diſtance between them is $b2Ca2$. We are therefore to prove that $b2Ca2$ the angular diſtance between $a2$ and $b2$ is greater than $b1Ca1$ the angular diſtance between the two preceeding images $a1$ and $b1$, by twice the angle of inclination BCA. The difference between $a1Cb1$ and $a2Cb2$ is $a1Cb2+b1Ca2$, this difference is equal to 2BCA; for it is equal to $b1Ca1$, and $b1Ca1$ has been proved to be equal to 2BCA, in propoſition 249. That $a1Cb2+b1Ca2=b1Ca1$ may be thus ſhewn. Firſt

$a1Cb2=b1CS$: for $b2CA=b1CA$, by propoſition 235; conſequently, if from theſe equal angles equal angles are taken, the remainders will be equal. But $ACa1=ACS$, by propoſition 235. Therefore $b2CA-ACa1=b1CA-ACS$. But $b2CA-ACa1=a1Cb2$ and $b1CA-ACS=b1CS$. Therefore $a1Cb2=b1CS$. Secondly $b1Ca2=a1CS$: for $a2CB=a1CB$; and $BCb1=BCS$, by propoſition 235; conſequently $a2CB-BCb1=a1CB-BCS$. But $a2CB-BCb1=b1Ca2$, and $a1CB-BCS=a1CS$. Therefore $b1Ca2=a1CS$. Now by the firſt ſtep, $a1Cb2=b1CS$, and by the ſecond ſtep, $b1Ca2=a1CS$. Therefore thirdly $a1Cb2+b1Ca2=b1CS+a1CS=b1Ca1=2BCA$. In the ſame manner we might ſhew that $a3Cb2+b3CA2=a1Cb2+b1Ca2=2BCA$, or that the difference between the diſtances of the third images in each ſet and the diſtances of the ſecond images in each ſet is equal to twice the angle of inclination. And the ſame may be ſhewn in the ſame manner of all the images in the whole circumference.

251. *When two plane mirrours are inclined to each other, as many images may be ſeen in both of them together as there are angles contained in a whole circle equal to the angle of inclination.*

If the angle of inclination BCA, Plat. XIV. fig. 7, is 30 degrees, there are 12 angles of 30 degrees contained in a whole circle, and conſequently in both the mirrours BC and CA 12 images may be ſeen. The diſtance between the images of each ſet encreaſes by double the angle of inclination, as appears from propoſition 250. Therefore in both ſets taken together there will in the whole circumference of the circle be two images for every double angle of inclination, or one image for every angle of inclination.

If the angle of inclination was 90 degrees, by this rule there would be 4 images; if it was 40 degrees there would be 9 images; if 10 degrees there would be 36 images. The rule is indeed liable to ſome reſtrictions according as the angle or double the angle of inclination is or is not exactly commenſurate to a circle, and according as the object is placed in the middle between the mirrour or nearer to one of them than it is to the other. But it will not be worth the while to trouble the reader with theſe reſtrictions. One of them however we may take notice of. If the double angle of inclination is exactly commenſurate to a circle, the diſtance between the two laſt images of each ſet will be the circumference of a circle, and conſequently theſe two images will be in the ſame place. Thus if the angle of inclination is 30 degrees the double angle of inclination is 60 degrees: the firſt two images $a1$ and $b1$ are 60 degrees aſunder;

der; the two next *a*2 and *b*2 are 60+60=120 degrees aſunder; the two next *a*3 and *b*3 are 120+60=180 degrees aſunder; the two next *a*4 and *b*4 are 180+60=240 degrees aſunder, the two next *a*5 and *b*5 are 240+60=300 degrees aſunder; and the two laſt *a*6 and *b*6 are 300+60=360 degrees aſunder. Therefore *a*6 and *b*6 are at *ab*6 the ſame point of the circle, and conſequently coincide, ſo that though there are 12 images only 11 will be ſeen in either mirrour.

252. *If two plane mirrours are parallel to one another and an object is placed between them, innumerable images of that object may be ſeen in each of them; and all the images will ſtand in a right line.*

This propoſition may be deduced from propoſitions 251, 247. In two plane mirrours BC and AC, Plat. XIV. fig. 7, that are inclined to one another, as many images may be ſeen as there are angles of inclination contained in a whole circle, and all theſe images will ſtand in the circumference of a circle, whoſe ſemidiameter is CS the diſtance of the object S from the vertex C of the angle of inclination BCA. Now if the two mirrours were to continue at the ſame diſtance from one another at B and A, but were to be opened a little at C ſo as to make a ſmaller angle, or to be leſs inclined, or to be nearer parallel to one another; by this means the number of images will be encreaſed, by propoſition 251; becauſe the angle contained between the mirrours is diminiſhed: and at the ſame time the circumference of the circle, in which the images are placed, will be enlarged, by propoſition 247; becauſe, when the mirrours continue at the ſame diſtance from one another at A and B, and are opened wider at the other end C, the vertex of the angle contained between them will be more remote than C, and conſequently the object will be more remote from the vertex, or SC will be encreaſed; but SC is the ſemidiameter of the circle in which the images are placed, therefore as this ſemidiameter is encreaſed the circumference will be encreaſed. By opening the mirrours ſtill farther at the end towards C, the number of images will keep encreaſing, and ſo will the circumference of the circle, in which theſe images are placed. And we muſt farther obſerve that, when the mirrours are far aſunder towards C, the images in that part of the circle, which is beyond C, will not appear: becauſe, unleſs the mirrours are cloſe together at that end, there will be no reflecting ſurface to make them viſible. From hence then it follows, that the many images, which are ſeen, ſtand in a part of the circumference of a circle, and that the arcs, in which they ſtand, are parts of a greater circle in proportion as the mirrours are nearer parallel to one another. Now the greater the circumference of any circle is, the leſs is the curvature of any given part of it. Conſequently as the mirrours, by opening them towards C, become nearer parallel to one another, the curvature of the lines in which the images are placed, will be leſs. But when the mirrours are parallel, the angle contained between them is nothing; ſo that innumerable ſuch angles are contained in a circle, therefore there will be innumerable images, by propoſition 251. And ſince, when the mirrours are parallel, the vertex C is at an infinite diſtance. SC the diſtance of the object or the ſemidiameter, and conſequently the circumference of the circle, in which the images are placed, will be infinite. But only a part of this circumference will be ſeen in each mirrour; and any finite arc of an infinite circle has no curvature or is a right line. Therefore the images in each mirrour will ſtand in a right line.

253. *In one ſingle plane mirrour, that is made of thick glaſs with quickſilver behind it, many images of any bright object may be ſeen.*

Suppoſe the object to be a candle. Some of the rays, which fall upon the glaſs, will be reflected from the firſt ſurface without entering it, by propoſition 198. Theſe rays produce the firſt image, which will not be a very bright one. But moſt of the rays will enter the glaſs, and when they are reflected from the quickſilver behind the glaſs, they will come out of the glaſs again through the firſt ſurface, and produce the ſecond image, which will be the brighteſt of them all. But when the rays are thus reflected from the quickſilver, though moſt of them come out of the glaſs, yet thoſe which are in a fit of eaſy reflection will not come out there, but will be returned back to the quickſilver: and when they are reflected from thence a ſecond time, they will produce a third image. Neither will all the rays, that are thus reflected a ſecond time from the quickſilver, come out at the firſt ſurface of the glaſs, and produce this third image; for ſuch of them, as are in a fit of eaſy reflection, when they come to the firſt ſurface, will be returned again to the quickſilver and being reflected from thence a third time will produce a fourth image. As often as this happens the image will be repeated, ſo that, if the eye and the candle are oblique to the glaſs nine or ten images may be ſeen. But as there are fewer rays left to produce every image after the ſecond, all after this grow fainter, till the laſt of them are ſo faint as to be ſcarcely viſible.

CHAP.

CHAP. XVIII.

Of reflected viſion in concave and convex mirrours.

254. *In ſpherical mirrours, whether concave or convex, when the place of the laſt image is determinate, the object and the laſt image are in the ſame ſituation if they are both on the ſame ſide of the center, but in contrary ſituations, if they are on oppoſite ſides.*

LET *abc*, Plat. XIV. fig. 2, be an object placed nearer to the concave mirrour *gil* than its principal focus; and let *k* be the center of the concavity. Becauſe the radiants *a*, *b*, *c*, &c. in the object are nearer, by the ſuppoſition, than the mirrours principal focus, the reflected rays will diverge, by propoſition 220, and the diſtances of the correſponding imaginary radiants *d*, *e*, *f*, &c. may be determined, by propoſition 221. Now as *k* is the center of the concavity, the real radiants in the object *abc* are on the ſame ſide of the center *k* with the imaginary radiants in the laſt image *d*, *e*, *f*. In this caſe the object and image will be in the ſame poſition. For the imaginary radiant, which correſponds to the real one *a*, is in the cathetus of incidence drawn from *a* to the place of the image; and ſo likewiſe the imaginary radiant, which correſponds to *c* is in the cathetus drawn from *c* to the place of the image, by propoſition 234. But the cathetus of incidence, becauſe it is perpendicular to the mirrours ſurface, always paſſes through the center; ſo that *kad* is the cathetus of one extreme radiant, and *kcf* is the cathetus of the other extreme radiant. Since therefore theſe cathetuſes are all drawn from the center, they croſs each other no where but at the center; and conſequently if the object *abc* and the image *def* are on the ſame ſide of the center, that is, if the center *k* is not between them, the extreme cathetus *ad* cannot croſs the other extreme cathetus *cf* in any place between the object and the image. Upon this account the two extreme cathetuſes, as they have not croſſed, will be in the ſame poſition in reſpect of each other both at the object and at the image; that, which was the higheſt at the object, will be the higheſt at the image, and that, which was the loweſt at one, will be the loweſt at the other. But every point of the object appears in its cathetus of incidence at the image, by propoſition 234. Therefore the higheſt point in the object will appear the higheſt in the image, and the loweſt point in the object will appear the loweſt in the image, that is, the object and image will be in the ſame ſituation.

In like manner if the object *abc*, fig. 4, is placed before a convex mirrour *gil*, the rays, which diverge at their incidence, will diverge after re-

reflection, by propoſition 228, and the place of the imaginary radiants *def*, which compoſe the laſt image is determined by propoſition 229. And here each point of the object appears at its cathetus of incidence in the image, by propoſition 234. Therefore if the object *abc* and the image *def* are both on the ſame ſide of *k* the center, the extreme cathetuſes *ak* and *ck*, which meet at the center but croſs each other no where elſe, will not croſs between the place of the object and the place of the image; and conſequently that cathetus, which is the higheſt or the loweſt at the object, will be the higheſt or the loweſt at the image. Therefore if *a* is the higheſt point of the object, it will appear at *d* the higheſt in the image; and *c*, if it is the loweſt point at the object, will appear at *f* the loweſt in the image, ſo that the image will be in the ſame ſituation with the object.

But if the object *abc*, fig. 3, is farther from the concave mirrour *gil* than its principal focus, the reflected rays will converge, by propoſition 213, and the place of the focuſes, to which they converge, is determined, by propoſition 214. Let *fed* be the place of the ſeveral focuſes, to which the rays, that come from the radiants *a*, *b*, *c*, &c. converge after reflection, then if a white paper is held at that place a diſtinct picture of the object will be painted upon the paper, by propoſition 84. For the effect in this reſpect is planely the ſame whether the rays are collected into their reſpective focuſes by refraction or by reflection. But if the eye is more remote from the mirrour than the place of the diſtinct picture, and the paper is taken away then the rays, that are collected into the ſeveral focuſes *def*, as they paſs on from thence, will diverge from theſe focuſes as from ſo many radiants, and the diſtinct picture will become the laſt image, as in propoſition 138. Now the radiants *a*, *b*, *c*, which compoſe the object, and the focuſes *d*, *e*, *f*, which compoſe the diſtinct picture, are always on contrary ſides of the center *k*, by propoſition 219. Upon which account the picture or laſt image will be inverted in reſpect of the object. For here likewiſe each point of the object, as *a* or *c*, will appear in its cathetus of incidence, by propoſition 234. The extreme cathetuſes *gad* and *lcf* croſs each other at *k*, which is the center of the concavity and, by the ſuppoſition, is between the object and the image. But, when the cathetuſes have croſſed each other between the object and the image, that, which is the higheſt at one, will be the loweſt at the other: and conſequently, if *a* is the higheſt point of the object, it will appear at *d* the loweſt in the image, and for the ſame reaſon if *c* is the loweſt point of the object, it will appear at *f* the higheſt in the image, ſo that the image in reſpect of the object will be inverted.

If *def* was the object, then *abc* would be the image, that is, if the real radiants were at *d*, *e*, *f*, the focuſes would be at *a*, *b*, *c*, by propoſition 219. Therefore in this caſe likewiſe the object and image would be on contrary ſides of the center, and the image *abc* would be inverted in reſpect of the object *def*.

255. *In ſpherical mirrours, whether convex or concave, the diameter of the object is to the diameter of the laſt image as the diſtance of the object from the center to the diſtance of the image from the center.*

The length of the object *ac*, Plat. XIV. fig. 2, 3, 4, bears the ſame proportion to the length of the image *df*, that *ak* the diſtance of the object from the center of the ſpherical ſurface, bears to *dk* the diſtance of the image from the ſame center. If the eye is any where in the line *ib* or in that line continued, the length of the object and the length of the image conſidered as viſible extenſions are parallel to one another: for in this poſition of the eye *bi* is the optic axis. Therefore the viſible length of both is proportional to a ſubtenſe of the optic angle perpendicular to *bi*, by propoſition 129. But if their lengths conſidered as viſible extenſions are perpendicular to the ſame right line, they muſt be parallel to one another. The ſame thing would be equally true, if the eye was in the line *lc* or *ga* or any other line perpendicular to the mirrours ſurface. We may therefore ſay univerſally, that the viſible length of the object is parallel to the viſible length of the image. The angle *akc* is equal to the angle *dkf*. They muſt be equal in fig. 2, and 4, becauſe they are one and the ſame angle; and in fig. 3 they are equal, becauſe they are vertical. Euc. b. I. prop. 15. The angle *kac* is likewiſe equal to the angle *kdf*. For, in fig. 2, 4, one is the external angle and the other is the internal oppoſite one on the ſame ſide made by the right line *kd* croſſing the two parallel lines *ac* and *df*; and in fig. 3, they are the alternate angles. Euc. b. I. prop. 29. For the ſame reaſon the angle *kca* is equal to the angle *kfd*. Since therefore the three angles of the triangle *ack* are reſpectively equal to thoſe of the triangle *dkf*; the ſides of theſe triangles are proportional, Euc. b. VI. prop. 4. *ac* the viſible length of the object is to *df* the viſible length of the image, as *ak* the diſtance of the object from the center to *dk* the diſtance of the image from the center.

256. *In ſpherical mirrours, whether concave or convex, the diameter of the object is to the diameter of the image, as the diſtance of the object from the ſurface of the mirrour to the diſtance of the image from the ſame ſurface.*

When the rays, that are reflected from a concave or convex mirrour, diverge, as in fig. 2, 4, Plat. XIV, the diſtance of the real radiant from the ſurface of the mirrour is to its diſtance from the center, as the diſtance of the imaginary radiant from the ſurface to its diſtance from the center, by propoſitions 221, 228. And when the rays that are reflected from a concave mirrour, converge, as in fig. 3, the diſtance of the radiant from the ſurface is to its diſtance from the center, as the diſtance of the focus from the ſurface to its diſtance from the center, by propoſition 214. Now the object *abc* conſiſts of real radiants, and the laſt image *def* either of imaginary radiants or of focuſes, as was obſerved under propoſition 254. Therefore *bi* the diſtance of the object from the ſurface is to *bk* the diſtance of the object from the center, as *ei* the diſtance of the image from the ſurface is to *ek* the diſtance of the image from the center. And conſequently the diſtance of the object from the ſurface is to the diſtance of the image from the ſurface, as the diſtance of the object from the center is to the diſtance of the image from the center; that is, ſince as *bi* to *bk*, ſo *ei* to *ek*, it will likewiſe be as antecedent to antecedent ſo conſequent to conſequent, or as *bi* to *ei* ſo *bk* to *ek*. Euc. b.V. prop. 16. But the diameter of the object is to the diameter of the image, as the diſtance of the object from the center to the diſtance of the image from the center, or *ac* is to *df* as *bk* to *ek*, by propoſition 254. Therefore the diameter of the object is likewiſe to the diameter of the image as the diſtance of the object from the ſurface to the diſtance of the image from the ſurface, or *ac* is to *df*, as *bi* to *ei*. We may bring the parts of this demonſtration nearer together in this manner. *bi* is to *bk*, as *ei* to *ek*, by propoſitions 221, 228, 214. Therefore *bi* is to *ei*, as *bk* to *ek*, Euc. b.V. prop. 16. But *ac* is to *df* as *bk* to *ek*, by propoſition 254. Therefore *ac* is likewiſe to *df* as *bi* to *ei*.

257. *If the eye is cloſe either to a concave or convex-mirrour, the apparent diameter of the object is equal to the apparent diameter of the laſt image.*

The apparent diameters of viſible objects are equal, when their real diameters are as their diſtances from the eye, by propoſition 128. But the real diameters of the object and image are as their reſpective diſtances from the ſurface of the mirrour, by propoſition 256. Therefore if, the eye is contiguous to the mirrours ſurface, their real diameters will be to one another as their reſpective diſtances from the eye, and conſequently their apparent diameters will be equal. Thus if the eye is at *i*, Plat. XIV.

fig. 2, 3, 4, the apparent length of the object *ac* will be equal to the apparent length of the image *df*, or both of them will ſubtend the equal angles *aic* and *dif*.

258. *If the eye is placed in the center of a concave mirrour, it can ſee nothing in the mirrour but its own image.*

The apparent length of the object and image ſeen from the center of a ſpherical mirrour, whether concave or convex, ought to be equal, by propoſition 128: becauſe their real diameters are as their reſpective diſtances from the center, by propoſition 254, and conſequently as their reſpective diſtances from the eye, if the eye could be placed in that center, or could ſee the image of the object in the mirrour, when it is placed there. But *k* the center of any convex mirrour *gil*, Plat. XIV. fig. 4. is behind it, and conſequently if the eye was placed there, it could ſee nothing at all by reflection from the mirrour: and though *k*, fig. 2. 3, the center of the concave mirrour *gil* is before it, yet the eye, when placed in that center, will ſee nothing in the mirrour but its own image. For in this ſituation of the eye, all the rays, which diverge from it, will converge to it after reflection, by propoſition 218, and no other rays from any part of the mirrour can enter it but thoſe which come from itſelf: upon which account it will ſee its own image ſpread over the whole ſurface of the mirrour.

Or otherwiſe. When the eye is in the center of the concavity, the diſtinct picture or laſt image of the eye will be there too, by propoſition 218. Conſequently the eye being at the place of its own diſtinct picture will ſee itſelf infinitely magnifyed, and ſpread over the whole ſurface of the mirrour. That any object, and conſequently that the eye when it is its own object, will appear infinitely magnifyed to an eye placed at the diſtinct picture has already been proved, in propoſition 153, where this picture is produced by refraction. But the ſame reaſoning is equally applicable, whether the picture is produced by refraction or by reflection.

259. *If an object is nearer to a concave mirrour than its principal focus, but does not touch the mirrour; then the image appears behind the mirrour, is farther diſtant from the mirrour and greater than the object, and is erect and diſtinct.*

If any object *abc*, Plat. XIV. fig. 2, is nearer to the concave mirrour *gil* than the mirrours principal focus; the rays, which diverge from each point in the object before the mirrour, will diverge after reflection from as many points behind it, but leſs than they did at their incidence, by propoſition 220. The imaginary radiants *d*, *e*, *f*, or the image, will be farther from the mirrour than the real radiants *a*, *b*, *c*; by propoſition 29; becauſe the reflected rays diverge leſs than the incident ones. And the place of the laſt image *def* is determined, by propoſition 221. But if the image is farther from the mirrour than the object is, the image will be bigger than the object, by propoſition 256. Since the object, by the ſuppoſition, is nearer than the principal focus, it muſt be nearer likewiſe than the center; conſequently the object *abc* and the image *def* are both on the ſame ſide of the center *k*; and upon this account the image will be erect or in the ſame poſition with the object, by propoſition 254. Laſtly the image *def* will be ſeen diſtinctly, becauſe the rays from it diverge but not very much, juſt as thoſe do that come from objects, which are at a moderate diſtance and which the naked eye can ſee diſtinctly.

260. *If an object touches a concave mirrour the image will touch it likewiſe and will be equal to the object.*

When the real radiants, or object *abc*, Plat. XIV. fig. 2, come cloſe to the ſurface of a concave mirrour, the imaginary radiants, or laſt image *def*, will likewiſe be cloſe to it, by propoſition 225. And when they are both of them cloſe to the ſurface of the mirrour, their diameters will be equal, by propoſition 256.

261. *If an object is placed in the principal focus of a concave mirrour, the image is at an infinite diſtance behind the mirrour, is bigger than the object, and is erect and diſtinct.*

When the object *abc*, Plat. XIV. fig. 2, which conſiſts of real radiants, is in the principal focus of the concave mirrour *gil*, the laſt image *def*, which conſiſts of imaginary radiants, is at an infinite diſtance, by propoſition 211. The image will be larger than the object upon account of its remoteneſs, by propoſition 256. It will be ſeen erect, becauſe it is on the ſame ſide of the center with the object, by propoſition 254. And ſince the reflected rays of each beam are parallel to one another, by propoſition 211, it will be ſeen as diſtinctly as the naked eye ſees any remote objects.

262. *If an object is farther from a concave mirrour than its principal focus, and the eye is nearer than the place of the diſtinct picture; the image will be confuſed, will be behind the mirrour, will appear erect and will be larger than the object.*

Let *abc*, Plat. XIV. fig. 3, be an object, more remote from the concave mirrour *gil* than the mirrours principal focus; then the rays of each beam, that diverge from the ſeveral radiants *a*, *b*, *c*, &c. in the object, will be collected into as many focuſes *d*, *e*. *f*, &c. by propoſition 213, and if a paper is placed at *def*, the inverted image of the object will be painted there, in the ſame manner as theſe focuſes, if the rays were brought together by refraction, would produce ſuch a picture, by propoſitions 84, 85. Now if the eye is any where between *c* and *i*, that is, if it is nearer to the mirrour than the diſtinct picture the rays of every beam are converging, when they come from the mirrour to the eye, and conſequently the image, which appears by reflection in the mirrour, will be confuſed. For ſince no rays of the ſame beam converge naturally, but either are parallel or elſe diverge, nature has not given us a power of changing the figure of the eye ſo as to ſee diſtinctly by ſuch rays. Now in this poſition of the object *abc*, the rays *ag* and *ai*, diverge from the ſame point *a* when they fall upon the mirrour, and *ag*, becauſe it is perpendicular to the ſurface will be reflected in the line *ga*, and *ai* will be reflected in the line *id*, ſo that both of them will converge to *d*. But if the eye is nearer to the mirrour than *d*, the reflected ray *id* will not croſs its cathetus of incidence *gd* in any place before the eye; for if theſe rays *dg* and *di* are continued ever ſo far before the eye, inſtead of croſſing one another they would be farther aſunder: conſequently, by propoſition 234, the apparent place of this or any other point of the image ſeen in the mirrour will be indeterminate: and upon this account we can apply neither propoſition 254 to find out whether the image will be erect or inverted, nor propoſition 256 to find out whether it will be ſmaller or larger than the object. However we may ſhew that it will appear behind the mirrour at an indefinite diſtance. For if the diſtance of the image was finite, the rays that come from it would diverge; but the rays that come from it converge; therefore the apparent diſtance of it is indefinitely great, and conſequently it will ſeem more remote than the mirrour or will appear to be behind the mirrour. We may likewiſe ſhew that the image ſeen in the mirrour will be erect. For *fed* is the inverted picture of the object, and, if it was received upon a paper, would be diſtinct; but if the eye is nearer than the place of this picture, then, inſtead of being painted upon the paper, it will be painted upon the retina, and though it would be diſtinct upon the paper it will be confuſed upon the retina, by propoſition 113, becauſe the viſion in this caſe is confuſed, as has been already proved. But though the picture is confuſed upon the retina, it will ſtill be inverted. Therefore, by propoſition 115, the viſion will be erect; or the image in the mirrour, which is ſeen by means of this picture upon the retina, will appear in the ſame poſition with the object. Laſtly we may ſhew that the image will be larger than the object. For the object *abc* conſiſts of radiant points: but the rays of each beam, as *gd*, *id*, or *lf if*, converge after reflection; and conſequently the rays of any reflected beam will appear to come from a circular ſpot inſtead of a point. And ſince ſuch circular ſpots are bigger than points, the object of reflected viſion, or the image, which conſiſts of ſuch ſpots, will be bigger than the object itſelf, which is only made up of as many radiant points.

263. *When an object is placed in the principal focus of a concave mirrour, the apparent magnitude of the image ſeen in the mirrour is not altered by the departure of the eye. When the object is nearer than the principal focus, the apparent magnitude of the image decreaſes as the eye departs. When the object is more remote than the principal focus, then as the eye departs the apparent magnitude of the image encreaſes; provided it does not depart beyond the place of the diſtinct picture.*

Here we muſt recollect what has been proved in propoſitions 152, 153, in the caſe of refracted viſion: for it is equally true in reflected viſion. If the eye departs from the laſt image, its apparent magnitude will decreaſe. If the eye approaches to the diſtinct picture, the apparent magnitude of the laſt image will encreaſe. Firſt then, if the object is in the principal focus of the mirrour, the reflected rays of each beam are parallel to one another, by propoſition 211. Therefore the diſtance of the imaginary radiants, or laſt image, behind the mirrour, and of the focuſes, or diſtinct picture, before it, will be infinite, by propoſitions 30, 31. For this reaſon as the eye departs from the mirrour, it neither gets farther from the laſt image, nor nearer to the diſtinct picture: and conſequently the apparent magnitude of the laſt image will neither decreaſe nor encreaſe. Secondly, if the object is nearer than the principal focus of the mirrour, the reflected rays of each beam diverge from an imaginary radiant at a determinate diſtance behind the mirrour, ſo that the place of the laſt image which conſiſts of theſe radiants is determinate, by propoſitions 220, 221. But becauſe the reflected rays diverge, there will be no focuſes, or no diſtinct picture before the mirrour. Conſequently as the eye departs from the mirrour, it gets farther from the laſt image but no nearer to the diſtinct picture: and upon this account the apparent magnitude of the laſt image will decreaſe in the eyes departure, without encreaſing at all. Thirdly, if the object is more remote than the principal focus, the reflected rays of each beam converge to focuſes at a determinate diſtance, by propoſitions 213, 214. But the laſt image is at an indefinite diſtance, as was obſerved in propoſition 262. Therefore as the eye departs from the mirrour, its gets nearer to the diſtinct picture but no farther from the infinitely diſtant laſt image: and upon this account in the departure of the eye the apparent magnitude of the laſt image will keep encreaſing, without decreaſing at all. This will be the caſe, till the eye gets to the place of the diſtinct picture, and then the laſt image ſeen in the mirrour will be infinite, as was obſerved in a ſimilar caſe of refracted viſion, propoſition 153.

264. *When the eye is in the principal focus of a concave mirrour, the apparent magnitude of the laſt image is not altered by the departure of the object from the mirrour. When the eye is nearer than the principal focus, as the object departs, the apparent magnitude of the laſt image will decreaſe. When the eye is more remote than the principal focus, as the object departs, the apparent magnitude of the laſt image will encreaſe, provided it does not depart ſo far as to bring the diſtinct picture between the eye and the mirrour.*

This is proved in the ſame manner with the ſimilar caſes of refracted viſion through a convex lens, propoſition 156.

265. *If an object is farther from a concave mirrour than its principal focus, and the eye is farther from the mirrour than the place of the diſtinct picture, the image appears before the mirrour, is inverted, and may be ſeen diſtinctly.*

When the object *abc*, Plat. XIV. fig. 3, is at any diſtance *bi*, which is greater than the principal focal diſtance of the mirrour *gil*, the rays, that come from each point in the object *abc*, will converge to as many points *def*, by propoſition 213, 214; and upon a white paper held there would paint the inverted picture of the object, by propoſitions 83, 84. Now if the paper was taken away, the rays of each beam, after they had met at the ſeveral focuſes, *d*, *e*, *f*, would go ſtrait forwards and diverge from thoſe focuſes, by propoſition 15. Therefore to the eye placed any where beyond *def*, the diſtinct picture will be the laſt image, by propoſition 233. But the diſtinct picture is before the mirrour, and is inverted: conſequently the image, in this poſition of the eye and object, will likewiſe be before the mirrour and will be inverted. The rays of each beam, after they have paſſed their focus, diverge from their focus as from a radiant: conſequently if the eye is not too near the image, the rays, that come from it, will diverge, when they fall upon the eye, juſt as rays do that come from the ſeveral radiants in an object, when we view that object at a moderate diſtance with the naked eye: ſo that as objects appear diſtinct to the naked eye at a moderate diſtance, this image before a concave mirrour will likewiſe appear diſtinct, if the eye is not too near it.

266. *When the image appears before a concave mirrour; if the object is nearer to the mirrour than the center of its concavity, the image will be larger than the object; if the object is more remote than the center, the image will be ſmaller than the object; and if the object is in the center, the image will be equal to the object.*

What poſition of the object and eye is requiſite in order to make the image appear before the mirrour, has already been ſhewn in propoſition 265. Now if the object *abc*, Plat. XIV. fig. 3, is between *k* the center of the concavity and *gil* the ſurface of the mirrour, ſo that the diſtance *ib* is more than half *ik*, it will then be nearer than the center but more remote than the principal focus, by propoſition 210. But when the object *abc* conſiſting of real radiants is on one ſide of the center, the diſtinct picture or laſt image *def* conſiſting of focuſes will be on the contrary ſide of the center, by propoſition 219. Therefore the image will be farther from the mirrour than the object is, and conſequently will be bigger than the object, by propoſition 256. If the object was at *fed* more remote from the mirrour than *k* the center of its concavity, then the diſtinct picture or image would be at *abc* on the contrary ſide of the center, by propoſition 219. Therefore the image in this caſe will be leſs than the object, becauſe it is nearer than the object to the ſurface of the mirrour, by propoſition 256. Laſtly if the object is in the center of the concavity at *k*, the image will be there too, by propoſition 218. Conſequently the object and image will be equal to each other, by propoſition 256, ſince they are both of them at the ſame diſtance from the mirrours ſurface.

In the motion of the object it will be eaſy to ſee both the motion of the image, and the alterations that are made in the magnitude of the image whilſt it moves. If a perſon ſtanding at ſome diſtance and looking at a concave mirrour *gil* ſtretches out his arm ſo that his hand is at *e*, he may ſee the image of his arm come out from the mirrour towards the place where he ſtands, and the image of his hand at *b* will be leſs than his hand. Whilſt he moves his hand from *e* towards the mirrour, the image of his hand will come forwards from *b* to meet it: and this image, as it comes forwards, will keep encreaſing in magnitude. When his hand gets to *k*, the image of it will meet it at *k*, and will then be equal to it. If he moves his hand ſtill farther towards the mirrour, the image of it will then be nearer to him than his hand itſelf is; for the hand and image will always be on the contrary ſides of the center *k*; ſo that if his hand was at *b* the image of it would be at *e*, and would then be bigger than his hand.

267. *When the image appears before a concave mirrour; if the eye moves either to the right hand or to the left, the image will appear to move the contrary way; but if the mirrour moves either to the right hand or to the left, the image will appear to move the ſame way.*

This propoſition in reflected viſion from a concave mirrour is the ſame and is proved in the ſame manner with propoſition 162 in refracted viſion through a convex lens.

268. *The inverted pictures of objects may be repreſented in a dark room by means of a concave mirrour.*

If *def*, Plat. XIV. fig. 3, is an object abroad and the rays which come from the ſeveral points in that object paſs through a hole in a window-ſhutter into a dark room, and then fall upon a concave mirrour *gil* within the room, they will be collected into as many points or focuſes *abc*; and a paper placed at *abc* to receive theſe focuſes will have the inverted picture of the object *def* repreſented upon it. This has frequently been ſhewn already in the foregoing propoſitions.

269. *The rays of the ſun, when they are reflected from a concave mirrour, will kindle fire.*

Becauſe the ſun is very remote, the rays of each beam are parallel to one another; and conſequently after reflection from a concave mirrour they will converge to the principal focus of the mirrour, in the ſame manner as they are collected by refraction in the principal focus of a convex lens. See propoſition 53. And the heat produced by this means will be ſufficient to ſet fire to ſuch bodies as are placed before the mirrour at the principal focal diſtance, which is half a ſemidiameter of the mirrours concavity.

The different degrees of heat at the focus of different concave mirrours is eſtimated in the ſame manner as we have already eſtimated the different degrees of heat at the focus of different convex lenſes, in propoſitions 107, 108, 109.

The ancients made uſe of concave mirrours to rekindle the Veſtal fires. Plutarch in the life of Numa ſays that the inſtruments uſed for this purpoſe

poſe were σκαφεῖα, or diſhes, which were made hollow by the ſide of a right-angled triangle, and that this triangle, for ſo I would underſtand him, converges from a circumference to a center. The diſh ſo formed is, he ſays, placed oppoſite to the ſun, and the combuſtible matter is placed in the center. Let us try what we can make out from this perplexed deſcription. The quadrant or mixed triangle ACB, Plat. XIV. fig. 8, agrees very well with the deſcription of the triangle, by which theſe burning-diſhes were made hollow. It is a right-angled triangle, for ACB is a right angle; it is not indeed a plane triangle or does not conſiſt of three right lines, and Plutarch does not deſcribe it as a plane triangle, but ſays that it converged from a circumference to a center; and thus from a circumference, or rather part of a circumference ADB, the two ſides AC and BC do converge to one and the ſame point C, which is the center of the ſide ADB. Now if the quadrant ACB was made of ſteel, and the edge ADB was ſharp, by turning this edge round in a thick plate of braſs or other metal, the braſs would be ſcraped away and the plate would be made hollow. The concavity ſo made in the plate would be a concave ſpherical mirrour: and ſuch a mirrour placed oppoſite to the ſun would collect the rays into a focus and would anſwer the purpoſe, for which Plutarch ſays thoſe diſhes were uſed. The only difficulty, which remains, is, that Plutarch tells us the combuſtible matter was placed in the center; whereas the principal focus of a concave mirrour is not in the center, or at the diſtance of a ſemidiameter of the concavity, but at the diſtance of half a ſemidiameter. But we may well ſuppoſe that Plutarch was miſtaken in this particular: he may ſay that the combuſtible matter was placed in the center of the concavity, not becauſe it was placed there, but becauſe he imagined this point to be the mirrours principal focus. For the author of a treatiſe of catoptrics, which goes under Euclids name, has fallen into the ſame miſtake, and ſupppoſes the center of a mirrours concavity to be the burning point. Euc. catopt. prop. XXXI. And though we may be very certain that the author of this treatiſe was not Euclid the geometrician, yet, whoever he was, he underſtood mathematics in all probability as well as Plutarch. So that if he could make this miſtake, Plutarch might make it too.

270. *When any object is placed before a convex mirrour, the image of that object appears behind the mirrour, is nearer to the mirrour than the object, is leſs than the object, is ſeen diſtinctly, and is erect.*

If the object *abc*, Plat. XIV. fig. 4, is placed before the convex mirrour *gil*, the rays, that diverge before reflection from each radiant

a,

a, *b*, *c*, &c. in the object will diverge after reflection from as many radiants *d*, *e*, *f*, &c. behind the mirrour, by propoſitions 228, 229, and theſe imaginary radiants compoſe the image that is ſeen in the mirrour, by propoſition 233. The reflected rays diverge more than the incident ones, by propoſition 228, and conſequently, by propoſition 28, the imaginary radiants or image *def* will be nearer to the mirrour than the real radiants or object *abc*. But if the image is nearer to the ſurface of the mirrour it will likewiſe be leſs than the object, by propoſition 256. And ſince the reflected rays diverge from the ſeveral points in the image *def*, juſt as they would from any viſible object placed there, this image will be ſeen as diſtinctly, as any object could be ſeen at the ſame diſtance from the eye. Laſtly the image will be erect, by propoſition 254, becauſe it is on the ſame ſide of the center *k* with the object. That the image will always be on the ſame ſide of the center with the object may be eaſily ſhewn. For ſuppoſe the ſeveral radiants or object *abc* was at an infinite diſtance, ſo that the rays of each beam were parallel, then the imaginary radiants or image *def* would be in the principal focus of the mirrour, that is, it would be in the middle between *k* the center and *i* the ſurface, by propoſition 28. Therefore in the greateſt diſtance of the object, the image will be on the ſame ſide of the center *k* that the object is on. But as the real radiants or object approaches to the mirrour, the imaginary radiants or image will likewiſe approach to it, by propoſition 230. Conſequently at any leſſer diſtance of the object the image will be nearer to the ſurface at *i* and farther from the center *k*. Therefore at any leſſer diſtance of the object, it will ſtill be on the ſame ſide of the center with the object.

171. *The image in a convex mirrour is equal to the object, if the object is cloſe to the mirrour.*

If the object *abc*, or all the real radiants which compoſe that object, moves towards the ſurface of the mirrour *gil*, then the image *def*, or all the imaginary radiants which compoſe that image, will move towards the ſurface at the ſame time, by propoſition 230: and when the object is cloſe to the mirrour the image will likewiſe be cloſe to it, by propoſition 231. But when the object and image are both of them cloſe to the mirrour their diameters will be equal, by propoſition 256.

272. *When either the eye or the object departs from a convex mirrour, the apparent diameter of the image decreaſes.*

When the object *abc*, Plat. XIV. fig. 4, is repreſented at *def* in the convex mirrour *gil*, if the object continues fixed in its place, the image *def* will always be at the ſame diſtance from the mirrour, by propoſition 229, and the real diameter of the image will be invariable, by propoſition 256, Conſequently the apparent diameter of the image will be inverſely as the eyes diſtance, by propoſition 126, ſo that, as the diſtance of the eye from the mirrour encreaſes, the apparent diameter of the image will decreaſe.

If the object *abc* depart from the mirrour, that is, if *ib* the diſtance of the object encreaſes, the proportion, which *ib* bears to *ie* the diſtance of the image, will encreaſe. Therefore by propoſition 256, the proportion, which the diameter of the object bears to the diameter of the image, will likewiſe encreaſe, or the image will be leſs in reſpect of the object. But when the eye continues in the ſame place, the apparent diameter of the image will be directly as its real diameter, by propoſition 127. Conſequently ſince in the departure of the object the real diameter of the image decreaſes, its apparent diameter will likewiſe decreaſe.

CHAP. XIX.

Of reflecting teleſcopes and ſome other optical inſtruments.

273. *The portable camera obſcura conſiſts of a convex lens and a plane mirrour.*

ALL the optical part of a portable camera obſcura is repreſented, Plat. XIV. fig. 9. it conſiſts of CD a convex lens, and HK a plane mirrour inclined at half a right angle. The lens CD is fitted into a tube, that conſiſts of two joynts, one of which ſlides into the other like the tube of a common perſpective glaſs. One joynt of this tube is fixed in a hole at one end of a box, ſo that as the other joynt into which the lens is fitted ſlides in it, the lens may be moved nearer to or farther from EG the oppoſite end of the box. If we conſider this box as a dark room, a diſtinct inverted picture of any object AB may, by means of the lens CD, be painted as EG, if the lens is drawn out to a proper diſtance, ſo as to make the focuſes of each pencil of rays proceeding from the object fall at EG, by propoſitions 84, 85. But if the rays of each pencil, before they come to their ſeveral focuſes at EG fall upon the plane mirrour KH, which makes an angle of 45 degrees with the bottom of the box and conſequently with EG the end of the box; theſe rays will be reflected. The rays, which were converging to the focus E behind the mirrour, will converge after reflection to the focus M at an equal diſtance before it; thoſe, which were converging to the focus G behind the mirrour,

rour, will after reflection converge to the focus N before it, by propoſition 206. The ſame is true of every other pencil of rays: the reflection turns the rays out of the way, before they arrive at the imaginary focuſes behind the mirrour; and makes them converge to as many real focuſes at the ſame diſtances before it. Therefore whatever is the diſtance of all the imaginary focuſes from E to G behind the mirrour, the ſame will be the diſtances of all the real focuſes from M to N before it: and conſequently ſince EG, by the conſtruction, makes an angle of 45 degrees with the mirrour on one ſide, the picture MN, when reflected, will make with it an angle of 45 degrees on the other ſide. Therefore EG and MN will make an angle of 90 or twice 45 degrees with one another; ſo that as the picture would be at EG the end of the box, if the mirrour was out of the way, the reflection of the mirrour will throw it upon the top of the box. Now if the box, inſtead of being covered with a lid, is covered with a plate of plane glaſs that is left unpoliſhed on one ſide, or with a ſheet of oyled paper, the picture NM will be ſeen through the glaſs or the paper; and will be ſo bright and diſtinct that it may be traced out with a black-lead pencil upon the glaſs or the paper, if care is taken to ſcreen it from the open day-light.

274. *An opera-glaſs conſiſts of a common perſpective and a plane mirrour.*

This inſtrument, which is repreſented, Plat. XIV. fig. 10, is ſo contrived, that you may look at a perſon in a public place, and no one can tell who it is you look at. One part of this inſtrument is *ihkl* a common perſpective or ſhort Galilean teleſcope conſiſting of a convex object-glaſs *ibh*, and a concave eye-glaſs *kal*, ſuch as is deſcribed in propoſition 183. The other part is a plane mirrour *fh*, which is fixed in a tube *fgih*, that ſcrews on to the perſpective at *ih*. This mirrour has its reflecting ſurface towards the object-glaſs *ibh*, and is placed ſo as to make half a right-angle with *abc* the axis of the perſpective. In the tube *fghi* there are three holes, one on the ſide at *o*, another in the oppoſite ſide at *t*, and a third in the end at *x*. The hole at *o* is principally to be conſidered: the uſe of the other two ſhall be mentioned preſently. If there is any object PQR, which is parallel to *abc* the axis of the perſpective, and rays come from it through the hole *o*, this object may be ſeen by reflection through the perſpective. For ſince the mirrour *fh* makes an angle of 45 degrees with the axis of the perſpective, and the object PQR is parallel to the ſame axis; *hy* the plane of the mirrour continued will likewiſe make an angle of 45 degrees with the object. Therefore when the rays P*d*, Q*c*, R*e* are reflected, they will appear to have come from an image *pqr*, which will make an angle of 45 degrees with the mirrour behind it, as was ſhewn in a like caſe in propoſition 245. Now ſince the object PQR makes an angle of 45 degrees with the mirrour and is before it, and ſince the image *pqr*, which is ſeen by reflection, makes an angle of 45 degrees with the mirrour and is behind it, the object and image make an angle of twice 45 or 90 degrees with one another. Therefore as the object PQR is parallel to *abc* the axis of the perſpective, the image *pqr* will be at right angles with this axis, and conſequently will be placed directly before the perſpective ſo as to be ſeen through it. This makes it difficult to know what perſon we are looking at, becauſe in common perſpectives, the perſons we are viewing muſt be directly before the perſpective, and in this reflecting perſpective, they muſt be on one ſide. The holes *x* and *t* will ſerve to make this ſtill more difficult, and are likewiſe deſigned to anſwer another purpoſe. If an object is at S ſo near to us, that we could not ſee it diſtinctly through the perſpective *ihkl*, then inſtead of turning the hole *o* towards it, we turn the hole *t* towards it and the mirrour *fh* has another reflecting ſurface on the ſide *v*, ſo that if we look through the hole *x*, we may ſee the object S by reflection in the mirrour, and the rays have no perſpective to paſs through before they come to the eye. The hole *x* has commonly a plane glaſs fixed in it, to make the whole inſtrument as like a common perſpective as may be; and becauſe the hole *o* is oppoſite to the hole *t*, and we may ſee an object through either hole according as we look through either the end *kal* or the end *x*; no one can eaſily tell on which hand of us the perſon ſtands, that we deſign to look at.

275. *The pictures, that are produced by a magic lantern, are to be explaned upon the ſame principles with thoſe, that are produced in a dark room.*

ABCD, Plat. XIV. fig. 11, is a tin lantern with a tube *nklm* fixed in the ſide of it. This tube conſiſts of two joynts, one of which ſlips into the other: and by drawing this joynt out or puſhing it in, the tube may be made longer or ſhorter. At *kl* in the end of the moveable joynt of the tube a convex lens is fixed, and an object painted with tranſparent colours upon a piece of thin glaſs is placed at *de* ſomewhere in the immoveable joynt of the tube; ſo that as the tube is lengthened or ſhortened, the lens will be either at a grearer or a leſs diſtance from this tranſparent object. In the ſide of the lantern there is a very convex lens *bhc*, which ſerves to caſt a very ſtrong light from the candle within the lantern upon the object *de*. Now when the rays, which ſhine through the object *de*, diverge from the ſeveral points as *d*, *e*, &c. in the object, and fall upon the lens *kl*, they will be made to converge to as many points *f*, *g*, &c. on the other ſide of the lens, and will paint an inverted picture of the object at *fg* upon a white wall, a ſheet, or a ſcreen of white paper, provided the object is farther from the lens than its principal focus, by propoſitions 83, 84, 85. To make this picture appear diſtinct and bright; it muſt have no other light fall upon it, but what comes through the lens *kl*, and for this reaſon the whole apparatus is to be placed in a dark room EFGH. The lens *kl* muſt be very convex ſo that the object *de* may be very near to it and yet not be nearer than its principal focus: for by this means, as the object is near to the lens, the picture *fg* will be at a great diſtance from it, by propoſition 87, and conſequently the picture will be much bigger than the object, by propoſition 88. Since the picture is inverted in reſpect of the object, in order to make the picture appear with the right end upwards, it is neceſſary that the object *de* ſhould be placed with the wrong end upwards.

From what has been ſaid about the magic lantern, we may be able to underſtand the uſe of the ſolar microſcope. If inſtead of the tranſparent picture *de*, any other tranſparent object was placed there, the diſtinct picture of ſuch an object would be repreſented upon the white ſcreen. Now ſmall animals, and thin ſmall parts of moſt ſubſtances, when they are ſtrongly enlightened, are in ſome degree tranſparent. Therefore if any ſmall animal is placed at *de* and a ſtrong light ſhines upon it from the ſide where the lantern is placed in the figure, the rays that paſs through it will paint the picture of this animal at *fg*. The picture will be larger than the animal, as has been ſhewn already, and by changing the lens *kl* for a more convex one the picture *fg* may be made as much larger than the object *de* as we pleaſe. For if the lens is more convex than *kl*, the object *de* muſt be brought nearer to it in order to keep the diſtinct picture upon the ſcreen, by propoſition 97. Now the diameter of the object *de* is to the diameter of the diſtinct picture *fg*, as the diſtance of the object from the lens to the diſtance of the diſtinct picture from the lens, by propoſition 88. Therefore the leſs the diſtance of the object is in proportion to the diſtance of the picture, the leſs will the diameter of the object be in proportion to the diameter of the picture. Now as the object is nearer to the lens, its diſtance from the lens bears a leſs proportion to the diſtance of the picture, ſince the diſtance of the picture is a given quantity, becauſe it is always repreſented upon the ſcreen ſtanding in the ſame place. Therefore the nearer the object is to the lens, the leſs proportion the diameter of the object will bear to the diameter of the picture. But the diameter of the object is a given quantity; and conſequently the leſs proportion its given diameter bears to the diameter of the picture, the greater muſt be the diameter of the picture. Therefore the nearer the object is to the lens, ſo much greater is the diameter of the picture, or the diameter of the picture, when its diſtance from the lens is given, is inverſely as the diſtance of the object from the lens. Thus we have ſhewn how a tranſparent object at *de* may be magnifyed in any proportion, that we pleaſe, by uſing a lens of ſufficient convexity, provided there is a very ſtrong light ſhining through the object. The object-glaſſes of common microſcopes will ſerve for this purpoſe. In order to procure a ſtrong light the end *nm* of the tube *knlm* or of ſome other tube of the ſame ſort with a microſcopic lens at *kl*, inſtead of being fixed in a lantern, muſt be placed in the hole of a window-ſhutter of a dark room, ſo that the lens *kl* may be within the room; and then if the ſun ſhines directly through the hole, any ſmall object placed at *de* will be ſufficiently enlightened, and a diſtinct picture of that object of any ſize that we chuſe, may be repreſented upon a ſcreen of white paper placed within the room at a proper diſtance. Though the ſun is not directly before the window or is ſo high as not to ſhine directly through a hole in the ſhutter, yet a plane mirrour may be fixed in ſuch a poſition on the outſide of the ſhutter as to throw the reflected rays directly through the hole upon the object, and theſe rays will enlighten it ſufficiently.

276. *Reflecting teleſcopes of Sir Iſaac Newtons invention conſiſt of one concave mirrour, one plane mirrour and a convex lens.*

GH is the concave mirrour, Plat. XV. fig. 1. KK is the plane mirrour. LL is the convex lens. All theſe parts are fixed in their proper places in a tube ABCD. What their proper places are we will next conſider. The concave mirrour GH is placed towards the end BC of the tube ABCD, and is perpendicular to DC the ſide of the tube. The open end AC of the tube is turned towards the object PR. If the object PR is at ſo great a diſtance that any rays as E, *e*, that come from one and the ſame point in it, may be conſidered as parallel to one another, and that the rays F, *f*, that come from any other point are likewiſe parallel amongſt themſelves, by propoſition 30; then theſe or any other beams, when they fall upon the concave mirrour at *g*, *h*, and are reflected from it, will converge to *mn* at the principal focus of the mirrour, and would repreſent an inverted picture of the object upon a paper held at that place, by propoſition 268. But if the converging rays before they come to the place of the diſtinct picture, fall upon the plane mirrour KK, which is inclined ſo as to make an angle of 45 degrees with DC the ſide of the tube,

PLATE XIV.

Fig. 1.

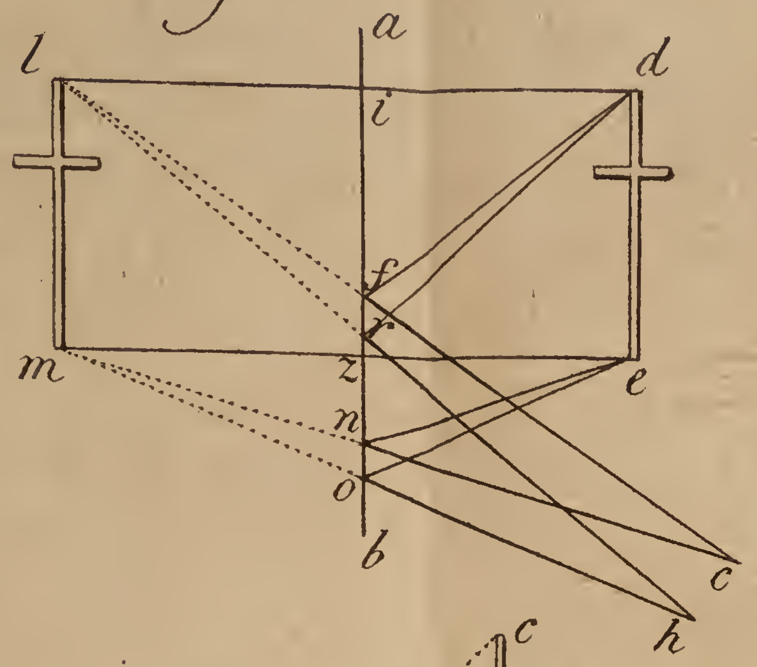


Fig. 2.

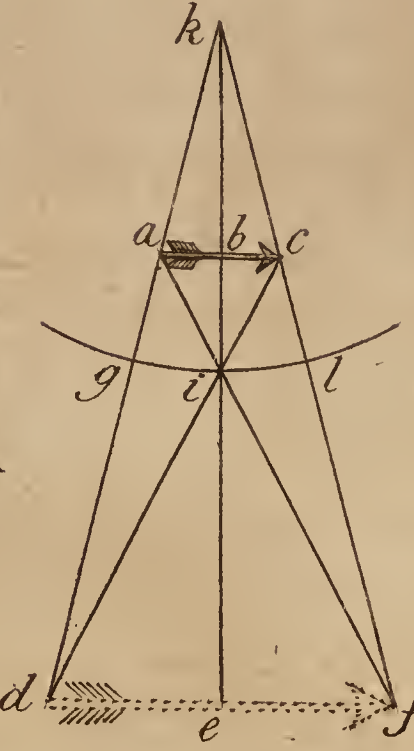


Fig. 3.

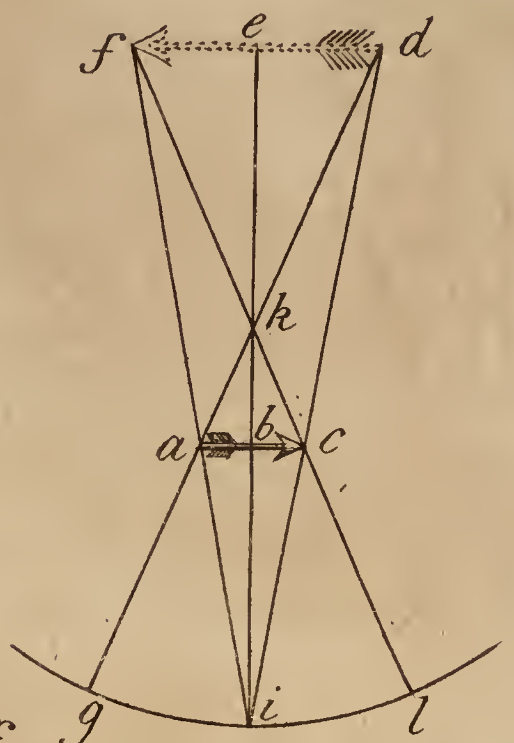


Fig. 4.

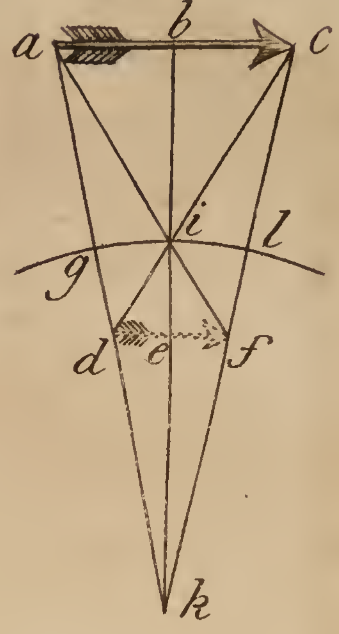


Fig. 5.

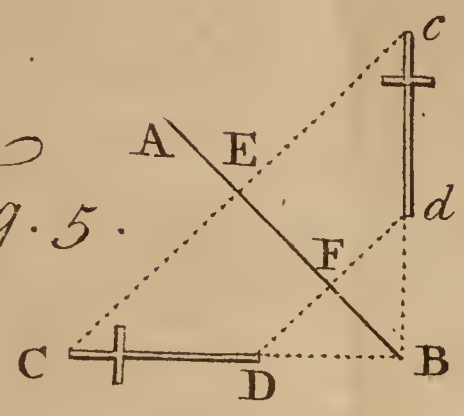


Fig. 6.

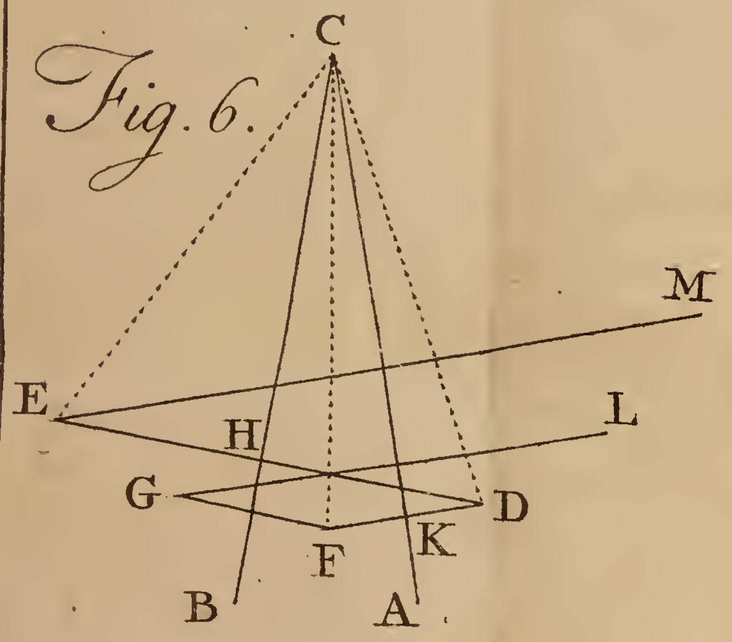


Fig. 7.

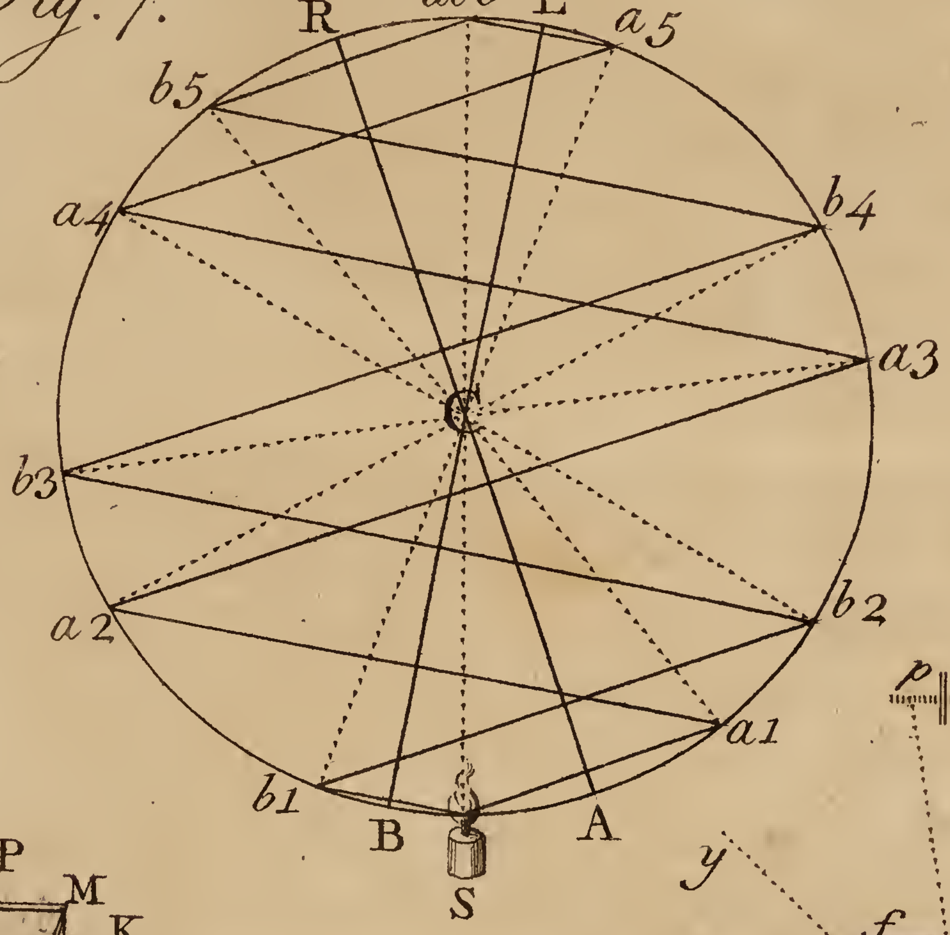


Fig. 8.

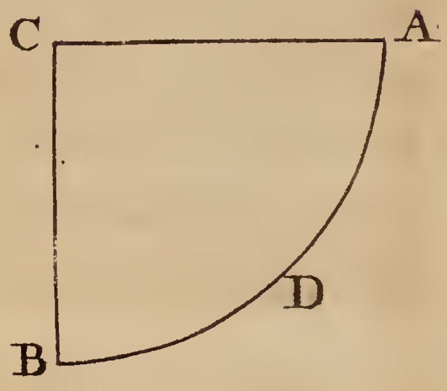


Fig. 9.

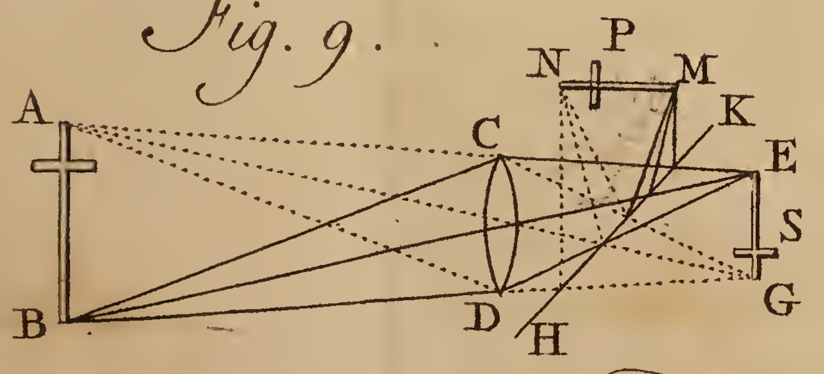


Fig. 10.

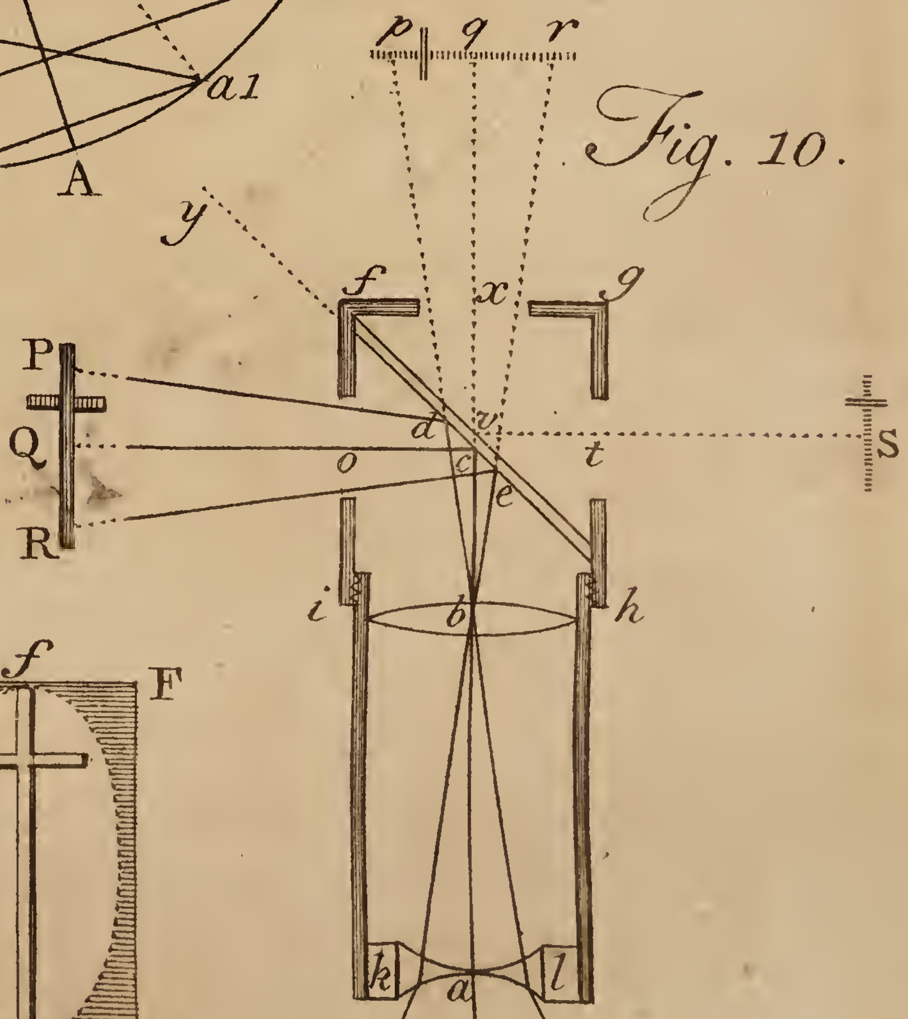
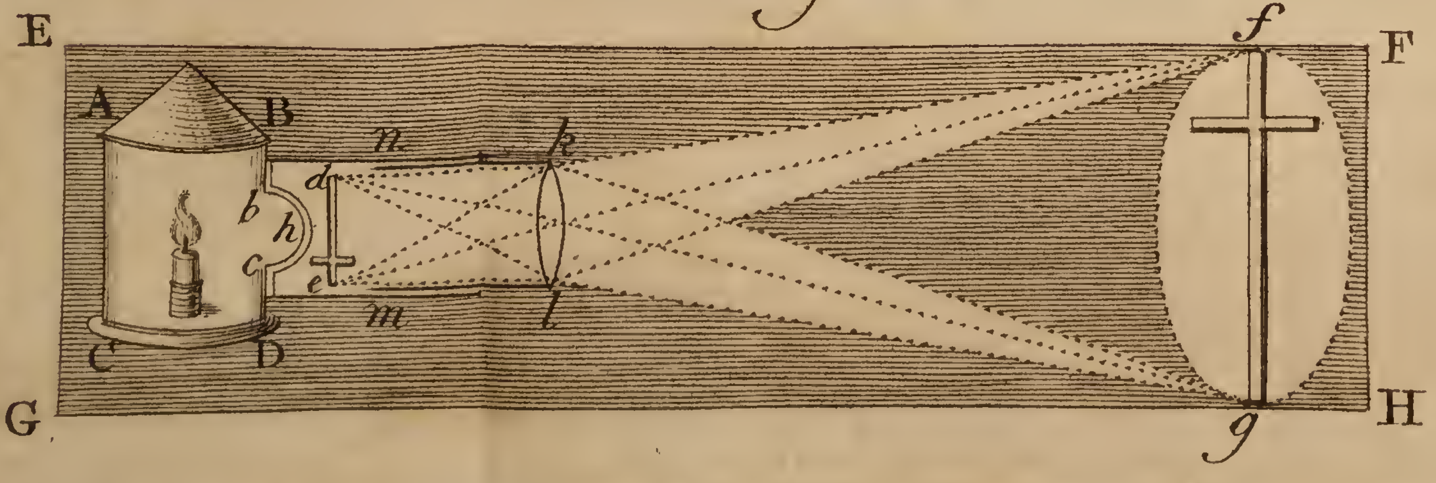


Fig. 11.



tube, or rather with the axis of the teleſcope; the rays will be reflected from thence, and inſtead of meeting in their reſpective focuſes at *mn*, they will meet before the mirrour at *qs*. Now ſince the picture *mn* would be perpendicular to DC, becauſe the concave mirrour GH is ſo, and conſequently, ſince this picture would be behind the plane mirrour KK and by the poſition of the mirrour would make with it an angle of 45 degrees: it follows that when the rays, inſtead of going on ſo as to form the picture *mn*, are reflected and form the picture *qs*, this picture *qs* will be before the mirrour and will make with it an angle of 45 degrees. Therefore *mn* and *qs* make with each other an angle of twice 45 or of 90 degrees, and conſequently, ſince *mn* is perpendicular to the ſide of the tube or to the axis of the teleſcope, the reflected picture *qs* will be parallel to it. All this will be obvious from a ſimilar caſe already explaned in propoſition 273. Now if this picture *qs* is in the principal focus of the convex lens LL; this lens will be placed in reſpect of the picture juſt as the eye-glaſs is in an aſtronomical teleſcope, by propoſition 172, and the eye O will ſee the picture diſtinctly through it.

The magnifying power of ſuch a reflecting teleſcope as this, is determined in the ſame manner with the magnifying power of an aſtronomical teleſcope, in propoſition 173. For the apparent diameter of the object ſeen with the teleſcope is to its apparent diameter ſeen without it, as the diſtance of the diſtinct picture from the object-mirrour to the diſtance of it from the eye-glaſs. This can want but little proof, ſince we have already ſhewn in propoſition 257, that the picture *mn* ſeen by the eye contiguous to the concave mirrour GH would appear of the ſame ſize with the object. And by what has been ſaid in propoſition 173 it follows, that the picture *qs*=*mn* when ſeen from the ſtation of the object-glaſs, either with the glaſs or without it, will appear as much bigger than it does, when ſeen from the concave mirrour, that is, as much bigger than the object appears to the naked eye, as the diſtance of the picture from the eye-glaſs is leſs than its diſtance from the mirrour, or as its diſtance from the mirrour is greater than its diſtance from the lens.

277. *Gregory's reflecting teleſcope, as it is now made, conſiſts of two concave mirrours, a plano-convex lens, and a meniſcus.*

In the tube TYTY, Plat. XV. fig. 2, a concave mirrour LD is placed. Through TT the open end of the tube, rays, as IL, *il*, and CD, *cd*, are admitted from ſome diſtant object AB. The rays that are of the ſame beam, or that come from one and the ſame point in the object, are parallel, upon account of the remoteneſs of the object, by propoſition 30.

Therefore KH the diſtinct inverted picture of the object will be formed in the principal focus of the mirrour LL, by propoſitions 210, 268. If *f* is the principal focus of the ſmaller concave mirrour EF, then KH is more remote from this mirrour than its principal focus, and the rays, which diverge from KH and fall at G and R upon the mirrour EF, will converge after refraction, by propoſition 213; and conſequently, if they paſs on through a hole X in the firſt mirrour LD, they will form a ſecond picture, which will be inverted in reſpect of the firſt, or which will be in the ſame poſition with the object AB. This picture would be ſomewhere about SS, if the rays that form it were made to converge only by being reflected from the mirrour EF: but before they come thither, whilſt they are converging, they paſs through the plano-convex lens MN, which is placed in a ſmaller tube joyned to the larger TYTY: the refraction of this lens makes the rays converge more, by propoſition 74, and conſequently they will meet in their reſpective focuſes or will form the ſecond picture at PV, before they come to SS. This erect picture is in the principal focus of the meniſcal eye-glaſs SS, the convexity of which is greater than its concavity, ſo that its effect is the ſame with that of a convex eye-glaſs, by propoſition 80, and conſequently to an eye at O the erect picture PV will appear as a picture of the ſame ſort does, in the common perſpective of four glaſſes deſcribed in propoſition 181.

CHAP. XX.

Of the different refrangibility of light.

WE have hitherto ſuppoſed that a particle of light, as it comes from the ſun, is the leaſt particle, into which light can be ſeparated. But we muſt now correct this ſuppoſition by ſhewing that a particle of light, as it comes from the ſun, is properly ſpeaking a bundle of rays, which may be ſeparated from one another. Therefore for the future by a ray of light we muſt be underſtood to mean not that collection of particles, which we have hitherto called by this name, but the leaſt particles, into which light can be ſeparated.

278. *Rays of light are ſaid to be differently refrangible, when, at the ſame or equal angles of incidence, ſome are more turned out of the way than others.*

279. *Rays are ſaid to be differently reflexible, if ſome are more eaſily reflected than others.*

280. *Light is called homogeneous, when all the rays are equally refrangible: it is called heterogeneous, when ſome rays are more refrangible than others.*

281. *The colours of homogeneous light are called primary or ſimple colours; and thoſe of heterogeneous light are called ſecondary or mixed.*

282. *The rays of the ſun are not all equally refrangible: and thoſe rays, which have a different degree of refrangibility, have likewiſe a different colour.*

If a beam of light SF, Plat. XV. fig. 3, that comes from the ſun, paſſes into a dark room through F a round hole in a window-ſhutter EG; this beam proceeding ſtrait forwards and falling upon a paper at Y would there make a round picture of the ſun, by propoſition 105. This picture will be a confuſed one indeed, if the hole is a large one and there is no lens in the hole. However as this round ſpot of light is a picture of the ſun, we ſhall hereafter, notwithſtanding its confuſion, call it by this name. Now if a glaſs priſm ABC is placed between the hole in the window-ſhutter and the paper at Y, the rays of this beam by the refraction, which they ſuffer in the priſm, will be bent from their ſtrait courſe, and inſtead of going on ſo as to fall upon the paper at Y they will be turned upwards, and the picture of the ſun produced by them will fall upon a paper MN that is placed above Y. If all the rays were equally bent upwards the picture would be a round one upon the paper MN, after the rays have been refracted, as well as when they paſſed ſtrait forwards and fell upon a paper at Y. But this refracted picture PT is found to be oblong. The horizontal diameter or breadth of this oblong picture is equal to the diameter of the circular one Y: but the perpendicular diameter or height of the picture PT is much greater than its breadth. The refraction is made upwards and not in a horizontal direction: therefore no alteration ought to be made in the breadth or horizontal diameter of the picture; becauſe no refraction, that is not in the direction of that diameter, can make the picture either broader or narrower. And if all the rays were equally refracted upwards, ſuch a refraction would not change the length of the picture, as it is round, when it falls at Y, ſo it would be round, when it is refracted upwards by the priſm and falls at PT. This oblong picture conſiſts therefore of rays, which are differently refrangible: they all fall at equal angles of obliquity upon BC the firſt ſide of the priſm, but in the refraction, ſome are more turned out of the way than others; thoſe rays, which go to P the upper part of the picture are the moſt refrangible, and thoſe, which go to T the lower part of it, are leaſt refrangible; the reſt, which fall between P and T have intermediate degrees of refrangibility.

This oblong picture is of different colours in different parts of it. The moſt refrangible rays at P are violet, the leaſt refrangible at T are red; the rays of intermediate refrangibility from the violet downwards to the red are indigo-coloured, blue, green, yellow, and orange. So that the whole picture is made up of rays of theſe ſeven different colours. We may from hence ſee the reaſon why the coloured picture conſiſting of differently refrangible rays ſhould be oblong in ſuch a manner that the two ſides of it are right lines and the two ends ſemicircles. For it conſiſts, as in fig. 4, of ſeven circles, the higheſt of which PAGQ is violet, the loweſt STN is red, the five intermediate ones, BH, CI, DK, EL, OM, are indigo-coloured, blue, green, yellow, and orange. The white round picture Y, fig. 3, is formed by heterogeneous rays, that are of ſeven different ſorts diſtinguiſhed from one another by their different degrees of refrangibility and different colours. The refraction of the priſm ſeparates theſe rays from one another by refracting ſome of them more and others leſs. And conſequently the refracted picture will conſiſt of ſeven round pictures one below another. Theſe round pictures are ſo near to each other that the higheſt of them APGR will mix itſelf with ſome of thoſe below it, as with BH and CQI. This nearneſs of theſe ſeveral round pictures to each other will prevent their colours from being diſtinctly ſeen, it will likewiſe make the ſides AS, GN, which are compoſed of ſmall arcs of circles very cloſe to one another, appear like right lines; but the two ends P and T will be ſemicircles.

If the centers of theſe circles continue at the ſame diſtance from one another and the circles themſelves are made leſs, as *apg*, *bh*, *ci*, *dk*, *el*, *om*, *stn*, they will then be diſtinct or will not mix with each other; and as the colours of the ſeveral parts will by this means be kept ſeparate, ſo the refracted rays inſtead of forming one continued oblong picture will form ſeven ſmall circular ones placed in a line perpendicular to the horizon. This ſeparation of the ſeveral parts in the refracted picture from each other is brought about, as in fig. 5, by making the hole F in the window-ſhutter very ſmall, and by collecting the rays that come through it with a convex lens MN. For this will make a very ſmall white picture of the ſun at L if there is no priſm *abc*: but the refraction of this priſm, if it is placed a little beyond the lens, will ſeparate the heterogeneous rays by refracting them upwards, and inſtead of one ſmall round and white picture at L, there will be ſeven ſmall round pictures at *pt*, of which *r* will be violet, *s* indigo, *u* blue, *x* green, *l* yellow, *y* orange, and *z* red.

That the priſm ABC, fig. 3, does not make the rays diverge, ſo as to ſpread over the ſpace PT, upon any other account but their different refrangibility, will be evident, if, as in fig. 6, a ſecond priſm DH is placed beyond the firſt *abc*. For if the rays, that come from S and paſs through the hole F of the window-ſhutter EG, were by the firſt priſm *abc* made to diverge and form the oblong picture PT upon any other account beſides their different refrangibility; then ſuppoſing a ſecond priſm DH to be placed at right angles to the former the effect muſt be this; the firſt priſm *abc* makes the rays diverge from one another in a line PT perpendicular to the horizon, and conſequently the ſecond priſm DH muſt make them diverge from one another in a line parallel to the horizon, ſo that the ſecond priſm would encreaſe the breadth of the picture, as much as the firſt encreaſed its length, and as one priſm alone makes the picture a long one both of them together would make it ſquare as *prst*. But this ſecond refraction does not alter the figure of the picture, but only the poſition of it: the ſecond priſm refracts the picture ſideways; and thoſe rays, which fell the higheſt at P after the firſt refraction, are refracted ſideways the moſt by the ſecond priſm, thoſe which fell the loweſt at T, are refracted ſideways the leaſt, by which means the picture, though it continues oblong, will not be perpendicular to the horizon as PT was, but will be inclined ſo as to lie in the poſition *pt*. This makes it evident that the ſpreading of the rays by the firſt refraction was owing to their different refrangibility and to no other cauſe. It muſt be owing to their different refrangibility, becauſe thoſe, which were moſt refracted upwards by the firſt priſm, are moſt refracted ſideways by the ſecond. It cannot be owing to any other cauſe, becauſe if it was, the ſecond priſm would ſpread the rays in breadth as much as the firſt priſm ſpreads them in length, and both priſms would make the picture ſquare.

283. *Thoſe rays of light, which are moſt refrangible, are likewiſe moſt reflexible.*

When a beam of light is admitted into a dark chamber through the hole F in the window-ſhutter EG, Plat. XV. fig. 7, and this beam falls upon a priſm ACB, the ſides of which AC and AB are equal, and the angle at A a right one; when the obliquity of theſe rays, as they are to paſs out of the priſm at its baſe BC, is leſs than 40 degrees, the greateſt part of the beam will paſs out, but ſome few rays will be reflected at the ſurface BC, by propoſition 200. The rays, which paſs through the baſe, form

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an oblong coloured picture HK, by propoſition 282, where MH is a more refrangible ray, and MK a leſs refrangible one. If the few rays of the beam, which are reflected from M in the direction MN, are made to paſs through another priſm XYV, they will likewiſe form an oblong coloured picture *pt*, where *p* is the moſt refrangible and *t* the leaſt refrangible ray. This picture will be a very faint one, becauſe there are but few rays reflected from M.

Now if the priſm ACB is turned ſlowly round upon its axis in the direction ACB, the obliquity of the rays EM to the baſe BC will keep encreaſing, till at laſt this obliquity may become ſo great, that no rays will paſs out at M, but all of them will be reflected, by propoſition 200. When this total reflection is made, the oblong picture *pt*, which was faint before, will become much brighter, becauſe then not only a few rays but all the beam will be reflected thither. This total reflection will not be made all at once, but as the priſm is turned ſlowly round upon its axis, the moſt refrangible rays MH will be firſt reflected, for the violet colour will diſappear in the oblong picture HK, whilſt all the other colours continue as bright as they were before; and when this colour diſappears at HK, the ſame colour at *p* will become bright, and all the other colours at *pt* will continue as faint as they were before. When the priſm is turned a little farther upon its axis, the indigo colour, which conſiſts of rays that have the next greateſt degree of refrangibility, will be reflected, ſo that this colour will diſappear at HK and will become bright at *pt*. The ſame thing will happen to all the rays in their order; as the priſm is turned round, each different ſort of rays will be reflected ſooner, as the rays have a greater degree of refrangibility or later as they have a leſs degree. The red rays at K, which are the leaſt refrangible of all will be reflected laſt of all. From hence therefore it appears, that the rays of the ſun are differently reflexible, and that thoſe, which are moſt refrangible are likewiſe moſt reflexible.

284. *Rays of light, which differ in colour, differ likewiſe in the degree of their refrangibility.*

We have already ſeen, in propoſition 282, that rays which are differently refrangible, are of different colours, the moſt refrangible rays of the ſuns light for inſtance, are violet, the leaſt refrangible are red. We are now to prove the converſe of this, that all violet rays whatever are more refrangible than all red ones. Let a ſtiff chart or piece of paſteboard DE, Plat. XV. fig. 8, be painted, half of it DG of a light purple colour, and the other half FE of a ſtrong red. Now if this paſteboard is viewed through a priſm BAC, *bac*, with its refracting angle at A upwards, the apparent place of the paſteboard ſeen through the priſm will be raiſed above its real place, as is evident from the demonſtration of propoſition 137, upon Plat. XII. fig. 4. And ſo likewiſe, if the refracting angle A of the priſm was downwards, the apparent place of the paſteboard ſeen through the priſm would be below its real place. Firſt therefore ſuppoſe the angle A to be upwards and the painted paſteboard, which is at DE, will appear at *de*, when it is ſeen through the priſm. In this caſe the paſteboard ſeems to be broken in the middle, and the purple part *dg* appears ſomething higher than the red part *fe*. But ſince the apparent place of the whole paper is raiſed by the refraction of the rays, as they paſs through the priſm, and the purple part is more raiſed than the red part, it follows that the purple rays are more refracted than the red ones. Secondly, ſuppoſe the angle A to be downwards, and then the painted paſteboard DE ſeen through the priſm will appear at $\delta\varepsilon$. And in this caſe again the paſteboard will ſeem to be broken in the middle, and the purple part $\delta\gamma$ will appear ſomething lower than the red part $\phi\varepsilon$. But ſince the apparent place of the whole paſteboard is depreſſed by the refraction of the rays, as they paſs through the priſm, and the purple part is more depreſſed than the red part, it follows that the purple rays, as before, are more refracted than the red ones.

285. *Homogeneous light is refracted regularly without any dilatation or ſcattering of the rays.*

When the rays of any one particular colour in the oblong picture of the ſun, as the green rays, for inſtance, are ſeparated from one another in the manner already deſcribed in propoſition 282; if ſome of theſe green rays which are homogeneal or are all equally refrangible, are tranſmitted through a very ſmall round hole in a ſtiff paſteboard, and are refracted by a priſm on the other ſide of the hole, the picture formed by theſe green rays after refraction upon a white paper held beyond the priſm will not be oblong but circular, as the hole is through which they paſſed. Therefore this homogeneous light is not dilated nor are the rays of it ſcattered by this refraction.

286. *The confuſed appearance of objects, when they are ſeen through refracting bodies, is owing to the different refrangibility of light.*

If flies, or the letters of a ſmall print, or any other minute objects are placed in heterogeneal light ſuch as a direct beam of the ſuns, which has never been ſeparated by any refraction into its homogeneous parts; theſe objects being viewed through a glaſs-priſm will be ſeen confuſedly, their edges will appear ſo miſty that the ſmaller parts of minute animals cannot eaſily be diſtinguiſhed from one another, and the letters of the ſmall print cannot be read. But if the ſame objects are placed in a beam of homogeneous light, which is ſeparated from all other rays of a different refrangibility in the manner already deſcribed in propoſition 282, they will appear as diſtinct through a priſm, as if they were viewed with the naked eye. Therefore we may conclude that this confuſion is owing to the different refrangibility of thoſe rays, which come from the objects: ſince objects never appear confuſed, when they are ſeen through refracting bodies, unleſs they are enlightened with ſeveral ſorts of rays, which have different degrees of refrangibility.

287. *Where the mediums are the ſame, in rays of one and the ſame ſort the ſine of incidence will always bear the ſame proportion to the ſine of the refracted angle, when the ray paſſes out of one of theſe given mediums into the other, whatever may be the obliquity of the incident ray.*

The violet rays, for inſtance, are more refrangible than the red ones; and conſequently, when both the violet rays and the red rays are paſſing out of glaſs into air, if they fall at equal obliquities upon the ſurface of the glaſs, the violet rays will be more refracted from a perpendicular than the red ones, or the refracted angle in the violet rays will be bigger than in the red ones, by propoſitions 282, 284, 38. But if the proportion is once found, which the ſine of incidence in each particular ſort of rays bears to the ſine of the refracted angle at any one obliquity as the rays are to paſs out of glaſs into air; the ſame will always be the proportion of theſe ſines to one another whatever is the obliquity or angle of incidence, when the rays are paſſing out of the ſame medium of glaſs into the ſame medium of air. Thus particularly, if at any ſmall obliquity the ſine of incidence is to the refracted ſine in the violet rays as 50 to 77; the ſine of incidence in the ſame ſort of rays will be to the refracted ſine as 50 to 77 at all greater obliquities. If the ſine of incidence is to the refracted ſine in the red rays as 50 to 78 at any one obliquity, this proportion will continue the ſame at all other obliquities whatever.

In propoſition 40, it has been ſhewn, that when a ray of the ſun is paſſing out of any given medium, as glaſs, into any other given medium, as air, the proportion of the ſine of incidence to the refracted ſine will not be changed by changing the obliquity of the incident ray. But then this was ſhewn upon a ſuppoſition that all rays as they come from the ſun are equally refrangible: and conſequently that propoſition is not exactly true, ſince we have now ſhewn, in propoſition 282, that rays of light, as they come from the ſun, are ſome more and ſome leſs refrangible. But ſtill propoſition 40 will be true and the demonſtration of it applicable to any one ſort of rays, to the violet ones, or to the red ones in particular, which are equally refrangible; for in each particular ſort of rays the ſuppoſition holds good, upon which that propoſition was demonſtrated. Therefore in each particular ſort of rays the ſine of incidence always bears the ſame proportion to the refracted ſine at all obliquities whatever, when the refracting mediums are given.

288. *It is probable that any ſingle ray of the leaſt refrangible ſort contains a greater quantity of matter than any ſingle ray of the moſt refrangible ſort.*

We have already ſeen, in propoſitions 282, 283, that at the ſame angles of incidence violet rays will be more refracted or more turned out of the way than red rays. And we have likewiſe ſeen, in propoſitions 37, 38, that rays are refracted when they paſs out of one medium into another by being either more or leſs attracted in one medium than they are in the other. Now ſince, when all other circumſtances are equal, when red rays and violet rays fall at equal obliquities and are to paſs out of glaſs into air, ſo that the mediums and conſequently the attractive force or cauſe of refraction is given, if the ſame cauſe can turn the violet rays more out of the way, or refract them more, than it does the red rays, theſe rays muſt have different moments; the moſt refrangible rays, or thoſe which are moſt eaſily turned out of the way, have the leaſt moment; and the leaſt refrangible rays, or thoſe which are moſt difficult to turn out of the way, have the greateſt moment. But if all ſorts of rays have the ſame velocity, as is probable from propoſition 33, their reſpective quantities of matter will be as their moments, by propoſition 13 of mechanics. And conſequently any ſingle ray of the moſt refrangible ſort contains a leſs quantity of matter, than any ſingle ray of the leaſt refrangible ſort.

It may be upon this account that a red colour or a pale purple is leſs pleaſant to the eye than a blue, a green, or a yellow. The red rays ſtrike the eye with ſo great a force as to be offenſive to it; and the ſmall force of the pale purple ones will produce too faint a ſenſation to be agreeable. The intermediate colours are therefore more pleaſant to the eye, as the force of the rays is neither too great to be offenſive, nor too ſmall to produce a quick and lively ſenſation.

289. *The colours of homogeneous light are ſo invariable, that neither any refraction nor any reflecton can alter them.*

If a beam of homogeneous light paſſes through a round hole in a paſteboard, as in propoſition 285, and then is refracted by a priſm on the other ſide of the hole, this refraction will make no alteration in the colour of the rays: if they were red, or whatever was their colour, before they entered the priſm, their colour will ſtill be the ſame, when they have paſſed through it, and fall upon a white paper held beyond the priſm. This proves the firſt part of the propoſition, that the colours of homogeneous light are not to be changed by any refraction.

Red lead, which, as the name denotes, is red, when it is viewed in open day-light, or when heterogeneous rays fall upon it, will likewiſe be red, if it is placed in homogeneous red light: but red lead, when it is placed in any other ſort of homogeneous light, will have the ſame colour with the rays that fall upon it and are reflected from it: if it is placed in yellow homogeneous light, it will be yellow, if in green light, it will be green, or if in blue light, it will be blue. Conſequently the reflection of the rays from the red lead makes no alteration in their colour: for if it did, rays of any ſort reflected from the lead would be of the ſame colour ſo that it would appear red in whatever ſort of light it was placed. The ſame that is here ſaid of red lead, is true of any other ſubſtance of any other colour. Graſs, which is green, either in open day-light or in homogeneous green light, will not change the colour of any homogeneous rays by reflecting them, but will itſelf have the ſame colour with the rays, in which it is placed; it will be red in red light, or blue in blue light, or yellow in yellow light.

From hence we may conclude, by the way, that a body is of any particular colour, not becauſe it reflects no other rays but thoſe of that particular colour, but becauſe it reflects thoſe more copiouſly and others more ſparingly. Red lead, as it appears red in red light, ſo in green light it appears green, or in blue light it appears blue: conſequently it reflects rays of theſe ſorts, and in the ſame manner it might be ſhewn to reflect all other ſorts of rays. But then the red colour of red lead, when it is placed in red light is much brighter, than any other colour will be that it puts on by being placed in another ſort of light: conſequently it reflects red rays more copiouſly than any other ſort of rays: and for this reaſon, when it is placed in open day-light, where it reflects all ſorts of rays at once, the red rays are ſo much more numerous than the reſt, as to make the whole mixture of their own colour.

PLATE XV.

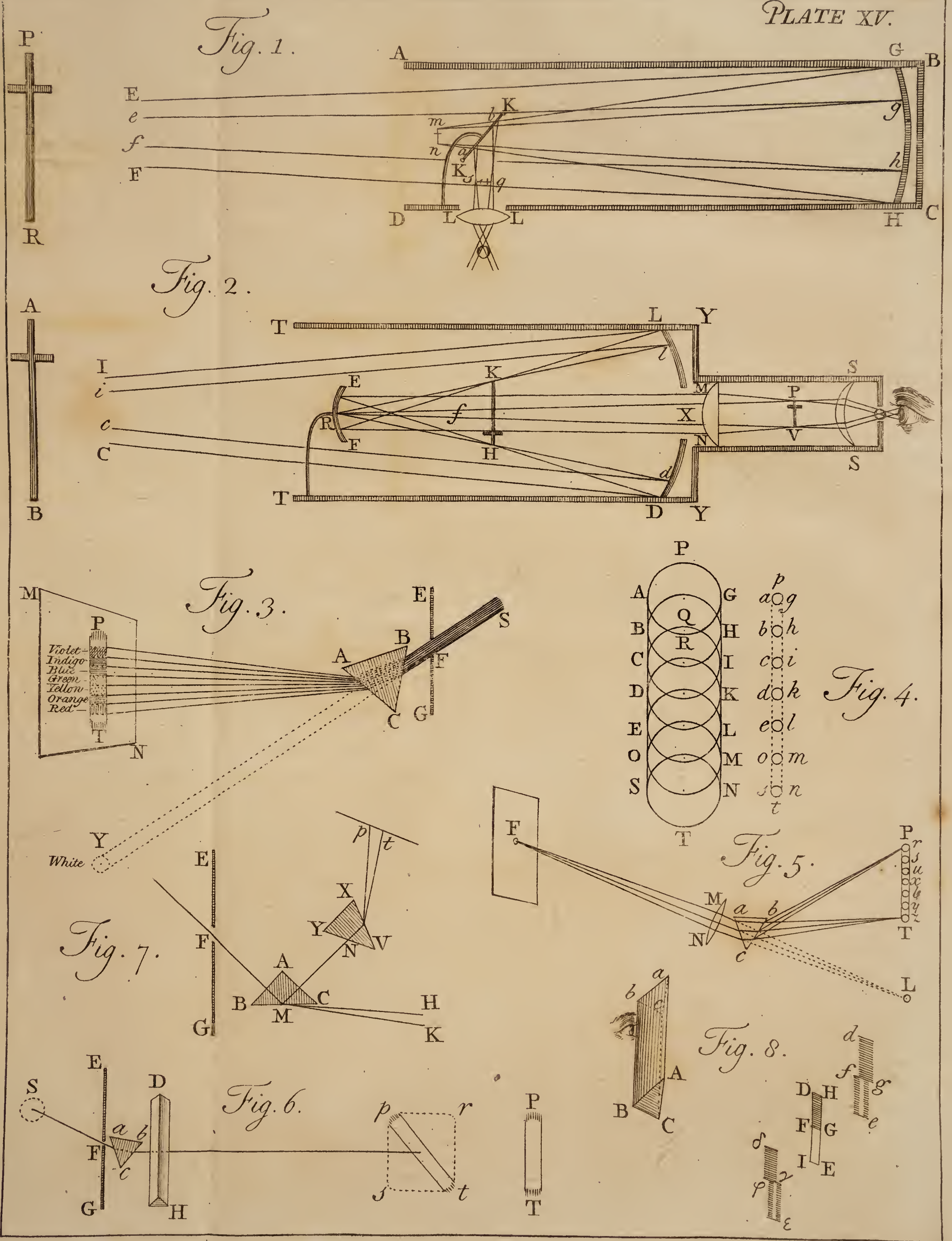
Fig. 1.
P
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G
B
E
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f
F
m
n
K
b
a
g
h
D
L
L
H
C
Fig. 2.
A
B
T
L
Y
I
i
c
C
E
R
F
K
f
H
l
M
X
N
d
P
V
S
S
T
D
Y
Fig. 3.
M
P
Violet
Indigo
Blue
Green
Yellow
Orange
Red
T
N
A
B
C
E
F
G
S
Y
White
Fig. 4.
P
A
B
C
D
E
O
S
Q
R
G
H
I
K
L
M
N
T
p
a
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c
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o
s
g
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k
l
m
n
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Fig. 5.
F
M
N
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Fig. 7.
E
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G
p
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V
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A
B
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C
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Fig. 8.
a
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F
G
I
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γ
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Fig. 6.
S
E
F
G
a
b
c
D
H
p
r
s
t
P
T

290. *Colours may be produced by compoſition, which ſhall in appearance be like the colours of homogeneous light: but then theſe compound colours will be altered by refraction.*

When, by means of two holes in the window-ſhutter of a dark room and of two priſms, two oblong coloured pictures are produced; if a circular piece of white paper is ſo placed that the red light of one picture and the yellow light of the other may fall upon it; this mixture will produce an orange colour, that in appearance will be like the primary orange colour. But between the ſimple and compound colour, though they are alike in appearance, there will be this difference: if the circular piece of paper, when it is enlightened with compound orange, is viewed through a priſm, the rays will be found to be differently refrangible, and they will by the refraction of the priſm be ſo ſeparated from one another, that the paper ſeen through it will appear as two circles, one of which will be red and the other yellow: whereas, if the ſame paper, when it is enlightened with ſimple or primary orange, is viewed in like manner through a priſm, the rays will be found to be equally refrangible, and the paper will appear through the priſm, as it does to the naked eye, to be one orange-coloured circle diſtinctly terminated all round. After the ſame manner other homogeneous colours, as blue and yellow, when mixed together, will produce a new compound colour like the intermediate homogeneous green colour in appearance. But then the rays of this compound green will not be all of them equally refrangible, as the rays of the ſimple or primary green colour are.

291. *The whiteneſs of the ſuns light is compounded of all the primary colours mixed in a due proportion.*

Let the oblong coloured picture PT, Plat. XVI. fig. 1, fall upon the convex lens MN; and then all the rays, which are ſeparated from one another at PT will be collected together by paſſing through the lens and will meet at its focus G, by propoſition 84, in ſuch a manner as to form a round picture of the ſun upon a white paper DE. This round picture, which conſiſts of rays of all ſorts, of red, orange, yellow, blue, green, indigo, and violet, is white. And this whiteneſs is compounded of all the primary colours mixed together. None of the rays change their colour by being mixed with the reſt; each ſort retains the ſame colour after it is mixed with the reſt that it had before: neither the red rays, nor the orange, nor the yellow, nor the blue, nor the green, nor the indigo, nor the violet, are made white by being mixed with the reſt at the focus; but though none of the parts are white, yet the whole mixture is white.

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That the whiteneſs at the focus G ariſes from a mixture of all the primary colours, is evident. For if any of the colours are intercepted at the lens, the focus loſes its whiteneſs, and becomes of that colour, which ariſes from a mixture of thoſe, which are not intercepted. Thus if all the rays at PT are intercepted except the yellow, the orange, and the red, the focus will not be white, but will be orange-coloured. If all the rays are intercepted at PT except the blue, the green, and the yellow, the focus will then be green. The orange in one caſe and the green in the other caſe is the compound colour ariſing from a mixture of thoſe rays, which are not intercepted. And in either caſe, if the rays, that were intercepted are again ſuffered to paſs through the lens, the focus will recover its whiteneſs.

It may be more difficult to ſhew that the rays, when they are all of them mixed at the focus, retain their proper colours, and are none of them white though the compound mixture is white. To make this out, let the paper be removed from DE, where all the rays are mixed upon it at G, to *de*, where it will receive the rays, after they have croſſed one another at the focus, and having got beyond it diverge again, by propoſition 15. In this poſition of the paper, becauſe the rays, that were mixed at the focus, have diverged from thence and are again ſeparated from one another, the oblong coloured picture will appear again at *tp*, ſo that the red colour T, which was the loweſt at the lens, will be the higheſt at the paper *de*. But though the colours are thus inverted by paſſing the focus, yet all of them appear at *tp*; which would have been impoſſible, if each ſort of rays by being mixed with the reſt at the focus had loſt their colour and had been made white. Nor indeed is the colour of any ſort of rays at all changed by being mixed with the reſt at the focus; but the ſame rays, that produced any particular colour in the oblong picture PT, are the rays that produce the ſame colour in the inverted picture *tp*; as would be evident from intercepting any particular colour at PT: for if the green rays, for inſtance, are intercepted at PT there will be no green at *tp*: or if the red are intercepted at PT, there will then appear no red colour at *tp*: and the ſame thing will happen upon intercepting the rays of any other colour at PT, for then that colour will vaniſh at *tp*.

292. *Colours may be produced by compoſition that are neither exactly like any of the primary ones nor fully white.*

If the red rays of one coloured picture are mixed with the violet rays of another; according to the various proportions, in which they are mixed, various purples will be produced, ſuch as are not like in appearance to the colour of any homogeneous light; and of theſe purples mixed with yellow and blue may be made other new colours.

By mixing the coloured powders, which painters uſe, though the powders themſelves reſemble the primary colours, yet the mixture may be grey, or dun, or ruſſet-brown, ſuch as are the colours of a mans nail, of a mouſe, of aſhes, of ordinary ſtones, of mortar, of duſt and dirt in the high-ways. Thus one part of red lead and five of viride æris compoſe a dun colour like that of a mouſe. If to orpiment, which is yellow, a full bright purple powder uſed by painters is added, the mixture may be made of a pale red; and with the addition of a little viride æris, which, as the name imports, is green, and of a little blue biſe, this pale red will change to grey or pale white, ſuch as is the colour of aſhes, or of wood newly cut, or of a mans ſkin.

Theſe grey, dun, and ruſſet colours are only imperfect ſorts of white. And we may underſtand, why the mixture of theſe coloured powders ſhould produce an imperfect white, and not a full bright one, from the following obſervation. All coloured powders ſuppreſs and ſtop great part of the light, that falls upon them: they reflect more of thoſe rays from whence their colour ariſes than of any other ſort; but they reflect even thoſe more ſparingly than white bodies do. Red lead, for inſtance, reflects fewer red rays than white paper does: for if red lead and white paper are both of them placed in homogeneous red light, the paper will appear of a brighter red than the lead. But if red lead ſuppreſſes many red rays, it may well be ſuppoſed to ſuppreſs many more rays of other colours. Since its redneſs is owing to its reflecting red rays more copiouſly and all other rays more ſparingly, as was obſerved under propoſition 289. From hence it follows, that, in a mixture of coloured powders, though they reflect rays of all ſorts in a due proportion ſo that the compound light will not be more of one colour than another, but will be white; yet the whiteneſs will be much leſs bright than that of paper: becauſe the mixture of powders ſuppreſſes and ſtops many rays, whereas the paper reflects almoſt all the rays, that fall upon it, and ſuppreſſes ſcarce any. Thus the whiteneſs in the mixture of powders and the whiteneſs in the paper are both of the ſame ſort, and differ from one another only in degree or in the quantity of light. Therefore if ſome of this mixture of powders is placed in bright ſun-ſhine and a piece of white paper is placed in the ſhade, the mixture by thus encreaſing the light and the paper by thus diminiſhing it may be made to appear equally white.

293. *The colours of all bodies are either the ſimple colours of homogeneous light, or ſuch compound colours as ariſe from a mixture of homogeneous light.*

Each ſort of light has a peculiar colour of its own, which no refraction or reflection can change, by propoſition 289. Therefore the colour of no natural body can be any other than either the colour of ſome ſort of homogeneous light, or a compound colour ariſing from a mixture of the ſeveral ſorts. For bodies appear coloured only by reflecting light; and no reflection can give any other colours to the rays but what they had before.

When in the foregoing propoſitions, or in any of thoſe which follow, we ſpeak of light and rays as coloured or endued with colours, we muſt be underſtood to ſpeak not philoſophically but according to vulgar conceptions. For properly the rays are not coloured. In them there is nothing elſe but a certain power and diſpoſition to raiſe in the mind the idea of this or that colour. For as ſound in a bell is nothing but a trembling motion, in the air it is nothing but that motion propagated from the bell, and in the mind it is an idea which this motion, when conveyed to the ſenſorium, excites in us: ſo colours in the object are nothing but ſuch a modification, figure, and contexture of its parts, as makes it fit to reflect this or that ſort of rays more copiouſly than others; in the rays they are nothing but the diſpoſitions or powers of thoſe rays to propagate this or that motion to the ſenſorium, and in the mind they are the ideas, which thoſe motions excite there.

294. *The different refrangibility of light makes refracting teleſcopes imperfect.*

If we look through a refracting teleſcope at any object, the picture of this object ſhould be diſtinct in the principal focus of the eye-glaſs, by propoſition 172. But in a refracting teleſcope there will be no one diſtinct picture of the object any where. For ſuppoſe we were looking at the moon, the light, that comes from the moon, conſiſts of different ſorts of rays, and each ſort has a different degree of refrangibility. The violet rays are refracted moſt, and the red rays leaſt of all. Therefore, in paſſing through the object-glaſs, the violet rays will be made to converge more and will meet in their reſpective focuſes, or will form a diſtinct picture, nearer to the object-glaſs than the red ones. Thus if there are ſeven different ſorts of rays, inſtead of one picture of the moon, there will be ſeven; and each picture will be in a different place: the violet picture will be neareſt to the object-glaſs, the red one will be fartheſt from it, and thoſe of the intermediate colours will be between them. Now to ſee the moon perfectly through the teleſcope; the rays of all the ſeveral ſorts of colours ſhould be mixed together, and the picture compoſed of this mixture ſhould be diſtinct. But there is no diſtinct picture where all the colours are mixed. At the place of the green picture, for inſtance, that particular picture will be diſtinct, and the rays of all the colours will be mixed there: but then the picture made from the mixture will be confuſed. For the violet rays, and all thoſe, which are more refrangible than the green ones, will have met in their reſpective focuſes and will have begun to diverge from thence, before they come to the place of the green picture, and conſequently at the place of the green picture, all the other pictures of the more refrangible colours will be confuſed. The red rays, and all thoſe, which are leſs refrangible than the green ones, will be converging and will not be come to their reſpective focuſes, when they get to the place of the green picture; and conſequently, at the place of the green picture, all the other pictures of the leſs refrangible colours will be confuſed. The ſame is equally true in any other place. If the picture made by one ſort of rays is diſtinct, thoſe made by all the other ſorts will be confuſed. Therefore the compound picture will always be imperfect, and the moon, or any other object for the ſame reaſon, will be ſeen imperfectly or confuſedly through a refracting teleſcope.

295. *When two refracting teleſcopes of different lengths magnify equally, the ſhorter teleſcope of the two will be more imperfect than the longer.*

If the principal focal diſtance of the object-glaſs is 20 inches and that of the eye-glaſs is ½ inch, ſuch a teleſcope will magnify an object as much, as another teleſcope, in which the principal focal diſtance of the object-glaſs is 20 feet and that of the eye-glaſs is ½ foot, by propoſition 175. Now ſince the object appears equally magnifyed, through either teleſcope, it is plane that the diſtinct picture in each of them appears of the ſame ſize, when it is viewed through the eye-glaſs of its reſpective teleſcope. But the diſtinct picture, though it appears equal in each, is bigger in one of the teleſcopes than in the other. For the diſtinct picture in the longer teleſcope is 20 feet from the object-glaſs, and in the ſhorter is only 20 inches from it, by the conſtruction: and ſince as 20 feet is to 20 inches, ſo is 12 to 1, the diameter of the diſtinct picture in the longer teleſcope will be 12 times the diameter of the diſtinct picture in the ſhorter, by prop. 88. But each picture through its reſpective eye-glaſs appears of the ſame ſize. Therefore the picture in the ſhorter teleſcope is magnifyed 12 times as much as that in the longer: for ſince it is 12 times leſs in itſelf it could not appear of the ſame ſize, unleſs it was 12 times as much magnifyed. Both theſe pictures, which I have hitherto called diſtinct, would be ſo, if the rays of light were all equally refrangible: but as the different rays are differently refrangible, they will both be confuſed or imperfect, by propoſition 294. And this imperfection will appear the more through the eye-glaſs in proportion as the eye-glaſs magnifyes the picture more. Therefore the picture will appear more imperfect in the ſhorter teleſcope than in the longer: for which reaſon the object ſeen through the ſhorter teloſcope will be ſeen more confuſedly or more imperfectly than through the longer.

This is the reaſon referred to in propoſition 175, why refracting teleſcopes that magnify an object very much will be uſeleſs if they are too ſhort.

296. *Reflecting teleſcopes are more perfect than refracting ones.*

We have ſeen, in propoſition 294, that the picture of an object produced by a refracting object-glaſs will be imperfect; upon account of the different refrangibility of light. But a picture produced by a reflecting mirrour, will not be liable to the ſame imperfections. For though light, by propoſition 283, conſiſts of rays that are differently reflexible, yet reflection will never ſeparate theſe rays from one another, unleſs where ſome of them are reflected, whilſt others are tranſmitted in the manner deſcribed in proving propoſition 283. But when all the rays, the red, orange, yellow, green, blue, indigo, and violet, fall together upon the concave ſurface of a poliſhed mirrour, as they all have equal angles of incidence and are all reflected, they will all have equal angles of reflection, by propoſition 201, and conſequently will all meet in a focus at the ſame diſtance.

From hence we ſee the reaſon why though refracting teleſcopes, that magnify much, muſt be long, by propoſition 295, yet reflecting teleſcopes, that magnify as much, may be ſhorter. For the picture in a refracting teleſcope is an imperfect one, and will not bear magnifying much by the eye-glaſs, as was proved in propoſition 295: but the picture in a reflecting teleſcope is more perfect; and conſequently though it is ſmall, as it will be, by propoſition 256, when the concave object-mirrour has a ſhort principal focus, yet it will bear magnifying by the eye-glaſs as much as we pleaſe.

CHAP. XXI.

Of the colours of thin tranſparent plates.

297. *Water, air, glaſs, or any other tranſparent ſubſtance, when drawn out into thin plates, become coloured.*

WATER, when it is made tenacious by having ſoap mixed with it, may be blown up into a bubble A, Plat. XVI. fig. 2, ſuch as children play with. If this bubble is ſet under a glaſs, ſo that the motion of the air may not affect it, then, as the water glides down the ſides of it, and the top of it at A grows thinner, ſeveral colours will ſucceſſively appear at A, and will ſpread themſelves from thence in rings ſurrounding A and deſcending farther and farther down the ſides of the bubble, till they vaniſh at BC in the ſame order in which they appeared. Thus, for inſtance, the firſt colour that appears at A the top of the bubble is red: this red ſpot ſpreads itſelf into a circular ring round A, and then the top of the bubble A becomes blue: this blue ſpot ſpreads itſelf in the ſame manner round A, and then A becomes red a ſecond time. Before we go on to conſider what other colours ariſe at A, we will obſerve what becomes of thoſe, which ariſe firſt. The red, which firſt appeared at A, ſpreads itſelf into a circular ring round A; this ring grows larger, as the water glides down the ſides of the bubble; ſo that the coloured ring glides down the bubble along with the water, till it ſinks at laſt to BC and there encompaſſes the bubble. In like manner the blue, which ariſes at A after the red, ſpreads itſelf and deſcends down the bubble, as the red ring did. The colour, which ariſes next at A, is red a ſecond time; this ſpreads itſelf in the ſame manner, and is ſucceeded by blue a ſecond time. Theſe are followed by a great variety of colours, which appear ſucceſſively at A and ſpread themſelves from thence in this order. Red, yellow, green, blue, purple; then again red, yellow, green, blue, violet; and laſtly, red, yellow, white, blue. This laſt blue colour is ſucceeded at A by a black ſpot, which reflects ſcarce any light: this ſpot dilates itſelf, but not into a circular ring as the colours had done; it becomes broader and broader, till the bubble breaks.

A thin plate of water of the ſame ſort with this bubble, but more laſting, may be otherwiſe procured. If a piece of plane poliſhed glaſs is placed upon the object-glaſs of a long teleſcope; as in Plat. XVI. fig. 3, the plane ſurface of one glaſs and the convex one of the other will touch one another only at a ſingle point; and if the interval between them is filled with water, as the glaſſes are preſſed together, the ſame colours ariſe at the point of contact and ſpread themſelves in circular rings round it in the ſame order, as in the ſoap-bubble. If BC, Plat. XVI. fig. 4, is a ſection of the plane glaſs, and DAE a ſection of the convex one; when they are preſſed cloſe together the thin plate of water, that fills the interval between them, will have a black ſpot at A, and this ſpot will be encompaſſed with rings of colours, in the ſame order, that they ſtand in that figure upon the line BC, on each ſide of A. If the colours are reckoned in the order, in which they ſtand on the plate of water, after the black ſpot appears at A; and we reckon them from the ſpot A towards the edges of the plate at B and C, then we muſt call blue the firſt colour. But if we reckon them in the order, in which they aroſe at A and ſpread themſelves, then we muſt begin from B or C the edges of the plate and go on towards A, and in this reckoning we muſt call red the firſt colour.

If there is no water between the two glaſſes, then the interval will be filled with air, and this thin plate of air will have the ſame colours, that the plate of water had; with this difference only, that each of the coloured rings is larger in the plate of air than in the plate of water.

When glaſs is blown very thin at a lamp-furnace; thin plates of it thus formed will exhibit colours: and ſo likewiſe will thin plates of Muſcovy-glaſs. Metals, when they are heated, ſend out to their ſurfaces ſcoria or vitrified parts, which cover the metals in form of a thin ſkin: and theſe ſcoria or thin plates cauſe colours upon the ſurface of the metal, ſuch as are made to appear on poliſhed ſteel by heating it, or on bell-metal by melting it firſt and then pouring it on the ground to cool in the air.

298. *When the thin plate is denſer than the medium that ſurrounds it, the colours are more vivid, than they are, when the plate is rarer than that medium.*

A thin bubble is a plate of water encompaſſed with air: where the ſubſtance of the plate, which is water, is denſer than the air, which is the medium that ſurrounds it. On the contrary, the plate of air between the two glaſſes BAC, DAE, Plat. XIV. fig. 4, is encompaſſed with glaſs: and here the ſubſtance of the plate is rarer than that of the circumambient medium. And the colours on the bubble of water are more vivid than thoſe on the thin plate of air.

299. *When thin tranſparent plates reflect one ſort of rays, they tranſmit the reſt.*

If the plate of air between the two glaſſes BAC, DAE, Plat. XVI. fig. 4, is viewed by reflected light, the colours of it are thoſe expreſſed on the upper part of the figure, from B to C, but if we look through it, that is, if we view it by tranſmitted light, or if the tranſmitted light falls upon a white paper, the colours that we ſee through the plate, or that fall on the paper, are thoſe expreſſed on the lower part of the figure. Now any of the tranſmitted colours are what would ariſe from a mixture of all the remaining rays, after thoſe of the reflected colour are ſeparated from the ſuns heterogeneal light. Thus for inſtance, the fourth reflected colour from the black ſpot A incluſively is yellow, the tranſmitted colour is violet. The yellow rays and ſome of the orange and green, are reflected here, ſo that the mixture of the reflected light will be yellow. The mixture of the tranſmitted light therefore will be violet or rather ſuch a purple, as is not exactly like any of the primary colours: for we obſerved, in propoſition 292, that from red rays, violet, and blue new purples may be produced.

This is the caſe in ſome natural bodies, as well as in theſe tranſparent artificial plates; for if leave gold, which is made thin enough to tranſmit light, is held againſt the ſtrong light of the ſuns rays, the gold, which is yellow, when ſeen by reflected light, will be blue, when thus ſeen by tranſmitted light.

The ſeventh reflected colour incluſively from the black ſpot is blue, the ſeventh tranſmitted colour is yellow. When the rays, which make the blue colour, are taken out of the ſuns heterogeneous light, the remaining rays will be yellow. Thus it happens likewiſe in ſome natural bodies; for an infuſion of lignum nephriticum, which is blue, when ſeen by reflected light, is yellow, when ſeen by tranſmitted light.

The black ſpot A reflects ſcarce any light; and as rays of all colours are tranſmitted there, the tranſmitted colour is white, as it ought to be by propoſition 291. The third reflected colour from the black ſpot incluſively is white. Therefore ſince, by propoſition 291, all the rays are reflected there, no colour ought to be ſeen there, when we look through the plate; and accordingly that part of the plate is black.

Hence we ſee the reaſon why if there be two liquors of full colours in two different glaſs veſſels, ſuppoſe red and blue, though each is tranſparent, when we look through it ſeparately, yet we ſhould not be able to ſee through both of them together, if one was held behind the other. For if the blue liquor for inſtance is held towards the light and the red towards the eye; ſince only blue rays paſs through the firſt liquor, and come to the ſecond; and ſince the ſecond liquor will tranſmit no blue rays, but only red ones; it follows that no rays at all can come to the eye.

Indeed ſome tranſparent bodies appear of the ſame colour, whether we ſee them by reflected or tranſmitted light. Of this ſort is moſt painted glaſs. But when this is the caſe, the coloured rays are reflected from the ſecond ſurface of the body. Thus if a piece of painted glaſs is yellow, either when ſeen by reflected light or when ſeen by tranſmitted light; all the rays, but the yellow ones, are ſuppreſſed, as they paſs through the glaſs; of theſe yellow rays moſt are tranſmitted at the ſecond ſurface, the few which are reflected from thence will be ſufficient to tinge all the light yellow, which is reflected from the firſt ſurface. This will be evident from making the body thick and pitching it on the backſide; for by this means the reflected colour will be loſt: whereas if it had been reflected from the firſt ſurface, the pitch at the ſecond ſurface could not have altered it.

300. *The more denſe the ſubſtance is, out of which a thin plate is made, the leſs is the thickneſs of the plate, where it reflects any certain colour.*

The colours are the ſame whether there is air or water between the two glaſſes BAC, DAE, only the coloured circles are ſmaller in the plate of water than in the plate of air. Thus the yellow, for inſtance, which is the fourth coloured circle from the black ſpot, is a leſs circle, or is nearer to the black ſpot, when there is a plate of water between the glaſſes, than when there is a plate of air between them. But the leſs the diſtance from the point of contact A, the cloſer the glaſſes are to one another, and conſequently the thinner will be the plate that lies between them: conſequently that part of a plate of water, where this yellow appears, is thinner than that part of a plate of water, where the ſame colour appears. And the ſame holds good in any other colour. But water is more denſe than air. Therefore the more denſe the ſubſtance is, out of which a thin plate is made, the leſs is the thickneſs of the plate, where it reflects any certain colour.

301. *The ſort of colour, which is reflected from any part of a thin plate, depends only upon the thickneſs of the plate itſelf in that part: but the ſame colour will be made leſs vivid by encreaſing the denſity of the medium, with which the plate is encompaſſed.*

The colours upon any part of a thin plate of Muſcovy-glaſs are the ſame in ſort, whether the plate is dry or wetted with water. Therefore the ſort of colour in any part depends not upon the medium that encompaſſes

paſſes the plate, but upon the thickneſs of the plate itſelf; ſince the colours are the ſame, when the plate is dry and encompaſſed with air, or wet and ſo encompaſſed with water. But the ſame colours are more faint, when the plate is wet, than when it is dry; and conſequently, the brightneſs of the colours does depend upon the medium that encompaſſes the plate, and the denſer that medium is the fainter will be the colours; they are more faint when the plate is covered with water than when it is dry and ſo is ſurrounded with air.

302. *The rays of light have alternate fits of eaſy reflection and eaſy tranſmiſſion, which return at equal intervals.*

Let GF, Plat. XVI. fig. 5, be a beam of homogeneous light conſiſting all of one ſort of rays, as ſuppoſe all the rays that compoſe the beam were red ones. Then if theſe rays fall upon a thin plate of air between the two glaſſes BAC, DAE, at A there will be a dark ſpot, and all the rays will be tranſmitted; round this ſpot there will be a red ring, where all the rays are reflected; round this red ring there will be a dark ring, where all the rays are tranſmitted. And if the thickneſs of the plate, where all the rays are reflected in the ring neareſt to A, is called 1, the thickneſs, where the dark ring appears and all the rays are tranſmitted, will be 2. Again, at that part of the plate were the thickneſs is 3, all the rays will be tranſmitted: at the thickneſs 4, they will be all reflected. And thus alternately, as expreſſed by the lines in the figure, the rays will be reflected, in all parts of the plate, where the thickneſs is expreſſed by any of the uneven numbers 1, 3, 5, 7, 9, &c. and will be tranſmitted, where the thickneſs is expreſſed by any of the even numbers 2, 4, 6, 8, 10, &c.

Now as the plate is the ſame in all parts; the cauſe of this alternate reflection and tranſmiſſion muſt be in the rays themſelves. And their diſpoſitions to be thus alternately, reflected and tranſmitted are what we call fits of eaſy reflection and eaſy tranſmiſſion.

The rays that are in a fit of eaſy reflection penetrate as far as the ſecond ſurface of the plate. For if the ſecond ſurface of a thin plate of Muſcovy-glaſs is wetted, the colours cauſed by the alternate reflection grow fainter. Whereas if the reflection was made at the firſt ſurface, wetting the ſecond could not affect the colours. But ſince thoſe rays, which have paſſed from the firſt ſurface of the plate to the ſecond, where the thickneſs of it is 1, are reflected, and thoſe, which have paſſed from the firſt ſurface to the ſecond, where the thickneſs of it is 2, are tranſmitted; and then again thoſe, which have thus paſſed from one ſurface to the other, where the thickneſs is 3 are reflected; and thoſe, which

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have paſſed in the ſame manner, where the thickneſs is 4, are tranſmitted, it follows that the fits of eaſy reflection and tranſmiſſion return at equal intervals. So that, if a ray was to ſet out from A in the line AB, fig. 6, and was to be in a fit of eaſy reflection, when it had moved from A to *c*; it would be in a fit of eaſy tranſmiſſion, when it had moved to twice that diſtance from A, or when it was got to *d*: at *e*, or the diſtance 3 from A, it will be in a fit of eaſy reflection: at *f*, or the diſtance 4, in a fit of eaſy tranſmiſſion: at *g*, or 5, in a fit of eaſy reflection: at B, or 6, in a fit of eaſy tranſmiſſion, and thus in the farther progreſs of the ray the ſame fits will return at equal intervals.

Thus if the thickneſs of the plate of air, where the rays of any homogeneous colour are all reflected is equal to A*c* or 1, and the rays are in a fit of eaſy reflection, when they come to the ſecond ſurface of the plate; then where the thickneſs of the plate is A*d* or 2, the rays, will be in a fit of eaſy tranſmiſſion, when they come to the ſecond ſurface, and conſequently, will all paſs through that ſurface. Again where the thickneſs is A*e* or 3, the rays when they come to the ſecond ſurface will be in a fit of eaſy reflection, and will all be reflected, where the thickneſs is A*f* or 4, the fit of eaſy tranſmiſſion will be returned when the rays come to the ſecond ſurface, ſo that all of them will be tranſmitted. And in like manner, by ſuch fits returning at equal intervals, the rays will be reflected, where the thickneſs is expreſſed by the numbers 1, 3, 5, 7, 9, &c. and will be tranſmitted, where it is expreſſed by 2, 4, 6, 8, 10, &c.

303. *When a thin coloured plate is viewed obliquely the colours of every part in the plate will be altered.*

When a bubble of water or a plate of air between two glaſſes BAC, DAE, Plat. XVI. fig. 4, is viewed obliquely, the coloured rings dilate themſelves: and conſequently a ring of any one colour by being dilated gets into that part of the plate, where a ring of ſome other colour appeared, when the plate was viewed directly.

304. *If the plate is denſer than the medium that encompaſſes it, the colours of it, when viewed obliquely, change leſs, than they would, if the plate was rarer than the medium that encompaſſes it.*

A bubble of water is a thin plate denſer than the air, that encompaſſes it: and a plate of air between the two glaſſes BAC, DAE, Plat. XVI. fig. 4, is rarer than the glaſs that encompaſſes it. Upon viewing each of theſe thin plates obliquely, the coloured rings on the plate of water dilate leſs than thoſe on the plate of air. Therefore ſince, by propoſition 303, it is by this dilatation of the rings, that a ring of one colour gets into a part of the plate, where a ring of ſome other colour appeared, when the plate was viewed directly; that is, ſince it is by this dilatation of the rings that the ſeveral parts of the plates change their colours, it follows, that any part of a plate of water encompaſſed with air changes colour leſs, upon being viewed obliquely, than any part of a plate of air encompaſſed with glaſs.

305. *When the medium, which encompaſſes a coloured tranſparent plate, is given, the colours change leſs upon altering the ſituation of the eye, as the ſubſtance is more denſe out of which that plate is made.*

The matter, out of which a bubble of water is made, is not ſo denſe, as that out of which a bubble of glaſs is made, and glaſs is not ſo denſe as the ſcoria or glaſſy ſkin thrown out by metals, when they are heated. Now any of theſe plates either of water, or glaſs, or metalline ſubſtance, when they are encompaſſed with the ſame medium air, will change their colour a little upon being viewed obliquely. But the plate of water changes the moſt, the plate of glaſs leſs than that, and the ſcoria of metals leaſt of all.

CHAP. XXII.

Of the opakeneſs, tranſparency and colours of natural bodies.

306. *The opakeneſs of bodies is owing to the many reflections and refractions, which the rays of light ſuffer within thoſe bodies.*

THE ſmalleſt parts of almoſt all natural bodies are tranſparent, as will readily be granted by thoſe, who have been uſed to look through microſcopes. A piece of leaf-gold is tranſparent, if it is held up againſt the hole of a window-ſhutter in a dark room. And any other ſubſtance, however opake it may ſeem in the open air, will appear tranſparent by the ſame means, when it is made of a ſufficient thinneſs. Even metals become tranſparent, if they are diſſolved in a proper menſtruum, as gold in aqua regia, or ſilver in aqua fortis, and by being thus diſſolved are reduced to very ſmall particles. But ſince even in opake bodies every ſingle particle tranſmits light or is tranſparent, the whole would likewiſe tranſmit light, unleſs the rays, when they are to paſs through all the particles, which make up the whole, were ſo turned out of the way by in-

innumerable refractions and reflections, as to be ſtopped and ſuppreſſed in their paſſage. That this is the reaſon why bodies, that conſiſt of tranſparent particles, ſhould be opake, is evident; ſince opake bodies, when they are reduced to a ſufficient thinneſs become tranſparent: for then there will be but few particles lying beyond one another for the light to paſs through; and as the rays will ſuffer fewer refractions and reflections, ſome of them may get through a thin plate, though all of them would be ſuppreſſed in a thicker maſs of the ſame ſubſtance.

307. *The medium, with which the pores of opake bodies are filled, is not of the ſame denſity with the particles of thoſe bodies.*

Bodies conſiſt of tranſparent particles, and their opakeneſs is owing to the many reflexions and refractions, which the light ſuffers within them, by propoſition 306. Now if the interſtices between the particles of any body were filled with a medium of the ſame denſity with the particles, the light would neither be refracted nor reflected, as it paſſed out of the particles into the interſtices and out of the interſtices into the pores, by propoſitions 37, 38, 200, but would paſs through the body, and the body would be tranſparent. Conſequently in an opake body, where the light is ſuppreſſed by the refractions and reflections, which it ſuffers, the particles, that compoſe the body, and the medium, that fills the pores or interſtices between the particles, muſt be of different denſities.

Hence we may ſee the reaſon why paper when it has been dipped in water or oyl, is more tranſparent, than when it is dry. For when the paper is thoroughly wetted with water or oyl, the pores of it are filled with a medium that is nearly of the ſame denſity with its particles. On the contrary, though oyl of turpentine and water are both of them tranſparent, when they are ſeparate, yet if they are ſhaken together ſo as to mix but imperfectly, the mixture becomes much leſs tranſparent. Becauſe the parts of each fluid are ſeparated from one another, and thoſe of the other fluid, which are of a different denſity, get in between them.

308. *The parts of bodies and their interſtices muſt not be leſs than of a certain definite bigneſs to render them opake and coloured.*

The moſt opake bodies become tranſparent, when their particles are ſubtilly divided. As metals ſuch as gold or ſilver, which are opake in large maſſes, become tranſparent, when the former is diſſolved in aqua regia and the latter in aqua fortis. And, in propoſition 297, we obſerved that at the top of a bubble of water, where the water is extremely thin, there is a black ſpot, which reflects ſcarce any light at all; though the water is encompaſſed with air, which is a medium of a different denſity. Conſequently if the diameter of the particles, of which any natural ſubſtance conſiſts, was no greater than the thickneſs of the bubble, where it reflects no light, but tranſmits all, ſuch a body would be tranſparent notwithſtanding the interſtices, that are between its particles, were filled with a medium, the denſity of which is different from theirs.

In like manner, we obſerved, that when a thin plate of air lies between two pieces of glaſs BAC, DAE, Plat. XVI. fig. 4, there is a dark ſpot, which reflects no light and tranſmits all, not only at the point A where the glaſſes touch one another, but alſo round that point to ſome diſtance where the glaſſes are very near to one another. From hence we may conclude that though the particles of any natural ſubſtance were as denſe as glaſs, and the medium which fills their interſtices was as rare as air, yet if theſe interſtices were no bigger than the interval between the two glaſſes BAC, DAE, at that place where all light is tranſmitted, ſuch a body would be tranſparent.

The tranſparency of water ſeems to be owing to the cauſes here mentioned, to the ſmallneſs of its parts, or of its pores, or of both. For we are ſure that the pores of water are filled with air, becauſe the air may be drawn out from the water in an air pump; and conſequently, as the pores are filled with a medium of a different denſity from the parts, the mixture ought to be opake, like ſuch a mixture of water and oyl of turpentine, as was mentioned in propoſition 307. But the ſmallneſs either of the parts, or of the interſtices, or of both will prevent the mixture from being opake.

Since therefore all bodies will be tranſparent, if either their parts or their interſtices are too ſmall, it follows that the parts and likewiſe the pores of ſuch bodies, as are not tranſparent but opake and coloured, muſt not be leſs than of a certain and determinate bigneſs.

309. *The colours of natural bodies depend upon the ſize of their particles.*

Different parts of thin tranſparent plates, according to the different thickneſs of them, are of different colours, by propoſition 297. Now if any part of ſuch a thin plate of glaſs, for inſtance, where it appears of one uniform colour, ſhould be ſplit into threads or broken into ſmall particles; all theſe particles would make a heap of powder of the ſame colour. And the ſmall particles of natural bodies, ſince they are tranſparent, like ſo many fragments of a thin plate, muſt exhibit colours in the ſame manner.

310. *The parts of bodies, on which their colours depend, are much denſer than the medium, which fills their pores.*

For where the tranſparent plate or particle conſiſts of a rarer ſubſtance than the medium, that encompaſſes it, the colours are leſs vivid than thoſe of natural bodies commonly are, by propoſition 289. For this reaſon it is that the colours of ſilks or cloths, when they are wetted with oyl or water, become more faint; becauſe theſe liquors are more nearly of the ſame denſity with the particles, than the medium is, which fills the interſtices when they are dry. Beſides, the colours upon a tranſparent plate change very ſenſibly, unleſs the plate conſiſts of a ſubſtance much denſer than the medium that encompaſſes it, by propoſitions 304, 305. But moſt natural bodies are of the ſame colour in whatever poſition of the eye they are viewed. Therefore their tranſparent particles, upon which their colours depend, are much denſer than the medium, which encompaſſes thoſe particles or fills the interſtices between them.

Nor is the caſe otherwiſe even in thoſe bodies which do change colour upon being viewed obliquely, ſuch as changeable ſilks, or the feathers of a peacocks tail or of a pidgeons neck. For this change of colour, upon the ſituation of the eye being changed, is no reaſon for concluding that the medium, which fills the interſtices or pores, is more nearly of the ſame denſity with the particles, upon which the colours depend, in theſe bodies than in others. Since the change of colour is plainly owing to our ſeeing a different part of the body in different poſitions of the eye. Thus in changeable ſilks the warp is of one colour and the woof of another, and in one poſition of the eye more of the warp is ſeen, and in another poſition of it more of the woof is ſeen. In like manner, if a pidgeons neck appears blue in one poſition of the eye, and crimſon in another, it is becauſe in theſe different poſitions we ſee different parts of the ſame feathers.

311. *We cannot from the colour of a body make any conjecture about the ſize of the particles, upon which its colours depend.*

Suppoſe, from the appearance of the colour in any yellow body, that we had determined its yellow to be of the ſame ſort with that, which is next to the black ſpot in a plate of air or water, or glaſs. The thickneſs of a plate, where it appears of this colour, is different according to the different denſity of the ſubſtance out of which that plate is made, by propoſition 300. The thickneſs of a plate of air, where it appears of this colour, is greater than that of a plate of water, where it appears of the ſame colour, and much greater ſtill than that of a plate of glaſs. Suppoſe

PLATE XVI.

Fig. 1.

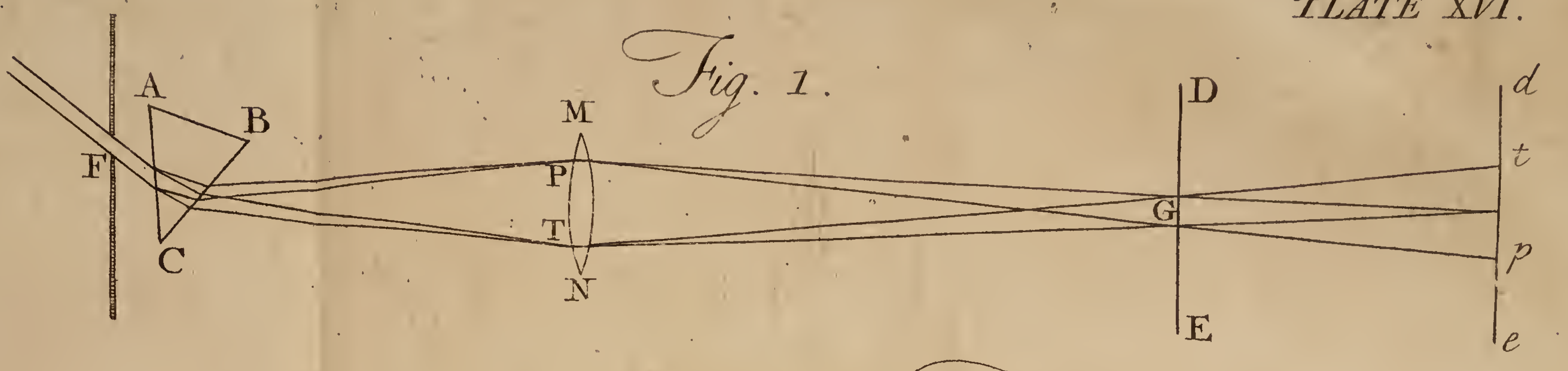


Fig. 2.

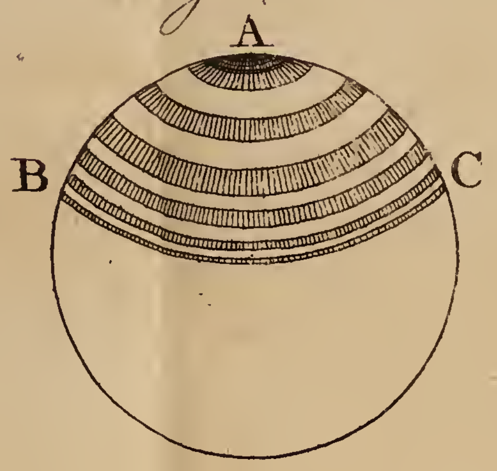


Fig. 3.

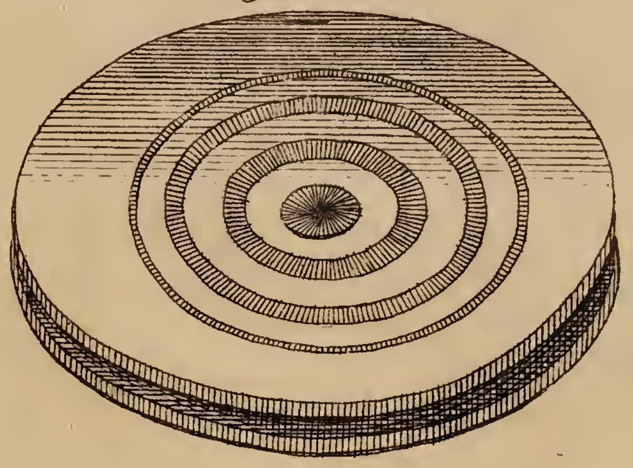


Fig. 4.

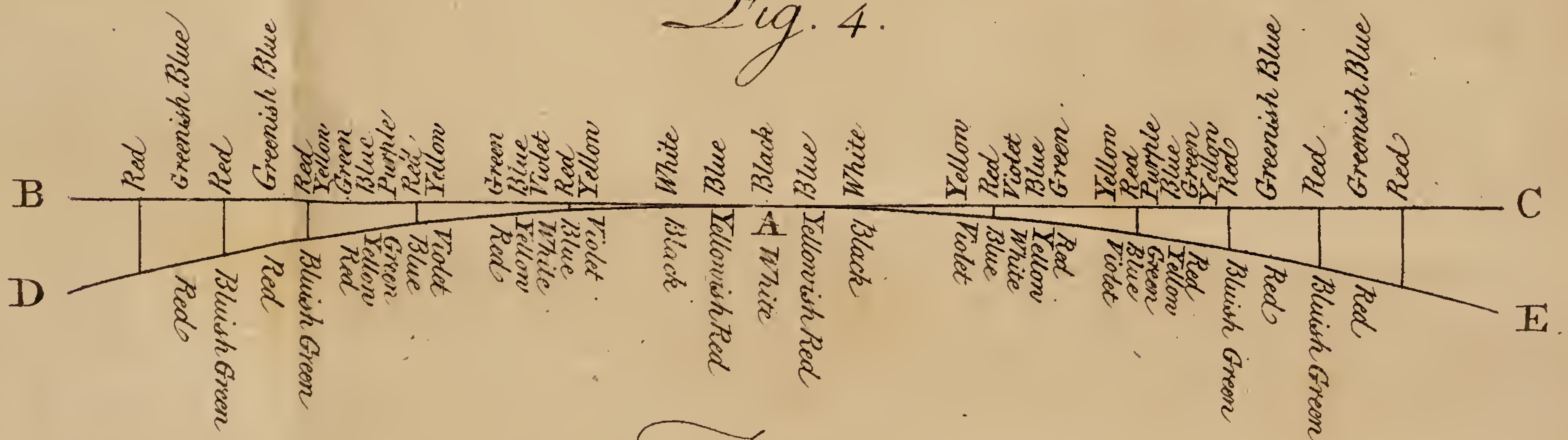


Fig. 5.

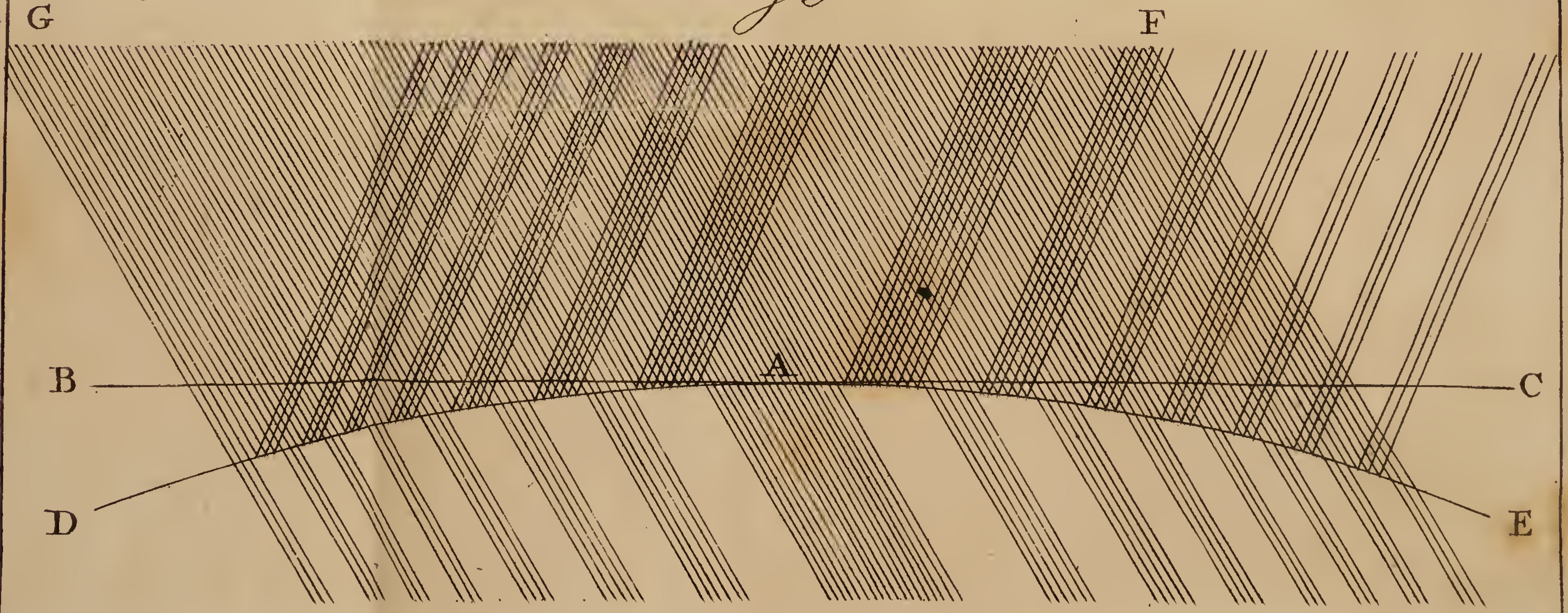
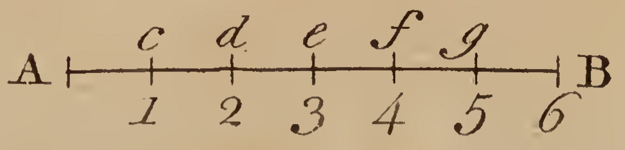


Fig. 6.



Cœlavit Gul. Stephens

poſe therefore farther, that we were able to determine exactly what is the thickneſs of a plate of air or water or glaſs, where each of them is tinged with the ſame yellow colour, that any natural body exhibits: yet we cannot determine whether the diameter of the particles, upon which this bodys colour depends, is equal to the thickneſs of the plate of air, or of water, or of glaſs, unleſs we could firſt determine whether the denſity of thoſe particles is equal to the denſity of air, or to that of water, or to that of glaſs: ſince the particles muſt be larger, if their denſity is equal to the denſity of air, than if it is equal to the denſity of water, and larger if it is equal to that of water, than if it is equal to that of glaſs. And indeed we have good reaſon to conclude that the denſity of the parts, upon which the colours of natural bodies depend, is greater even than that of glaſs; and conſequently that the diameter of thoſe parts is much leſs than the thickneſs of a plate of glaſs, where it appears of the ſame colour with the body. For upon being viewed obliquely thin plates of glaſs change colour, whereas natural bodies do not: and the colour of natural bodies is made more unchangeable than that of thin plates of glaſs by their particles being more denſe than glaſs is, by propoſitions 305, 310.

CHAP. XXIII.

Of the rainbow.

312. *When the rays of the ſun fall upon a drop of rain and enter into it, ſome of them after one reflection and two refractions may come to the eye of a ſpectator, who has his back towards the ſun and his face towards the drop.*

IF XY, Plat. XVII. fig. 1, is a drop of rain, and the ſun ſhines upon it in any lines *sf*, *sd*, *sa*, &c. moſt of the rays will enter into the drop; ſome few of them only will be reflected from the firſt ſurface, by propoſitions 198, 200. Thoſe rays, which are reflected from thence, do not come under our preſent conſideration, becauſe they are never refracted at all. The greateſt part of the rays then enter the drop, and thoſe paſſing on to the ſecond ſurface will moſt of them be tranſmitted through the drop, by propoſitions 198, 200. But neither do thoſe rays, which are thus tranſmitted, fall under our preſent conſideration, ſince they are not reflected. For the rays, which are deſcribed in the propoſition, are ſuch, as are twice refracted and once reflected. However, at the ſecond ſurface, or hinder part of the drop, at *pg* ſome few rays will be reflected, by propoſitions 198, 200, whilſt the reſt are tranſmitted: thoſe rays proceed in ſome ſuch lines as *nr*, *nq*, and coming out of the drop in the lines *rv*, *qt*, may fall upon the eye of a ſpectator, who is placed any where in thoſe lines, with his face towards the drop, and conſequently with his back towards the ſun, which is ſuppoſed to ſhine upon the drop in the lines *sf*, *sd*, *sa*, &c. Theſe rays are twice refracted and once reflected; they are refracted, when they paſs out of the air into the drop, they are reflected from the ſecond ſurface, and are refracted again, when they paſs out of the drop into the air.

313. *When rays of light reflected from a drop of rain come to the eye, thoſe are called effectual, which are able to excite a ſenſation.*

314. *When rays of light come out of a drop of rain, they will not be effectual, unleſs they are parallel and contiguous.*

There are but few rays, that can come to the eye at all, by propoſition 312. For the greateſt part of thoſe rays which enter the drop *xy*, Plat. XVII. fig. 1, between *x* and *a*, paſs out of the drop through the hinder ſurface *pg*; only few are reflected from thence and come out through the nearer ſurface between *a* and *y*. Now ſuch rays as emerge, or come out of the drop, between *a* and *y* will be ineffectual, unleſs they are parallel to one another as *rv* and *qt* are: becauſe ſuch rays as come out diverging from one another, will be ſo far aſunder, when they come to the eye, that all of them cannot enter the pupill; and the very few, that can enter it, will not be ſufficient to excite any ſenſation. But even rays, which are parallel, as *rv*, *qt*, will not be effectual, unleſs there are ſeveral of them contiguous or very near to one another. The two rays *rv* and *qt* alone will not be perceived, though both of them enter the eye: for ſo very few rays are not ſufficient to excite a ſenſation.

315. *When rays of light come out of a drop of rain after one reflection, thoſe will be effectual, which are reflected from the ſame point, and which entered the drop near to one another.*

Any rays as *sb* and *cd*, Plat. XVII. fig. 2, when they have paſſed out of the air into a drop of water will be refracted towards the perpendiculars *bl*, *dl*, by propoſition 37; and as the ray *sb* falls farther from the axis *av* than the ray *cd*, *sb* will be more refracted than *cd*, as was ſhewn in a ſimilar caſe in propoſition 57; ſo that theſe rays, though parallel to one another at their incidence may deſcribe the lines *be*, and *de* after refraction, and be both of them reflected from one and the ſame point *e*. Now all rays, which are thus reflected from one and the ſame point, when they have deſcribed the lines *ef*, *eg*, and after reflection emerge at *f* and *g*, will be ſo refracted, when they paſs out of the drop into the air, as to deſcribe the lines *fh*, *gi*, parallel to one another. If theſe rays were to return from *e* in the lines *eb*, *ed*, and were to emerge at *b* and *d*, they would be refracted into the lines of their incidence *bs*, *dc* by propoſition 39. But if theſe rays, inſtead of being returned in the lines *eb*, *ed*, are reflected from the ſame point *e* in the lines *eg*, *ef*, the lines of reflection *eg* and *ef* will be inclined both to one another and to the ſurface of the drop, juſt as much as the lines *eb* and *ed* are. Firſt *eb* and *eg* make juſt the ſame angle with the ſurface of the drop, for the angle *bex*, which *eb* makes with the ſurface of the drop, is the complement of incidence, and the angle *gey*, which *eg* makes with the ſurface, is the complement of reflection; and theſe two are equal to one another, by propoſition 201. In the ſame manner we might prove that *ed* and *ef* make equal angles with the ſurface of the drop. Secondly the angle *bed* is equal to the angle *feg*, or the reflected rays *eg*, *ef*, and the incident rays *be*, *de* are equally inclined to each other. For the angle of incidence *bel* is equal to the angle of reflection *gel*, and the angle of incidence *del* is equal to the angle of reflection *fel*, by propoſition 201. Conſequently the difference between the angles of incidence is equal to the difference between the angles of reflection, or $bel-del=gel-fel$ or $bed=gef$. Since therefore either the lines *eg*, *ef* or the lines *eb*, *ed* are equally inclined both to one another and to the ſurface of the drop; the rays will be refracted in the ſame manner, whether they were to return in the lines *eb*, *ed*, or are reflected in the lines *eg*, *ef*. But if they were to return in the lines *eb*, *ed*, the refraction, when they emerge at *b* and *d* would make them parallel. Therefore if they are reflected from one and the ſame point *e* in the lines *eg*, *ef*, the refraction, when they emerge at *g* and *f*, will likewiſe make them parallel.

But though ſuch rays, as are reflected from the ſame point in the hinder part of a drop of rain, are parallel to one another, when they emerge, and ſo have one condition that is requiſite towards making them effectual; yet there is another condition neceſſary: for, by propoſition 314, rays, that are effectual, muſt be contiguous, as well as parallel. And though rays, which enter the drop in different places, may be parallel, when they emerge, thoſe only will be contiguous, which enter it nearly at the ſame place.

Let *xy*, fig. 1, be a drop of rain, *ag* the axis or diameter of the drop and *sa* a ray of light, that comes from the ſun and enters the drop at the point *a*. This ray *sa*, becauſe it is perpendicular to both the ſurfaces, will paſs ſtrait through the drop in the line *agh* without being refracted, by propoſition 41. But any collateral rays that fall about *sb*, as they paſs through the drop, will be made to converge to their axis, and paſſing out at *n* will meet the axis at *h*, by propoſition 82. Rays which fall farther from the axis than *sb*, ſuch as thoſe which fall about *sc*, will likewiſe be made to converge; but then their focus will be nearer to the drop than *h*, by propoſition 57. Suppoſe therefore *i* to be the focus to which the rays that fall about *sc* will converge, any ray *sc*, when it is has deſcribed the line *co* within the drop and is tending to the focus *i*, will paſs out of the drop at the point *o*. The rays, that fall upon the drop about *sd* more remote ſtill from the axis, will converge to a focus ſtill nearer than *i*, as ſuppoſe at *k*, by propoſition 57. Theſe rays therefore go out of the drop at *p*. The rays, that fall ſtill more remote from the axis as *se*, will converge to a focus nearer than *k*, as ſuppoſe at *l*; and the ray *se*, when it has deſcribed the line *eo* within the drop and is tending to *l*, will paſs out at the point *o*. The rays, that fall ſtill more remote from the axis, will converge to a focus ſtill nearer. Thus the ray *sf* will after refraction converge to a focus at *m*, which is nearer than *l*, and having deſcribed the line *fn* within the drop, it will paſs out at the point *n*. Now here we may obſerve that as any rays *sb* or *sc*, fall farther above the axis *sa*, the points *n*, or *o*, where they paſs out behind the drop, will be farther above *g*, or that as the incident ray riſes from the axis *sa*, the arc *gno* encreaſes: till we come to ſome ray *sd*, which paſſes out of the drop at *p*, and this is the higheſt point where any ray, that falls upon the quadrant or quarter *ax*, can paſs out: for any rays *se*, or *sf*, that fall higher than *sd*, will not paſs out in any point above *p*, but at the points *o*, or *n*, which are below it. Conſequently though the arc *gnop* encreaſes, whilſt the diſtance of the incident ray from the axis *sa* encreaſed, till we came to the ray *sd*; yet afterwards the higher the ray falls above the axis *sa*, this arc *pong* will decreaſe.

We have hitherto ſpoken of the points on the hinder part of the drop, where the rays paſs out of it; but this was for the ſake of determining the points from whence thoſe rays are reflected, which do not paſs out behind the drop. For, in explaning the rain-bow, we have no farther reaſon to conſider thoſe rays, which go through the drop; ſince they can never come to the eye of a ſpectator placed any where in the lines *rv* or *qt* with his face towards the drop. Now as there are many rays, which paſs out of the drop between *g* and *p*, ſo, by propoſition 200, ſome few rays will be reflected from thence; and conſequently the ſeveral points between *g* and *p*, which are the points, where ſome of the rays paſs out of the drop, are likewiſe the points of reflection for the reſt, which do not paſs out. Therefore in reſpect of thoſe rays, which are reflected, we may call *gp* the arc of reflection, and may ſay that this arc of reflection encreaſes, as the diſtance of the incident ray from the axis *sa* encreaſes, till we come to the ray *sd*, the arc of reflection is *gn* for the ray *sb*, it is *go* for the ray *sc*, and *gp* for the ray *sd*. But after this, as the diſtance of the incident ray from the axis *sa* encreaſes, the arc of reflection decreaſes; for *og* leſs than *pg* is the arc of reflection for the ray *se*, and *ng* is the arc of reflection for the ray *sf*.

From hence it is obvious, that ſome one ray, which falls above *sd*, may be reflected from the ſame point with ſome other ray, which falls below *sd*. Thus for inſtance the ray *sb* will be reflected from the point *n*, and the ray *sf* will be reflected from the ſame point; and conſequently, when the reflected rays *nr*, *nq* are refracted as they paſs out of the drop at *r*, and *q*, they will be parallel, by what has been ſhewn in the former part of this propoſition. But ſince the intermediate rays, which enter the drop between *sf* and *sb*, are not reflected from the ſame point *n*, theſe two rays alone will be parallel to one another, when they come out of the drop; and the intermediate rays will not be parallel to them. And conſequently theſe rays *rv*, *qt*, though they are parallel, after they emerge at *r* and *q*, will not be contiguous, and for that reaſon will not be effectual, by propoſition 314. The ray *sd* is reflected from *p*, which has been ſhewn to be the limit of the arc of reflection; ſuch rays, as fall juſt above *sd* and juſt below *sd*, will be reflected from nearly the ſame point *p*, as appears from what has been already ſhewn. Theſe rays therefore, will be parallel, becauſe they are reflected from the ſame point *p*, and they will likewiſe be contiguous, becauſe they all of them enter the drop at one and the ſame place very near to *d*. Conſequently ſuch rays, as enter the drop at *d* and are reflected from *p* the limit of the arc of reflection, will be effectual, by propoſition 314, ſince when they emerge at the fore part of the drop between *a* and *y* they will be both parallel and contiguous.

If we can make out hereafter that the rain bow is produced, by the rays of the ſun, which are thus reflected from drops of rain, as they fall whilſt the ſun ſhines upon them, this propoſition may ſerve to ſhew us that this appearance is not produced by any rays, that fall upon any part and are reflected from any part of thoſe drops: ſince this appearance cannot be produced by any rays but thoſe, which are effectual; and effectual rays muſt always enter each drop at one certain place in the fore-part of it, and muſt likewiſe be reflected from one certain place in the hinder ſurface.

316;

316. *When rays, that are effectual, emerge from a drop of rain after one reflection and two refractions; thoſe, which are moſt refrangible, will, at their emerſion, make a leſs angle with the incident rays, than thoſe do, which are leaſt refrangible: and by this means the rays of different colours will be ſeparated from one another.*

Let *fh* and *gi*, Plat. XVII. fig. 2, be effectual violet rays emerging from the drop at *fg*, and *fn*, *gp*, effectual red rays emerging from the ſame drop at the ſame place. Now though all the violet rays are parallel to one another by propoſition 314, becauſe they are ſuppoſed effectual; and though all the red rays are likewiſe parallel to one another for the ſame reaſon; yet the violet rays will not be parallel to the red rays. Theſe rays, as they have different colours, and different degrees of refrangibility, will diverge from one another, any violet ray *gi*, which emerges at *g*, will diverge from any red ray *gp*, which emerges at the ſame place. Now both the violet ray *gi* and the red ray *gp*, as they paſs out of the drop of water into the air, will be refracted from the perpendicular *lo*. But the violet ray is more refrangible than the red one, by propoſition 282, and for that reaſon *gi*, or the refracted violet ray, will make a greater angle with the perpendicular than *gp* the refracted red ray; or the angle *igo* will be greater than the angle *pgo*. Suppoſe the incident ray *sb* to be continued in the direction *sk*, and the violet ray *ig* to be continued backwards in the direction *ik*, till it meets the incident ray at *k*. Suppoſe likewiſe the red ray *pg* to be continued backwards in the ſame manner, till it meets the incident ray at *w*. The angle *iks* is that which the violet ray, or moſt refrangible ray at its emerſion, makes with the incident ray: and the angle *pws* is that which the red ray, or leaſt refrangible ray at its emerſion, makes with the incident ray. The angle *iks* is leſs than the angle *pws*. For, in the triangle *gwk*, *gws* or *pws* is the external angle at the baſe, and *gkw* or *iks* is one of the internal oppoſite angles: and either internal oppoſite angle is leſs than the external angle at the baſe. Euc. b. I. prop. 16. What has been ſhewn to be true of the rays *gi* and *gp* might be ſhewn in the ſame manner of the rays *fh* and *fn*, or of any other rays that emerge reſpectively parallel to *gi* and *gp*. But all the effectual violet rays are parallel to *gi*, and all the effectual red rays are parallel to *gp*, by propoſition 314. Therefore the effectual violet rays at their emerſion make a leſs angle with the incident ones than the effectual red ones. And for the ſame reaſon in all the other ſorts of rays thoſe, which are moſt refrangible, at their emerſion from a drop of rain after one reflection, will make a leſs angle with the incident rays, than thoſe do, which are leſs refrangible.

Or otherwiſe. When the rays *gi* and *gp* emerge at the ſame point *g*, as they both come out of water into air and conſequently are refracted from a perpendicular, inſtead of going ſtrait forwards in the line *eg* continued, they will both be turned round upon the point *g* from the perpendicular *go*. Now it is eaſy to conceive that either of theſe lines might be turned in this manner upon the point *g* as upon a center, till they became parallel to *sb* the incident ray. But if either of theſe lines or rays were refracted ſo much from *go* as to become parallel to *sb*, the ray ſo much refracted would, after emerſion, make no angle with *sk*, becauſe it would be parallel to it. And conſequently that ray, which is moſt turned round upon the point *g*, or that ray, which is moſt refrangible, will after emerſion be neareſt parallel to the incident ray, or will make the leaſt angle with it. The ſame may be proved of all other rays emerging parallel to *gi* and *gp* reſpectively, or of all effectual rays; thoſe, which are moſt refrangible, will after emerſion make a leſs angle with the incident rays, than thoſe do, which are leaſt refrangible.

But ſince the effectual rays of different colours make different angles with *sk* at their emerſion, they will be ſeparated from one another: ſo that if the eye was placed in the beam *fghi* it would receive only rays of one colour from the drop *xagy*, and if it was placed in the beam *fgnp*, it would receive only rays of ſome other colour.

The angle *swp*, which the leaſt refrangible or red rays make with the incident ones, when they emerge ſo as to be effectual, is found by calculation to be 42 degrees 2 minutes. And the angle *ski*, which the moſt refrangible rays make with the incident ones, when they emerge ſo as to be effectual, is found to be 40 degrees 17 minutes. The rays, which have the intermediate degrees of refrangibility, make with the incident ones intermediate angles between 42 degrees 2 minutes and 40 degrees 17 minutes.

317. *If a line is ſuppoſed to be drawn from the center of the ſun through the eye of the ſpectator, the angle, which any effectual ray after two refractions and one reflection makes with the incident ray, will be equal to the angle, which it makes with that line.*

Let the eye of the ſpectator be at *i*, Plat. XVII. fig. 2, and let *qt* be the line ſuppoſed to be drawn from the center of the ſun through the eye of the ſpectator; the angle *git*, which any effectual ray makes with this line, will be equal to the angle *iks*, which the ſame ray makes with the incident ray *sb* or *sk*. If *sb* is a ray coming from the center of the ſun, then ſince *qt* is ſuppoſed to be drawn from the ſame point, theſe two lines, upon account of the remoteneſs of the point from whence they are drawn, may be looked upon as parallel to one another, as will eaſily appear from propoſition 30. But the right line *ki* croſſing theſe two parallel lines will make the alternate angles equal. Euc. b. I. prop. 29. Therefore *kit* or *git* is equal to *ski*.

318. *When the ſun ſhines upon the drops of rain as they are falling; the rays, that come from thoſe drops to the eye of a ſpectator after one reflection and two refractions, produce the primary rainbow.*

If the ſun ſhines upon the rain as it falls there are commonly ſeen two bows, as AFB, CHD, Plat. XVII. fig. 4, or if the cloud and rain does not reach over that whole ſide of the ſky where the bows appear, then only a part of one or of both bows is ſeen in that place, where the rain falls. Of theſe two bows the innermoſt AFB is the more vivid of the two, and this is called the primary bow. The outer part TFY of the primary bow is red, the inner part VEX is violet, the intermediate parts, reckoning from the red to the violet, are orange, yellow, green, blue, and indigo. Suppoſe the ſpectators eye to be at O, and let LOP be an imaginary line drawn from the center of the ſun through the eye of the ſpectator. If a beam of light S coming from the ſun falls upon any drop F; and the rays, that emerge at F in the line FO ſo as to be effectual, make an angle FOP of 42 degrees 2 minutes with the line LP, then theſe effectual rays make an angle of 42 degrees 2 minutes with the incident rays, by propoſition 317, and conſequently theſe rays will be red, by propoſition 316; ſo that the drop F will appear red. All the other rays, which emerge at F, and would be effectual if they fell upon the eye, are refracted more than the red ones and conſequently will paſs above the eye. If a beam of light S falls upon the drop E, and the rays, that emerge at E in the line EO ſo as to be effectual, make an angle EOP of 40 degrees 17 minutes with the line LP; then theſe effectual rays make likewiſe an angle of 40 degrees 17 minutes with the incident rays, by propoſition 317, and the drop E will appear of a violet colour, by propoſition 316. All the other rays, which emerge at E, and would be effectual if they came to the eye, are refracted leſs than the violet ones, and therefore paſs below the eye. The intermediate drops between F and E will for the ſame reaſons be of the intermediate colours.

Thus we have ſhewn why a ſet of drops from F to E, as they are falling, ſhould appear of the primary colours, red, orange, yellow, green, blue, indigo, and violet. It is not neceſſary that the ſeveral drops, which produce theſe colours, ſhould all of them fall at exactly the ſame diſtance from the eye. The angle FOP, for inſtance, is the ſame, whether the diſtance of the drop from the eye is OF, or whether it is in any other part of the line OF ſomething nearer to the eye. And whilſt the angle FOP is the ſame, the angle made by the emerging and incident rays, and conſequently the colour of the drop will be the ſame. This is equally true of any other drop. So that although in the figure the drops F and E are repreſented as falling perpendicularly one under the other, yet this is not neceſſary in order to produce the bow.

But the coloured line FE, which we have already accounted for, is only the breadth of the bow. It ſtill remains to be ſhewn, why not only the drop F ſhould appear red, but why all the other drops quite from A to B in the arc ATFYB ſhould appear of the ſame colour. Now it is evident that wherever a drop of rain is placed, if the angle, which the effectual rays make with the line LP, is equal to the angle FOP, that is, if the angle, which the effectual rays make with the incident rays, is 42 degrees 2 minutes, any of thoſe drops will be red for the ſame reaſon that the drop F is of this colour.

If FOP was to turn round upon the line OP, ſo that one end of this line ſhould always be at the eye and the other be at P oppoſite to the ſun; ſuch a motion of this figure would be like that of a pair of compaſſes turning round upon one of the legs OP with the opening FOP. In this revolution the drop F would deſcribe a circle, P would be the center and ATFYB would be an arc in this circle. Now ſince in this motion of the line and drop OF the angle made by FO with OP, that is, the angle FOP, continues the ſame; if the ſun was to ſhine upon this drop as it revolves, the effectual rays would make the ſame angle with the incident rays, in whatever part of the arc ATFYB the drop was to be, by propoſition 317. Therefore whether the drop is at A, or at T, or at Y, or at B, or wherever elſe it is in this whole arc, it would appear red as it does at F. The drops of rain, as they fall, are not indeed turned round in this manner: but then, as innumerable of them are falling at once in right lines from the cloud, whilſt one drop is at F, there will be others at Y, at T, at B, at A, and in every other part of the arc ATFYB: and all theſe drops will be red for the ſame reaſon that the drop F would have been red, if it had been in the ſame place. Therefore, when the ſun ſhines upon the rain as it falls, there will be a red arc ATFYB oppoſite to the ſun. In the ſame manner, becauſe the drop E is violet, we might prove that any other drop, which, whilſt it is falling, is in any part of the arc AVEXB, will be violet, and conſequently, at the ſame time that the red arc ATFYB appears, there will likewiſe be a violet arc AVEXB, below or within it. FE is

the

the diſtance between theſe two coloured arcs; and from what has been ſaid it follows that the intermediate ſpace between theſe two arcs will be filled up with arcs of the intermediate colours, orange, yellow, blue, green and indigo. All theſe coloured arcs together make up the primary rainbow.

319. *The primary rainbow is never a greater arc than a ſemicircle.*

Since the line LOP is drawn from the ſun through the eye of the ſpectator, and ſince P, Plat. XVII. fig. 4, as has been ſhewn in propoſition 318, is the center of the rainbow; it follows that the center of the rainbow is always oppoſite to the ſun. The angle FOP is an angle of 42 degrees 2 minutes, as was obſerved under propoſition 316; or F the higheſt part of the bow is 42 degrees 2 minutes from P the center of it. If the ſun is more than 42 degrees 2 minutes high, P the center of the rainbow, which is oppoſite to the ſun, will be more than 42 degrees 2 minutes below the horizon; and conſequently F the top of the bow, which is only 42 degrees 2 minutes from P, will be below the horizon; that is, when the ſun is more than 42 degrees 2 minutes high, no primary rainbow will be ſeen. If the ſun is ſomething leſs than 42 degrees 2 minutes high, then P will be ſomething leſs than 42 degrees 2 minutes below the horizon; and conſequently F, which is only 42 degrees 2 minutes from P, will be juſt above the horizon, that is, a ſmall part of the bow at this height of the ſun will appear cloſe to the ground oppoſite to the ſun. If the ſun is 20 degrees high, then P will be 20 degrees below the horizon; and F the top of the bow being 42 degrees 2 minutes from P will be 22 degrees 2 minutes above the horizon: therefore at this height of the ſun, the bow will be an arc of a circle, whoſe center is below the horizon; and conſequently that arc of the circle, which is above the horizon, or the bow, will be leſs than a ſemicircle. If the ſun is in the horizon, then P the center of the bow will be in the oppoſite part of the horizon, F the top of the bow will be 42 degrees 2 minutes above the horizon, and the bow itſelf, becauſe the horizon paſſes through the center of it will be a ſemicircle. More than a ſemicircle can never appear; becauſe if the bow was more than a ſemicircle, P the center of it muſt be above the horizon: but P is always oppoſite to the ſun, therefore P cannot be above the horizon, unleſs the ſun is below it: and when the ſun is ſet or is below the horizon, it cannot ſhine upon the drops of rain, as they fall, and conſequently when the ſun is below the horizon, no bow at all can be ſeen.

320, *When the rays of the ſun fall upon a drop of rain, ſome of them after two reflections and two refractions may come to the eye of a ſpectator, who has his back towards the ſun and his face towards the drop.*

If *bgw*, Plat. XVII, fig. 3, is a drop of rain, and parallel rays coming from the ſun, as *zv*, *yw*, fall upon the lower part of it, they will be refracted towards the perpendiculars *vl*, *wl*, as they enter into it, by propoſition 37, and will deſcribe ſome ſuch lines as *vh*, *wi*. At *h* and *i* great part of theſe rays will paſs out of the drop; but ſome of them will be reflected from thence in the lines *hf*, *ig*, by propoſition 200. At *f* and *g* again, great part of the rays, that were reflected thither, will paſs out of the drop, as thoſe do, which were deſcribed in propoſition 312. But theſe rays will not come to the eye of a ſpectator at *o*. However here again all the rays will not paſs out, but ſome few will be reflected from *f* and *g* in ſome ſuch lines as *fd*, *gb*, by propoſition 200, and theſe, when they emerge out of the drop of water into the air at *b* and *d*, will be refracted from the perpendiculars, and deſcribing the lines *dt*, *bo*, may come to the eye of a ſpectator, who has his back towards the ſun and his face towards the drop.

321. *Thoſe rays, which are parallel to one another after they have been once refracted and once reflected in a drop of rain, will be effectual, when they emerge after two refractions and two reflections.*

No rays can be effectual, by propoſition 314, unleſs they are contiguous, and parallel. From what was ſaid, under propoſition 315, it appears, that when rays come out of a drop of rain contiguous to one another, either after one or after two reflections, they muſt enter the drop nearly at one and the ſame place. And if ſuch rays, as are contiguous, are parallel after the firſt reflection, they will emerge parallel, and therefore will be effectual. Let *zv* and *yw* be contiguous rays, which come from the ſun and are parallel to one another, when they fall upon the lower part of the drop *bgw*, Plat. XVII. fig. 3. Suppoſe theſe rays to be refracted at *v* and *w* and to be reflected at *h* and *i*; if they are parallel to one another, as *hf*, *gi*, after this firſt reflection, then, after they are reflected a ſecond time from *f* and *g*, and refracted a ſecond time as they emerge at *d* and *b*, they will go out of the drop parallel to one another in the lines *dt* and *bo* and will therefore be effectual.

The rays *zv*, *yw*, are refracted towards the perpendiculars *vl*, *wl*, when they enter the drop, by propoſition 37, and will be made to converge, by propoſition 48. As theſe rays are very oblique, their focus will not be far from the ſurface *vw*. If this focus is at *k*, the rays, after they have paſſed the focus, will diverge from thence in the directions *kh*, *ki*, and if *ki* is the principal focal diſtance of the concave reflecting ſurface *hi*, the reflected rays *hf*, *ig*, will be parallel, by propoſition 211. Theſe rays *hf*, *ig*, are reflected again from the concave ſurface *fg*, and will meet in a focus at *e*, ſo that *ge*, will be the principal focal diſtance of this reflecting ſurface *fg*, by propoſition 210. And becauſe *hi* and *fg* are parts of the ſame ſphere the principal focal diſtances *ge*, and *ki* will be equal to one another. When the rays have paſſed the focus *e*, they will diverge from thence in the lines *ed*, *eb*: and we are to ſhew that, when they emerge at *d* and *b* and are refracted there, they will become parallel.

Now if the rays *vk*, *wk*, when they have met at *k*, were to be turned back again in the directions *kv*, *kw*, and were to emerge at *v* and *w*, they would be refracted into the lines of their incidence *vz*, *wy*, by propoſition 40, and therefore would be parallel. But ſince *ge* is equal to *ik*, as has already been ſhewn; the rays *ed*, *eb*, that diverge from *e*, fall in the ſame manner upon the drop at *d* and *b*, as the rays *kv*, *kw*, would fall upon it at *v* and *w*; and *ed eb* are juſt as much inclined to the refracting ſurface *db*, as *kv*, *kw*, would be to the ſurface *vw*. From hence it follows that the rays *ed*, *eb*, emerging at *d* and *b* will be refracted in the ſame manner, and will have the ſame direction in reſpect of one another, as *kv*, *kw*, would have. But *kv* and *kw* would be parallel after refraction. Therefore *ed* and *eb* will emerge in lines *dp*, *bo*, ſo as to be parallel to one another, and conſequently ſo as to be effectual.

322. *When rays, that are effectual, emerge from a drop of rain after two reflections and two refractions, thoſe, which are moſt refrangible, will at their emerſion make a greater angle with the incident rays, than thoſe do, which are leaſt refrangible: and by this means the rays of different colours will be ſeparated from one another.*

If rays of different colours, which, by propoſition 282, are differently refrangible, emerge at any point *b*, Plat. XVII. fig. 3, theſe rays will not be all of them equally refracted from the perpendicular. Thus if *bo* is a red ray, which is of all others the leaſt refrangible, and *bm* is a violet ray, which is of all others the moſt refrangible, when theſe two rays emerge at *b* the violet ray will be refracted more from the perpendicular *bx* than the red ray, and the refracted angle *xbm* will be greater than the refracted angle *xbo*. From hence it follows that theſe two rays after emerſion will diverge from one another. In like manner, the rays, that emerge at *d*, will diverge from one another; a red ray will emerge in the line *dp*, a violet ray in the line *dt*. So that though all the effectual red rays of the beam *bdmt* are parallel to one anothor, and all the effectual red rays of the beam *bdop* are likewiſe parallel to one another, yet the violet rays will not be parallel to the red ones, but the violet beam will diverge from the red beam. Thus the rays of different colours will be ſeparated from one another.

This will appear farther, if we conſider, what the propoſition affirms, that any violet or moſt refrangible ray will make a greater angle with the incident rays, than any red or leaſt refrangible ray makes with the ſame incident rays. Thus if *yw* is an incident ray, *bm* a violet ray emerging from the point *b*, and *bo* a red ray emerging from the ſame point; the angle, which the violet ray makes with the incident one, is *yrm*, and that, which the red ray makes with it, is *yso*. Now *yrm* is a greater angle than *yso*. For in the triangle *brs* the internal angle *brs* is leſs than *bsy* the external angle at the baſe. Euc. b. I. prop. 16. But *yrm* is the complement of *brs* or of *bry* to two right ones, and *yso* is the complement of *bsy* to two right ones. Therefore, ſince *bry* is leſs than *bsy*, the complement of *bry* to two right angles will be greater than the complement of *bsy* to two right angles; or *yrm* will be greater than *yso*.

Or otherwiſe. Both the rays *bo* and *bm*, when they are refracted in paſſing out of the drop at *b*, are turned round upon the point *b* from the perpendicular *bx*. Now either of theſe lines *bo* or *bm* might be turned round in this manner, till it made a right angle with *yw*. Conſequently that ray, which is moſt turned round upon *b* or which is moſt refracted, will make an angle with *yw* that will be nearer to a right one, than that ray makes with it, which is leaſt turned round upon *b* or which is leaſt refracted. Therefore that ray, which is moſt refracted will make a greater angle with the incident ray, than that which is leaſt refracted.

But ſince the emerging rays, as they are differently refrangible, make different angles with the ſame incident ray *yw*, the refraction, which they ſuffer at emerſion, will ſeparate them from one another.

The angle *yrm*, which the moſt refrangible or violet rays make with the incident ones, is found by calculation to be 54 degrees 7 minutes: and the angle *yso*, which the leaſt refrangible or red rays make with the incident ones, is found to be 50 degrees 57 minutes: the angles, which the rays of the intermediate colours, indigo, blue, green, yellow, and orange make with the incident rays, are intermediate angles between 54 degrees 7 minutes and 50 degrees 57 minutes.

323. *If a line is ſuppoſed to be drawn from the center of the ſun through the eye of the ſpectator; the angle, which after two refractions and two reflections any effectual ray makes with the incident ray, will be equal to the angle, which it makes with that line.*

If *yw*, Plat. XVII. fig. 3, is an incident ray, *bo* an effectual ray, and *qu* a line drawn from the center of the ſun through *o* the eye of the ſpectator; the angle *yso*, which the effectual ray makes with the incident ray, is equal to *ſon* the angle, which the ſame effectual ray makes with the line *qn*. For *yw* and *qn* conſidered as drawn from the center of the ſun are parallel, by propoſition 30, *bo* croſſes them, and conſequently makes the alternate angles *yſo*, *ſon* equal to one another. Euc. b. I. prop. 29.

324. *When the ſun ſhines upon the drops of rain as they are falling; the rays, that come from thoſe drops to the eye of a ſpectator after two reflections and two refractions, produce the ſecondary rainbow.*

The ſecondary rainbow is the outermoſt CHD, Plat. XVII. fig. 4. When the ſun ſhines upon a drop of rain H; and the rays HO, which emerge at H ſo as to be effectual, make an angle HOP of 54 degrees 7 minutes with LOP a line drawn from the ſun through the eye of the ſpectator, the ſame effectual rays will make likewiſe an angle of 54 degrees 7 minutes with the incident rays S, by propoſition 323; and the rays which emerge at this angle are violet ones, by what was obſerved under propoſition 322. Therefore, if the ſpectators eye is at O, none but violet rays will enter it: for as all the other rays make a leſs angle with OP, by propoſitions 322, 323, they will fall above the ſpectators eye. In like manner, if the effectual rays, that emerge from the drop G, make an angle of 50 degrees 57 minutes with the line OP, they will likewiſe make the ſame angle with the incident rays S, by propoſition 323, and conſequently from the drop G to the ſpectators eye at O no rays will come but red ones, by propoſition 322; for all the other rays making a greater angle with the line OP, by propoſitions 322, 323, will fall below the eye at O. For the ſame reaſon the rays emerging from the intermediate drops between H and G, and coming to the ſpectators eye at O, will emerge at intermediate angles, and therefore, by propoſition 322, will have the intermediate colours. Thus, if there are ſeven drops from H to G incluſively, their colours will be violet, indigo, blue, green, yellow, orange and red. This coloured line is the breadth of the ſecondary rainbow.

Now if HOP was to turn round upon the line OP, like a pair of compaſſes upon one of the legs OP with the opening HOP, it is plane from the ſuppoſition that, in ſuch a revolution of the drop H, the angle HOP would be the ſame, and conſequently the emerging rays would make the ſame angle with the incident ones. But in ſuch a revolution the drop would deſcribe a circle of which P would be the center and CNHRD an arc. Conſequently, ſince, when the drop is at N, or at R, or any where elſe in that arc, the emerging rays make the ſame angle with the incident ones, as when the drop is at H, the colour of the drop

PLATE XVII.

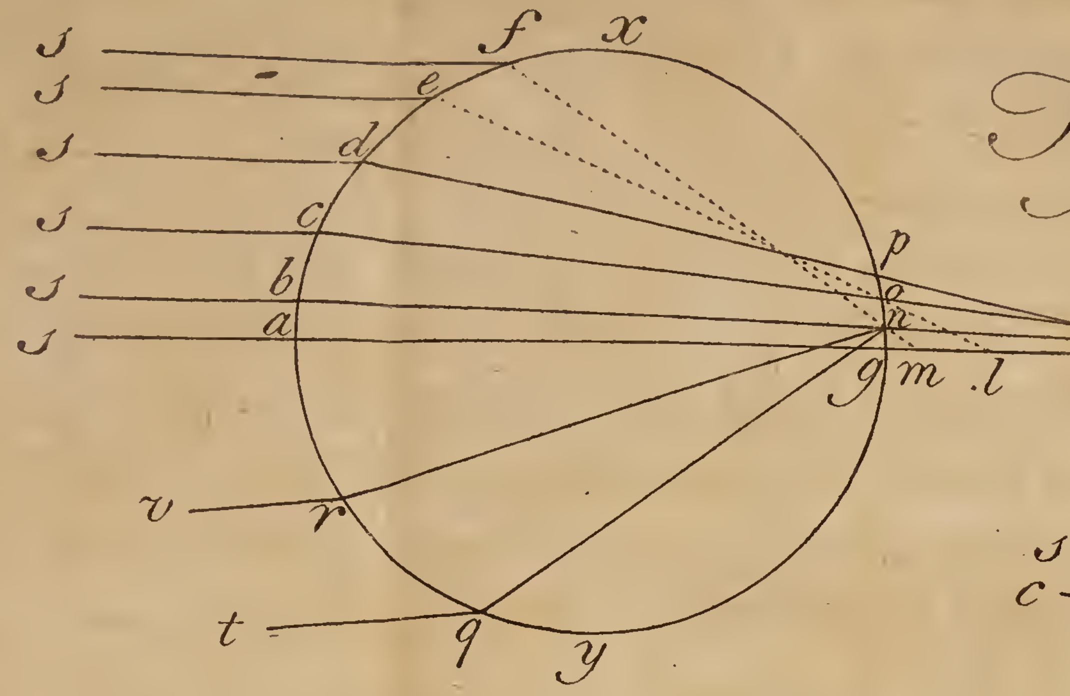


Fig. 1.

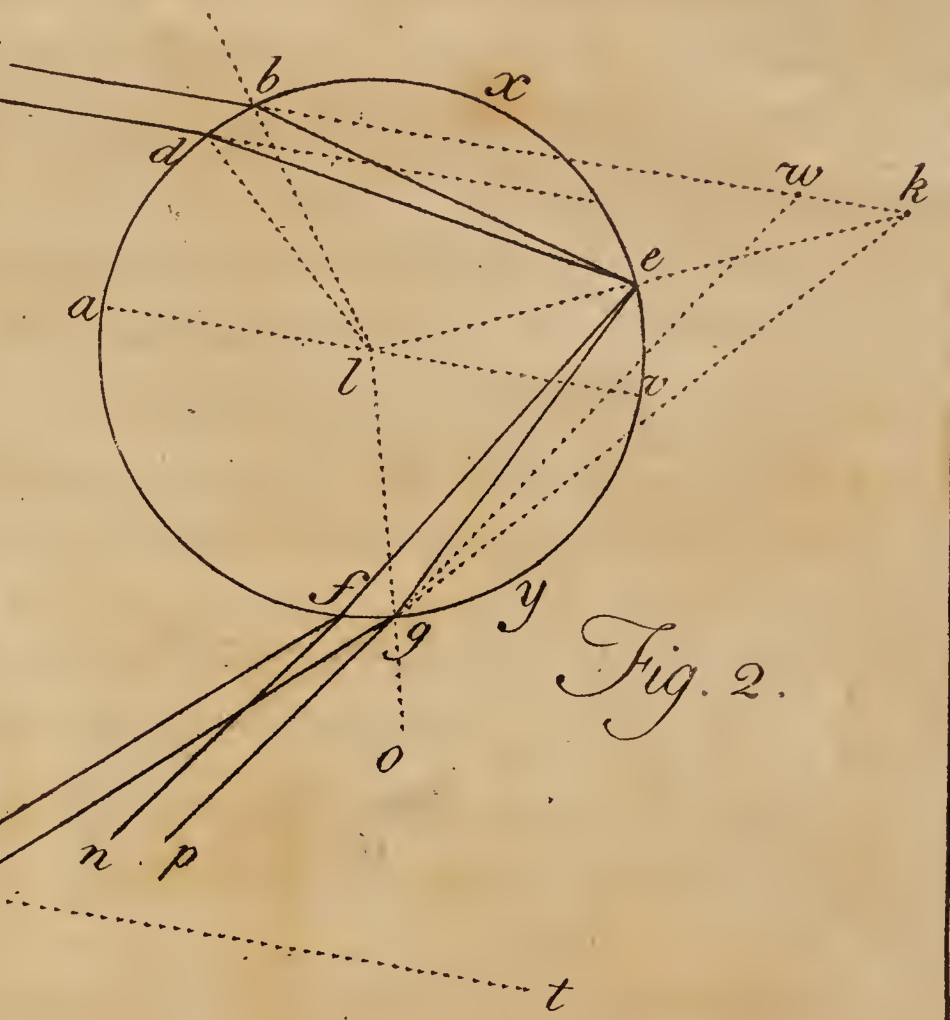


Fig. 2.

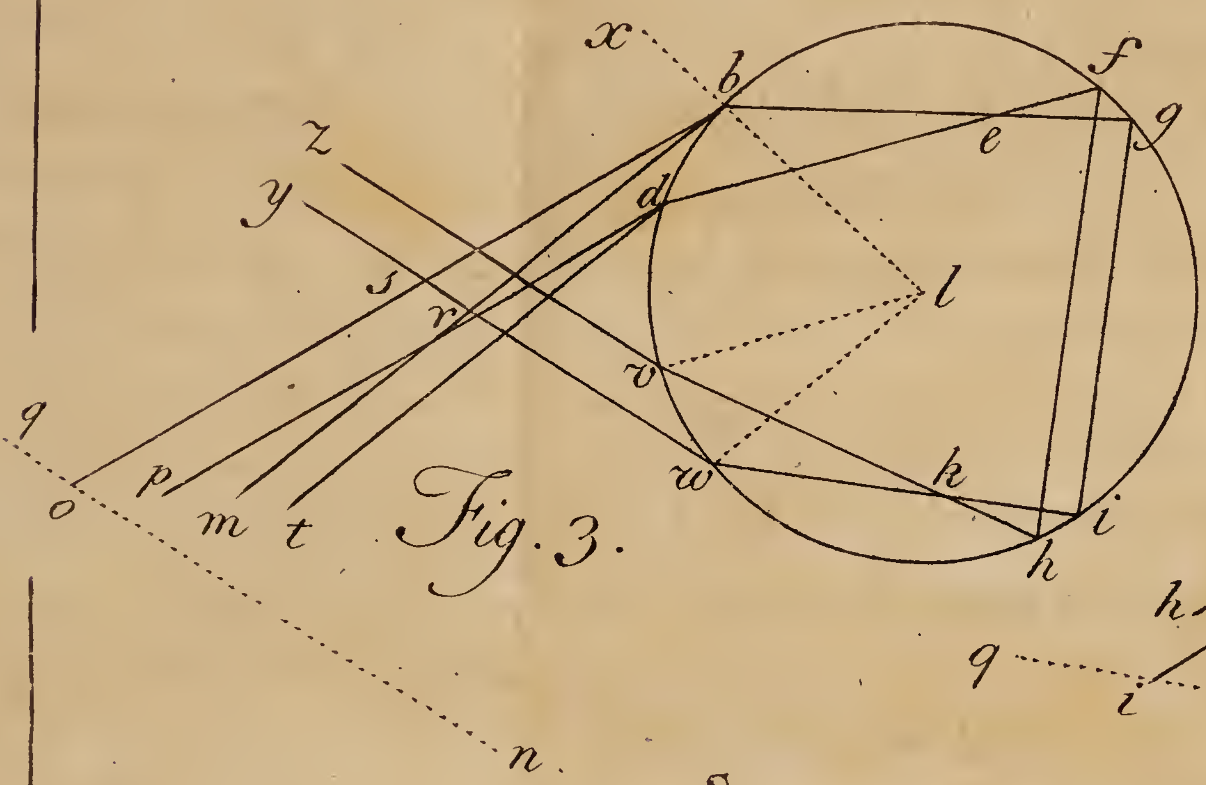


Fig. 3.

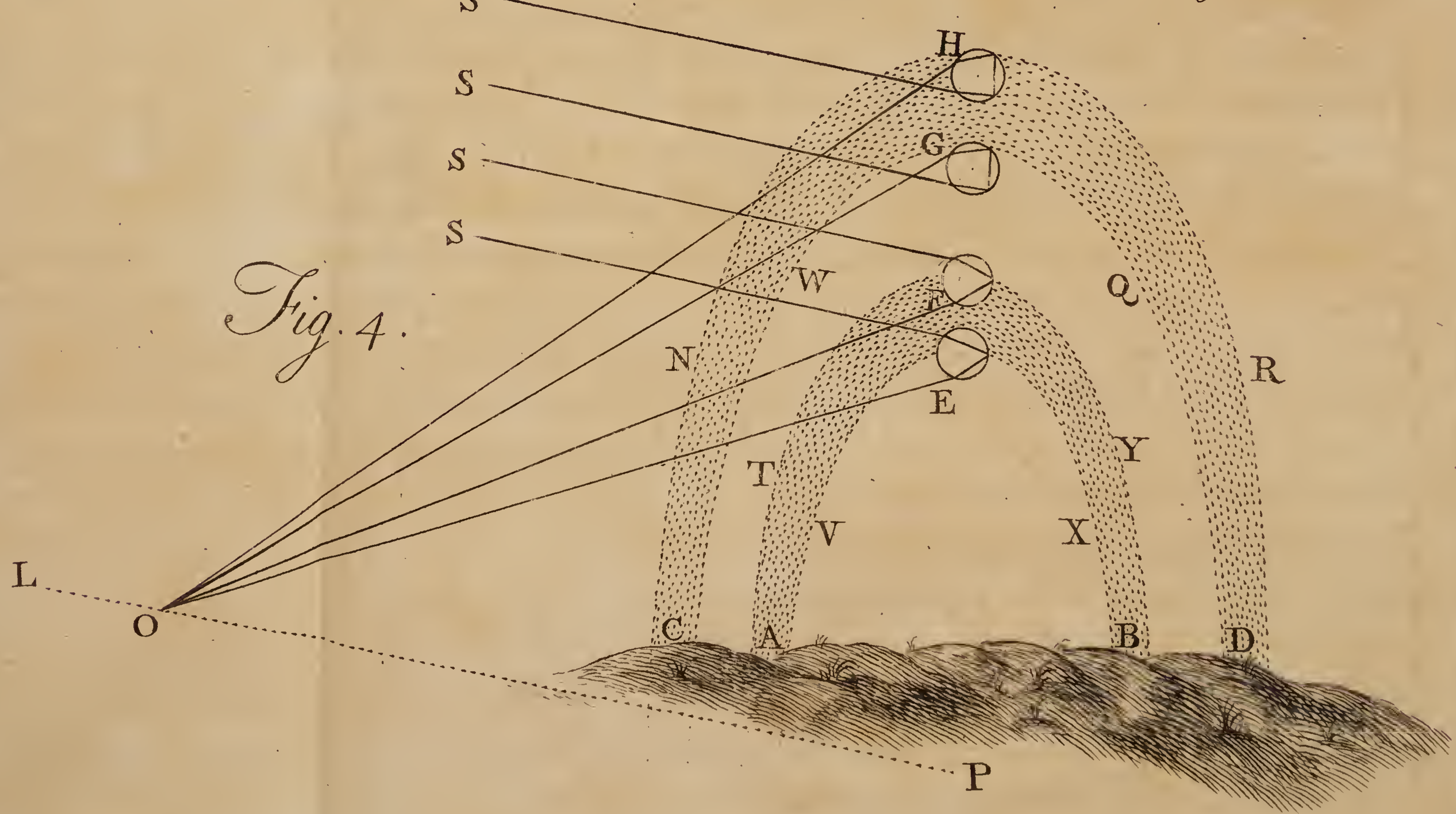


Fig. 4.

will be the ſame to an eye placed at O, whether the drop is at N, or at H, or at R, or any where elſe in that arc. Now though the drop does not thus turn round as it falls, and does not paſs through the ſeveral parts of this arc, yet, ſince there are drops of rain falling every where at the ſame time, when one drop is at H, there will be another at R, another at N, and others in all parts of the arc, and theſe drops will all of them be violet-coloured, for the ſame reaſon that the drop H would have been of this colour if it had been in any of thoſe places. In like manner as the drop G is red, when it is at G, it would likewiſe be red in any part of the arc CWGQD; and ſo will any other drop, when, as it is falling, it comes to any part of that arc. Thus as the ſun ſhines upon the rain, whilſt it falls, there will be two arcs produced, a violet-coloured one, CNHRD, and a red one CWGQD; and for the ſame reaſons the intermediate ſpace between thoſe two arcs will be filled up with arcs of the intermediate colours. All theſe arcs together make up the ſecondary rainbow.

325. *The colours of the ſecondary rainbow are fainter than thoſe of the primary rainbow; and are ranged in the contrary order.*

The primary rainbow is produced by ſuch rays, as have been only once reflected; the ſecondary rainbow is produced by ſuch rays, as have been twice reflected, by propoſitions 318, 324. But at every reflection ſome rays paſs out of the drop of rain without being reflected, by propoſition 200: ſo that the oftener the rays are reflected, the fewer of them are left. Therefore the colours of the ſecondary bow are produced by fewer rays, and conſequently will be fainter, than the colours of the primary bow.

In the primary bow, reckoning from the outſide of it, the colours are ranged in this order, red, orange, yellow, green, blue, indigo, violet. In the ſecondary bow, reckoning from the outſide, the colours are violet, indigo, blue, green, yellow, orange, red. So that the red, which is the outermoſt or higheſt colour in the primary bow, is the innermoſt or loweſt colour in the ſecondary one.

Now the violet rays, when they emerge ſo as to be effectual after one reflection, make a leſs angle with the incident rays, than the red ones, by propoſition 316; conſequently the violet rays make a leſs angle with the lines OP, Plat. XVII. fig. 4, than the red ones, by propoſition 317. But in the primary rainbow the rays are only once reflected, and the angle, which the effectual rays make with OP, is the diſtance of the coloured drop from P the center of the bow, by propoſition 318. Therefore the violet drops or violet arc in the primary bow will be nearer to the center of the bow, than the red drops or red arc; that is, the innermoſt colour in the primary bow will be violet, and the outermoſt colour will be red. And for the ſame reaſon through the whole primary bow every colour will be nearer to the center P, as the rays of that colour are more refrangible.

But the violet rays, when they emerge ſo as to be effectual after two reflections, make a greater angle with the incident rays, than the red ones, by propoſition 322; conſequently the violet rays will make a greater angle with the line OP, than the red ones, by propoſition 323. But in the ſecondary rainbow the rays are twice reflected, and the angle, which the effectual rays make with OP, is the diſtance of the coloured drop from P the center of the bow, by propoſition 324. Therefore the violet drops or violet arc in the ſecondary bow will be farther from the center of the bow, than the red drops or red arc; that is, the outermoſt colour in the ſecondary bow will be violet and the innermoſt colour will be red. And for the ſame reaſon through the whole ſecondary bow every colour will be farther from the center P, as the rays of that colour are more refrangible.

326. *The ſecondary rainbow is never a greater arc than a ſemicircle.*

This is proved in the ſame manner as propoſition 319. The only difference between the primary and ſecondary rainbows in this reſpect is this: in the primary bow the rays of the higheſt colour make an angle of 42 degrees 2 minutes with the line OP, or the higheſt colour is 42 degrees 2 minutes from P the center of the bow: and for this reaſon the bow does not begin to appear, till the ſuns height is leſs than 42 degees 2 minutes; and, when the ſun is in the horizon on one ſide, the top of the bow will be 42 degrees 2 minutes high on the oppoſite ſide of the heavens. But the rays of the higheſt colour in the ſecondary bow make an angle of 54 degrees 7 minutes with the line OP, or the higheſt colour is 54 degrees 7 minutes from P the center of the bow: for this reaſon the ſecondary bow will begin to appear, when the height of the ſun is leſs than 54 degrees 7 minutes; and, when the ſun is in the horizon on one ſide, the center of the ſecondary bow will be in the horizon and the top of the bow will be 54 degrees 7 minutes above it.

The end of optics.